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Forecasting the Polish Inflation Using Bayesian VAR Models with Seasonality

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Abstract

Bayesian VAR (BVAR) models offer a practical solution to the parameter proliferation concerns as they allow to introduce *a priori* information on seasonality and persistence of inflation in a multivariate framework. We investigate alternative prior specifications in the case of time series with a clear seasonal pattern. In the empirical part we forecast the monthly headline inflation in the Polish economy over the period 2011-2014 employing two popular BVAR frameworks: a steady-state reduced-form BVAR and just-identified structural BVAR model. To evaluate the forecast performance we use the pseudo real-time vintages of timely information from consumer and financial markets. We compare different models in terms of both point and density forecasts. Using formal testing procedure for density-based scores we provide the empirical evidence of superiority of the steady-state BVAR specifications with tight seasonal priors.

Keywords: Bayesian VAR models, seasonality, forecasting inflation, density-based scores

JEL Classification: C32, C53, E31, E37

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1 Introduction

Consumer prices are continuously under influence of various price factors including shocks from many markets (food, energy, commodities, and currency, to name the most important). Many of these price signals are observed within a current month or a quarter when the forecast is prepared. It is a common practice to take their persistent role into account by using a multivariate, dynamic framework like a vector autoregression (VAR) model. We argue that using monthly data in a regular short-term forecasting requires a careful treatment of their seasonality. Shrinkage of an autoregressive parameter space is also helpful in addressing the curse of dimensionality problem (see Bańbura *et al.*, 2010). A prior knowledge on these time-series properties of inflation brings an important input into both point and interval predictions.
Since Sims (1980) introduced VAR models into macroeconomics it took many years to develop the framework into a standard tool for a short-term forecasting (for a historical perspective see Geweke and Whiteman, 2006). The main drawback of empirical VARs relies in the increased number of parameters that need to be estimated. The excessive parameter uncertainty brings serious limitations in producing reliable out-of-sample density forecasts. In a VAR framework with seasonal time series these limitations become binding very fast. For example, at monthly data frequency a five-variate VAR model of lag order 12 requires as many as 360 parameters to be estimated including parameters at seasonal dummies and at lags of each variable. On the one hand, rich parametrization of seasonal VARs poses in-sample overfitting concerns and increases the risk of poor out-of-sample forecasting performance. On the other hand, once a big shock on any of important but omitted variables occurs then the mean and variance of the forecast could be severely affected.
To make VAR model useful in forecasting researchers introduce restrictions on parameter space in a Bayesian framework by means of prior distributions. Frequently the starting point is a reduced-form BVAR model with a system of prior distributions known as Minnesota-type priors. These ideas directly come from a seminal paper of Litterman (1979) and were further developed in Doan *et al.* (1984) and Litterman (1986). The Minnesota priors on parameters at consecutive lags of the dependent variables take the form of univariate Gaussian distributions with variances being combinations of a relatively small number of hyperparameters. The role of these hyperparameters consists of shrinking the dynamics of multivariate time series around *a priori* beliefs on the time-series properties of the data. The shrinkage for each of the variables is usually centred towards simple univariate stationary autoregressive (AR) or unit-root random walk (RW) processes.
In the applied research values of the Minnesota hyperparameters in BVAR models are commonly set by simple rules of thumb or by optimizing some out-of-sample forecasting criteria, e.g. RMSFE (root mean squared forecast error), in a training period. The most popular in this conventional approach are the following heuristic rules: harder shrinkage of VAR parameters towards zero for longer lags, scaling variances of parameters of other variables by standard deviations of errors from AR

model and decaying them faster towards zero than for their own lags. Following Litterman (1979) these priors are often combined with an assumption of a diagonal error covariance matrix with variances estimated in an 'empirical Bayes' fashion. Since then many methodological enhancements and refinements are proposed. Kadiyala and Karlsson (1993) report small improvements in forecasting accuracy from BVARs with conjugate normal-diffuse or normal-Wishart priors. The proper treatment of initial observations and inexact differencing, both using Theil mixed estimation approach, are also very popular (see Giannone *et al.*, 2014). An up-to-date review of BVAR methods and their forecasting performance for inflation and GDP in terms of mean squared error (MSE) is provided by Karlsson (2013).

The issue of Bayesian shrinkage becomes notably apparent while working with large VAR models. The more variables are included, the tighter priors should be considered (see Bańbura *et al.*, 2010). Overparameterization might also be mitigated by imposing *ad hoc* exclusion restrictions with hierarchical priors as in Bayesian Lasso by Belmonte *et al.* (2014) or by means of soft shrinkage as in stochastic search variable selection method by Koop (2013). If macroeconomic forecasting scenario focuses on a short term, the impact of possible policy changes is commonly assumed to be negligible. In the case of long-run analysis, notwithstanding extension of time-varying parameters (TVP) may be helpful. For a detailed study of TVP-BVAR model with a stochastic volatility see Primiceri (2005) with corrigendum by del Negro and Primiceri (2015), Koop (2012) and Koop (2013).

The studies of seasonal time series in a BVAR framework have been under-represented till 90s of the XXth century (see a short review in the next section). Most of the inflation forecasting exercises with BVAR models focus on regular (seasonally adjusted) or annualized inflation rates without much attention to the seasonal factors. Among not so many BVAR studies for inflation in Central and Eastern European countries rather simple frameworks are applied (e.g. Simionescu and Bilan, 2013). Some of these studies also include density forecasting (Franta *et al.*, 2014). Considering seasonal patterns in a BVAR framework is, however, quite important for obtaining accurate density and interval forecasts of monthly time series.

In this paper we investigate the ability of two popular BVAR specifications to forecast the monthly inflation in the Polish economy over the period 2011-2014. The selected approaches are: structural BVAR model with Sims and Zha (1998) prior distributions and reduced-form BVAR with steady-state assumptions on priors *à la* Villani (2009). The research is in line with current empirical studies in inflation forecasting: Giannone *et al.* (2015), Carriero *et al.* (2015), where Bayesian shrinkage is carefully explored as the cure for a curse of dimensionality.

As our target variable is Consumer Price Index (CPI) in month-over-month terms we consider different methods of introducing seasonality in these frameworks. To this end, we simultaneously apply two types of seasonality. The first one involves monthly seasonal dummies in BVAR model. We examine the role of informative vs uninformative priors for the seasonal parameters, as well as the priors on conditional seasonal means as in Sims and Zha (1998) vs unconditional (steady-state) priors as in Villani (2009). The second type of seasonality is introduced in both models via VAR lag order appropriate to the data frequency. The intuition is that these lags introduce some short-term (transitory) adjustments to a generally stable (long-run) seasonal pattern.

The selected BVAR models (described in Section 2) differ in many aspects including the treatment of seasonal factors (conditional vs unconditional means), prior distributions (more or less informative priors on seasonality in terms of mean and variance), model identification (structural vs reduced-form VAR), and the degree of shrinkage in Minnesota-type prior hyperparameters. From a theoretical point of view these approaches are hard to compare and they are not observationally equivalent. Technically, there is also an important distinction in terms of time for computing multi-period forecast distributions. The advantage of Sims and Zha (1998) priors approach is a closed approximate analytical formula for posterior distributions not like in Villani (2009) approach. The question is whether there is a forecasting performance payoff for computational difficulties and model specification burdens between the two BVAR specifications. We answer these questions in a pseudo real-time data experiments in Section 3. With data collected in the middle of each month we imitate the impact of information flowing from domestic consumption and financial markets. In Section 4 we evaluate the out-of-sample performance of forecasting models for CPI both in terms of point forecast criteria (RMSFE, MFE and MAFE) and in terms of density-based predictive scores (log score and CRPS). To show the advantages of Bayesian shrinkage in forecasting we also provide the comparisons to a simple benchmark i.e. a conventional VAR estimated by maximum likelihood method and apply Amisano and Giacomini (2007) test for that purpose. Finally, Section 5 presents the general conclusions from our forecasting experiments.

2 Forecasting models

The usefulness of VAR analysis in forecasting stems from the convenient features of a reduced-form model in which several endogenous variables are explained by lags only (own and of other variables). Firstly, in the case of the stationary time series the reduced-form VAR models are easy to estimate with standard methods (OLS or ML). Secondly, they offer a straightforward prescription for making iterated forecasts.

Introducing prior information into VAR model was postulated from the very beginning by Sims (1980). Bayes theorem is a straightforward way to challenge expert opinions on macroeconomic data with data themselves. It is a formal method of combining marginal prior distributions of parameters with likelihood function of data to obtain joint posterior distribution of parameters. Clearly, with contemporary simulation methods (MCMC) it is possible to produce multi-period density forecasts from BVAR models by drawing samples of the dependent variables from posterior (predictive) distributions (Karlsson, 2013).

The main question is how to incorporate prior beliefs into a system of unrestricted equations. Minnesota-type priors in the form of univariate RW for non-stationary variables or AR for stationary variables are the most popular basis for the informed beliefs on the time-series dynamics. There are many variations on this system of priors including different assumptions on distribution of error covariance matrix (diffuse or Wishart), initial observations or 'sum-of-coefficients' restrictions (for a review and in-deep treatment see Karlsson, 2013). The elements of Minnesota priors are also widely used in two popular BVAR frameworks: in the structural form by Sims and Zha (1998) and in the steady-state form by Villani (2009). We consider these specifications in the context of forecasting seasonal time series.
Introducing seasonality in BVAR framework is crucial in short-term inflation predictions. With predictive distribution at hand it explicitly enables to account for the part of forecast uncertainty of seasonal origin. Moreover, Bayesian methods under very general assumptions may produce interval forecasts, which are exact (not asymptotic) and consistent with *a priori* beliefs on seasonal properties of the data.
Although VAR framework is well suited for dealing with different types of seasonality in the data, only few studies follow this line of research. The forecasters (with a few exceptions) tend to adjust information set in advance and perform modelling on regular (seasonally adjusted) variables. This approach is also followed in the most elaborated study of forecasting performance of different BVAR models by Carriero *et al.* (2015). There are few important exceptions to the rule. Some of them are quite out-dated. Canova (1993) models stochastic seasonality with 'sum-of-coefficients' restrictions motivated by frequency domain approach. Raynauld and Simonato (1993) consider three types of prior distributions (all in the spirit of Minnesota-type priors): a prior with a multiplicative seasonal transformation and seasonal decay factor, RW prior with seasonal dummies, and seasonal RW. They indicate that a simple specification with seasonal dummies is supported by US macroeconomic data. This simple approach together with a lag order appropriate for the seasonal frequency (4 for quarterly data and 12 for monthly data) is repeated in many BVAR studies for inflation (e.g. Benalal *et al.*, 2004), although for monthly data it leads to an extensive number of parameters.
We start with a VAR model (1) in a structural form and a system of priors from Sims and Zha (1998) paper, henceforth S-Z priors or S-Z specification:

$$y_t' A_0 = \sum_{l=1}^{12} y_{t-l}' A_l + s_t' D + \epsilon_t' \quad \text{for each } t = 1, \ldots, T \tag{1}$$

where:

y_t – $m \times 1$ vector of endogenous variables,

A_0 – $m \times m$ full-rank, triangular contemporaneous matrix,

A_l – $m \times m$ matrices of parameters at y_{t-l} i.e. lags for $l = 1, ..., 12$,

s_t – 12×1 vector of 12 seasonal dummy variables,

D – $12 \times m$ matrices of seasonal parameters in each of m equations,

ϵ_t – $m \times 1$ vector of m independent shocks from standard Gaussian distributions $\mathrm{N}(0, 1)$.

The model (1) corresponds to a reduced-form VAR after a multiplication by an inverse of A_0 matrix. Cholesky decomposition of reduced-form error covariance matrix provides an exact identification of the VAR model. Notice that the ordering of variables in a structural VAR has some effect on the interpretation of seasonal parameters. It is hard to imagine that there is a precise prior knowledge on the distribution of seasonal factors of inflation conditional on the realizations of other variables in the system. Hence, in S-Z specification we order CPI at first place which enables us to interpret seasonal parameters independently from other contemporaneous endogenous variables.
To describe the prior distributions we write model (1) in a stacked matrix form:

$$YA_0 = X_+A_+ + E,$$

where Y and E are $T \times m$ matrices of observables and shocks, respectively, X_+ is a $T \times (12m + 12)$ matrix of observations on lagged dependent variables and seasonal dummies, and $A_+ = (A_1', \ldots, A_{12}', D')'$ is a stacked $(12m + 12) \times m$ matrix of VAR parameters.
The formula for a joint posterior distribution, $q(a|Y)$, of a vectorized stacked parameters matrix $a = vec(A)$, where $A = (A_0', A_+')'$, combines a multivariate normal likelihood function, $L(Y|a)$, with a factorized joint prior distribution:

$$q(a|Y) \propto L(Y|a) \times p(a_0) \times p(a_+|a_0) \tag{2}$$

where $p(a_0) \propto 1$ is an improper prior on $a_0 = vec(A_0)$, and $p(a_+|a_0)$ is a normal prior for right-hand side VAR parameters with a mean $\tilde{a}_+ = vec(A_0, \mathbf{0})$ and a diagonal covariance matrix $\tilde{\Psi}$. Both, prior means, being conditional on diagonal elements of A_0, and prior variances govern the shrinkage of VAR towards univariate random walks. Variances in $\tilde{\Psi}$ of parameters at lags l of variable y_{it} in equation j, $\tilde{\psi}_{a_l,i,j}$, and at seasonal dummies s_t, $\tilde{\psi}_s$, are calculated from hyperparameters $\lambda_0, \lambda_1, \lambda_3, \lambda_5$ according to:

$$\tilde{\psi}_{a_l,i,j} = \left(\frac{\lambda_0\lambda_1}{\sigma_i l^{\lambda_3}}\right)^2 \text{ and } \tilde{\psi}_s = (\lambda_0\lambda_5)^2. \tag{3}$$

The elements of $\tilde{\Psi}$ are common in every equation which facilitates the posterior inference. Prior variances differ between dependent variables y_{it} by a scaling factor σ_i which we set equal to the pre-sample standard deviations of variables in the system. Note that a prior dependence on the data of this type is a common property in applied modelling (see Karlsson, 2013). *A priori* we also give a preference to rather

slow, harmonic decay of variance with lags ($l = 1, \ldots, p$) of dependent variables $\lambda_3 = 1$ as in Sims and Zha (1998). We set the following values of other hyperparameters: $\lambda_0 = 0.2$ for overall tightness, $\lambda_1 = 0.9$ for a relative tightness around the random walk prior, and $\lambda_5 = 1.2 > \lambda_1$, giving relatively less informative prior on dispersion of seasonal factors. In a sensitivity check we also ask whether loose (less informative) priors on seasonal factors are as successful as informative (tight) priors.
With this set of S-Z priors under a triangular A_0 matrix we simulate draws from the marginal posterior of $\Phi = A_0 A_0'$ using Wishart distribution and then we generate draws of A_+ conditional on A_0. Next we calculate reduced-form VAR parameters $A_+ A_0^{-1}$ and recursively produce density forecasts simulating primitive shocks from zero-mean multivariate normal distribution with a covariance Φ^{-1}. For a detailed treatment and discussion we refer the reader to Karlsson (2013), section 4.
As a second approach we consider a stationary reduced-form BVAR of Villani (2009) defined for deviations from seasonal (unconditional) means:

$$A(L)(y_t - \Xi s_t) = \varepsilon_t, \tag{4}$$

where:

y_t – $m \times 1$ vector of endogenous variables,

s_t – 12×1 vector of seasonal dummies,

Ξ – $m \times 12$ matrices of seasonal means for each of m variables,

$A(L)$ – lag polynomial of deviations from seasonal means,

ε_t – $m \times 1$ vector of error terms having multivariate normal distribution ($\varepsilon_t \sim N(0, \Sigma)$), possibly correlated between equations but not in time.

Independent prior structure is assumed as in Villani (2009):

$$p(\Gamma, \Xi, \Sigma) = p(vec(\Gamma))p(vec(\Xi))p(\Sigma) \tag{5}$$

where:

$p(vec(\Gamma))$ is a normal prior distribution of stacked VAR parameters $\Gamma' = (A_1', \ldots, A_p')$,

$p(vec(\Xi))$ is a normal prior on steady-state seasonal factors,

$p(\Sigma)$ is an inverse Wishart for the error covariance matrix.

A priori we assume the mean of Ξ equal to seasonal factors calculated from the pre-sample data (for inflation) or zeros (for other variables). Because a prior on the steady state is supposed to be reasonably informative we set a common tightness hyperparameter on every seasonal factor in Ξ, $\xi = 0.0005$. What is more, tight priors on steady-state are reported to be notably successful in out-of-sample forecasting experiments (see Wright, 2013). The prior similar to the Minnesota prior concept is used for Γ as in Villani (2009). We centre our beliefs on VAR parameters around OLS estimates of VAR(12) on a pre-sample data as the usual random-walk prior is inconsistent with the steady-state prior assumption. Prior variances at the VAR parameters, κ_{A_l}, which are responsible for the shrinkage, are of a reduced and symmetric Minnesota form: $\kappa_{a_l} = \left(\frac{\kappa_1}{l^{\kappa_3}}\right)^2$. At first we set these hyperparameters in a loose prior fashion to $\kappa_1 = 0.1$ and $\kappa_3 = 1$ (harmonic decay) and check the sensitivity of forecasting results in respect to their variation.

As there is no closed formula for joint posterior distribution of non-linear model described by Equation (4), we use three-blocks Gibbs sampler to draw each of the three groups of model parameters Γ, Ξ, Σ conditional on the other two groups as described by Villani (2009). To forecast we draw 50 thousand realizations of parameters from their conditional posteriors including 10 thousand as burn-in subsample. It is well documented that tight priors for steady-state parameters are necessary for the convergence of Gibbs sampler for Villani model – see Villani (2009), Wright (2013). Nevertheless, we have performed carefully all convergence univariate diagnostics tests available in CODA package (Plummer *et al.*, 2006). There are no significant signs of any convergence problems in MCMC chains. Additionally, up to 250 thousand draws have been tested, however, as long as we are interested in average forecast performance, the differences are negligible.

These simulation methods are quite time consuming in five-equation seasonal BVAR(12) we apply. That was probably one of the important reasons to omit the Villani steady-state specification from the most elaborated exercise of BVAR specifications by Carriero *et al.* (2015). The empirical question whether there are any gains from steady-state assumptions on seasonal pattern, which justify these computational costs, is an important value added of this research.

It is worth noting that models (1) and (4) may be observationally equivalent if neither persistence nor seasonality is present in the data under consideration. If both features are observed then these approaches determine a different co-dependence of both elements in the corresponding VAR models. S-Z prior system leads to a standard linear VAR model but at the cost of difficulties in introducing conditional seasonality (conditional on any of right-hand side variables). While in S-Z approach unconditional (long-run) seasonal means are non-linear functions of other model parameters, in Villani approach they are explicitly inferred from the prior distribution but the corresponding forecasting model is non-linear in respect of the parameters. In a consequence sampling from a posterior conditional distributions with a three-block Gibbs sampler in Villani BVAR increases the computation costs of inference but it facilitates the way of introducing seasonal beliefs compared to VAR with S-Z priors.

In forecasting experiment performed in this paper we compare the aforementioned approaches of S-Z and Villani with a benchmark frequentist VAR model. The benchmark VAR model with seasonal dummies is of the same lag order and it is

estimated with the maximum likelihood method.
The code for estimating S-Z model is based on Brandt and Davis (2014) *MSBVAR: Markov-Switching, Bayesian, Vector Autoregression Models* and it is invoked in R (CRAN). Gibbs sampling routine of the steady-state BVAR uses Armadillo library (C++) and it is developed from O'Hara (2014) package *Bayesian Macroeconometrics in R*. In order to include prior structures on seasonal factors properly, all of the original routines are modified by Damian Stelmasiak.

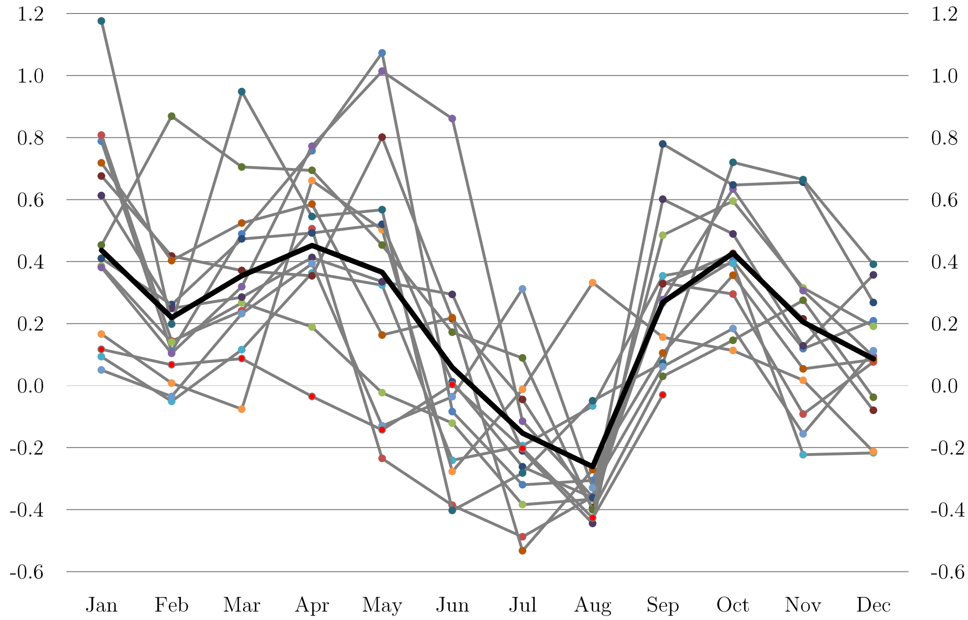
3 Data and forecasting exercise

We use a real-time dataset to forecast monthly index of consumer inflation in Poland from November 2011 to October 2014, which we call verification period (while the initial estimation period is from January 1999 till October 2011). Over both periods, the headline CPI inflation index in month-over-month terms reveals clear but non-deterministic seasonal pattern, which according to SARIMA model decomposition accounts for one third of overall inflation variance (see Figure 1). Hence seasonally adjusted CPI is a covariance stationary time series with quite persistent autoregressive pattern, it is straightforward to apply multivariate VAR frameworks with seasonal terms described in Section 2.
The other endogenous variables selected in the BVAR specifications are prices from representative consumption markets and from financial market. The prices are transformed to log monthly changes or yields, respectively. These are:

1. fuel prices (weekly data collected by `e-petrol.pl`) – 95 octane fuel (averaged over gas stations),
2. food prices (weekly data from the Common Agriculture Policy reports of the Polish Ministry of Agriculture and Rural Development) – represented by pasteurized milk price (averaged over products with different fat content),
3. daily data on nominal exchange rates (PLN/EUR),
4. and daily interest rates on 2Y bonds (log yields in annual terms).

The changes in these prices directly and indirectly translate into changes in the prices of other goods that constitute substantial part of consumer basket. These prices are also observed with higher frequency than inflation. Weekly and daily data releases are used as a source of timely information about current consumption market conditions and its future expectations (observed at financial markets). We apply the following approach to the mixed frequency dataset in pseudo real-time experiments. We produce monthly forecasts in the middle of the month (just after monthly CPI releases), when we already have some observations on current month market prices. We simply average over available daily or weekly observations from current month to obtain pseudo-monthly data. After the end of the month we substitute it with

Figure 1: The headline inflation index (month over month) in Poland over the period Jan1999-Sep2014. The black solid line shows a sample average for each month



a proper monthly averages. Hence, the last market price observations in each of 36 vintages of real-time dataset are pseudo-monthly data.

To produce out-of-sample forecasts $\hat{y}_{t+h}$ for $h = 1, 2, \ldots, 12$ months ahead the vertical alignment is performed. Because CPI is released in the middle of next month when we already have some current information on prices from consumer and financial markets, we put the data according to their publication date. It means that last vintage of data from mid October 2014 consists of CPI till September 2014 and pseudo-monthly prices till mid of October 2014. Real-time dataset also includes CPI revisions for January published in March when new consumer basket weights are released based on households expenditures survey from the previous year.

4 Forecast evaluation

Maximizing the precision of out-of-sample forecasts is formally not possible before observing the forecast realizations. In this section we analyse the forecast performance of different BVAR specifications in pseudo out-of-sample real-time exercises to shed some light on the usefulness of various assumptions behind them. Using the training

sample (evaluation period) may, however, result in model over-fitting (see Bańbura et al., 2010). The problem is specifically pronounced when researchers use only the measures of quality for the mean of forecast distribution (i.e. RMSFE, MAFE and MFE) and the distribution departs from normality. When the policy makers are explicitly interested in forecast uncertainty in terms of interval forecasts (commonly named fan charts) density-based measures are of a particular interest.
Hence our goal is to evaluate marginal density of out-of-sample forecasts, we provide a short review of density-based error measures (scores) for continuous random variables. Between many possible measures logarithmic score (log score, LS) and Continuous Ranked Probability Score (CRPS) are the most popular local scoring rules corresponding with MSFE and MAFE criteria, respectively. LS and CRPS also belong to the group of strictly proper rules. It means that for a given forecast density they minimize the expected loss at observed forecast realizations if the density is true (for technical details see Gneiting and Raftery, 2007). For comparability with other point forecast errors we define log scores as negatively oriented penalties (see Gneiting and Ranjan, 2011):

$$\mathrm{LS} = -\log(p_i(x^o)), \tag{6}$$

where $p(x^o)$ is a value of predictive density function of variable X at observed forecast realization x^o.
Log scores for analytical distribution do not give rise to difficulties and have interesting statistical interpretation as components of Bayes factor. Albeit, the measures from Monte Carlo simulation are reported to be sensitive to the choice of prior distribution in small samples (see Geweke and Amisano, 2010) and density approximation method (see Carriero *et al.*, 2015). Hence, log score applications are more popular for financial market forecasts when the number of events is relatively vast (see Weigend and Shi, 2000). Obviously, logarithmic transformation is severe for low probability events (Gneiting and Raftery, 2007). In the case of inflation forecasting approximation problems may occur when inflation rate is very low or exceptionally high.
Accordingly, alternative density-based score is considered. Let $F(x)$ denote a cumulative distribution function (CDF) of a density forecast. CRPS measures a squared departure of forecast CDF from empirical CDF. For a single observed value x^o CRPS is defined with the use of an indicator function $\mathbb{1}_{\{x^o \leq x\}}$:

$$\mathrm{CRPS}\,(F, x^o) = \int_{-\infty}^{+\infty} \left[F(x) - \mathbb{1}_{\{x^o \leq x\}}\right]^2 dx. \tag{7}$$

To avoid numerical integration, we consider a closed form proposed in [15]:

$$\mathrm{CRPS} = \mathrm{E}|X - x^o| - \frac{1}{2}\mathrm{E}|X - \tilde{X}|, \tag{8}$$

where $X \sim F$ and $\tilde{X} \sim F$ denote independent random draws from the same forecast CDF.

Equation (8) describes how to approximate CRPS at x^0 by simulating from the known forecast CDF F. Both CRPS and log score offer a symmetric loss function interpretation (Gneiting and Ranjan, 2011) but CRPS relatively rewards forecast realizations close to the middle of the forecast density. CRPS unlike the log score always takes positive and finite values. It is also reported to be less sensitive to outliers (see Gneiting and Raftery, 2007).

To compare forecasts from alternative VAR specifications we use traditional point forecasting performance measures (RMSFE, MAFE, MFE) and two density-based scores: LS and CRPS, averaged over 24 months. The evaluation of models with all analysed forecast errors defined as negatively oriented measures is straightforward then. The lower these scores are the more adequate forecast density is.

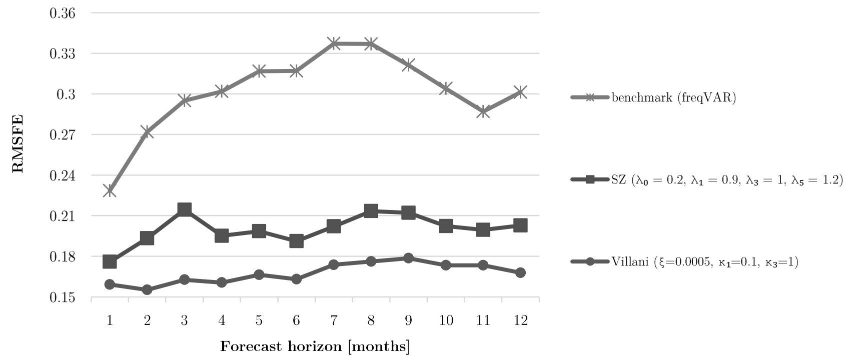
In terms of RMSFE, the best scores in our empirical study are obtained using the steady-state prior-structure *à la* Villani (see Equation (4)). For $h = 1$ (nowcasting) the most accurate specification is the one with variances harmonically decaying at consecutive lags ($\kappa_3 = 1$) and tight prior distribution for seasonals ($\xi = 0.0005$). Also for longer horizons the Villani-type priors do a good job in shrinking the parameter space. In this case faster decaying variances ($\kappa_3 = 2$) produce smaller forecast errors (see Figure 2(a) and Table 1). Nevertheless, Villani prior specifications outperforms (in terms of RMSFE) not only the benchmark, but also the S-Z specifications.

Table 1: RMSFE values for selected horizons

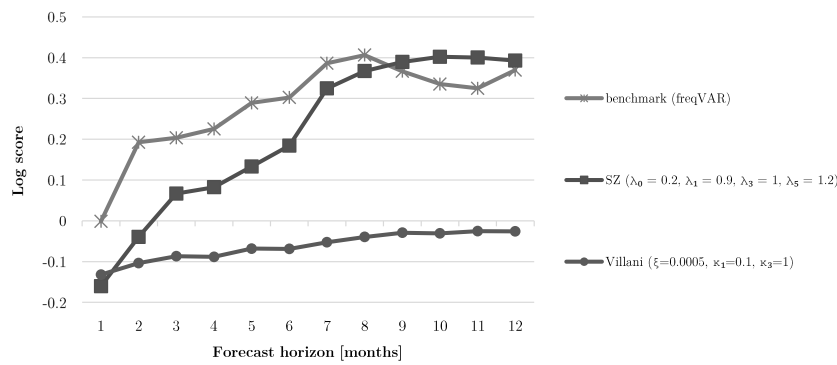
Model	**Specification**			**Forecast horizon (months)**				
(prior type)	ξ	κ_1	κ_3	**h=1**	**h=3**	**h=6**	**h=9**	**h=12**
	0.0005	**0.1**	**1**	**0.1592**	0.1627	0.1631	0.1786	0.1678
	0.0005	0.1	0.1	0.1702	0.1860	0.1898	0.2070	0.1988
	0.01	0.1	1	0.2029	0.2006	0.1843	0.2030	0.1907
	0.0005	0.001	1	0.1661	0.1608	0.1598	0.1612	0.1612
Villani	**0.0005**	**0.1**	**2**	0.1659	**0.1582**	**0.1500**	**0.1607**	**0.1591**
	1.2 (ZoS)	0.1	1	0.2942	0.2763	0.2495	0.2699	0.2594
	1	0.1	1	0.2866	0.2677	0.2536	0.2619	0.2507
	0.0001	0.1	1	0.1597	0.1610	0.1653	0.1776	0.1699
	0.0005 (ZoS)	0.1	1	0.2028	0.1855	0.1859	0.1921	0.1888
	λ_1	λ_3	λ_5					
	0.9	1	1.2	0.1759	0.2144	0.1911	0.2121	0.2028
	0.45	1	1.2	0.1889	0.1936	0.1901	0.2052	0.1987
Sims-Zha	0.9	3	1.2	0.1801	0.2035	0.1866	0.1986	0.1942
	0.9	1	0.5	0.1880	0.2226	0.1871	0.2086	0.1988
	0.9	0.5	1.2	0.1817	0.2271	0.2051	0.2196	0.2071
benchmark	*frequentist VAR*			0.2284	0.2950	0.3170	0.3212	0.3012

The best score at each horizon is bolded; ZoS stands for zeros on seasonal parameters. Overall tightness from S-Z specification $\lambda_0 = 0.2$ preserved for comparison.

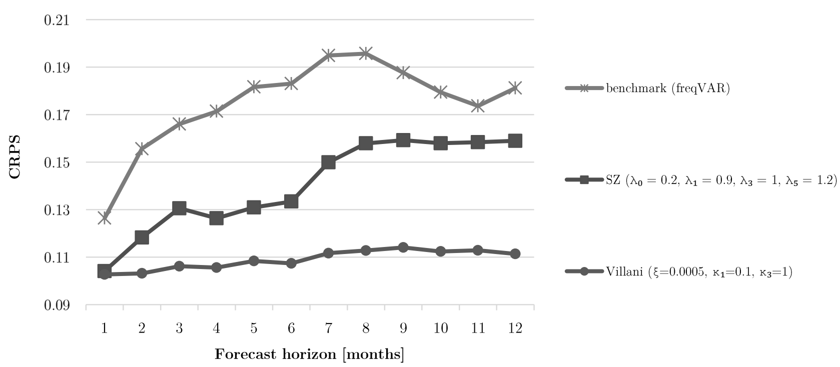
Figure 2: Forecast performance comparison among the models, for horizons $h = 1, 2, \ldots, 12$ months



(a) Root mean squared forecast errors



(b) Logarithmic scores



(c) Continuous Ranked Probability Scores

Evaluating forecast density performance with log scores the Villani specification is preferred, too (see Figure 2(b) and Table 2). For $h = 1$ the set of hyperparameters $\xi = 0.0005$, $\kappa_1 = 0.1$, $\kappa_3 = 0.1$ (slow-decaying) produces the best results, while for longer horizons a triad $\xi = 0.0001$, $\kappa_1 = 0.1$, $\kappa_3 = 1$ performs better. Please note, that even a totally misspecified prior structure in the Villani framework (e.g. very loose prior on seasonals, like $\xi = 1$) outperforms the benchmark and Sims-Zha approach for longer horizons.

Table 2: Log scores for selected horizons

Model (prior type)	**Specification** ξ	κ_1	κ_3	**Forecast horizon (months)** h=1	h=3	h=6	h=9	h=12
Villani	0.0005	0.1	1	-0.1321	-0.0867	-0.0689	-0.0293	-0.0258
	0.0005	**0.1**	**0.1**	**-0.1741**	-0.1083	-0.0861	-0.0327	-0.0256
	0.01	0.1	1	-0.0549	-0.0169	-0.0432	0.0010	-0.0119
	0.0005	0.001	1	-0.0094	0.0163	0.0217	0.0289	0.0348
	0.0005	0.1	2	-0.0611	-0.0113	-0.0136	0.0098	0.0170
	1.2 (ZoS)	0.1	1	0.1872	0.1526	0.1093	0.1544	0.1379
	1	0.1	1	0.1683	0.1383	0.1216	0.1386	0.1248
	0.0001	**0.1**	**1**	-0.1504	**-0.1147**	**-0.0890**	**-0.0513**	**-0.0487**
	0.0005 (ZoS)	0.1	1	0.0060	0.0382	0.0524	0.0785	0.0890
Sims-Zha	λ_1	λ_3	λ_5					
	0.9	1	1.2	-0.1601	0.0668	0.1841	0.3894	0.3926
	0.45	1	1.2	-0.0535	0.1737	0.3225	0.4841	0.5162
	0.9	3	1.2	-0.1039	0.1091	0.2185	0.3152	0.3315
	0.9	1	0.5	-0.0905	0.0950	0.2012	0.4151	0.4392
	0.9	0.5	1.2	-0.1651	0.0672	0.1807	0.3903	0.4115
benchmark	*frequentist VAR*			-0.0016	0.2033	0.3022	0.3665	0.3697

The best score at each horizon is bolded; ZoS stands for zeros on seasonal parameters. Overall tightness from S-Z specification $\lambda_0 = 0.2$ preserved for comparison.

Negative MFE values correspond to downward trend in CPI inflation at evaluation period. Thus, in terms of mean errors, almost all of the models overestimate CPI (see Table 3). Again, Villani priors stand out, clearly outperforming the benchmark (specifically for longer horizons). These remarks are also valid for MAFE criterion. Applying the Villani steady-state priors decreases MAFE up to 50% in relation to the benchmark (see Table 4). Only in nowcasting ($h = 1$) MAFE does not differentiate between Villani, Sims-Zha and benchmark. In terms of CRPS, the advantage of Villani framework is less pronounced (40% improvement over the benchmark, see Table 5) but the conclusions are similar to MAFE results. It seems that in the BVAR model steady-state assumptions on seasonal factors are more appropriate for modelling Polish inflation than the conditional mean assumptions as in S-Z approach. Forecasting performance based on Villani prior exhibits a greater sensitivity to the hyperparameter ξ, which represents *a priori* beliefs on seasonal factors tightness. Very loose prior ($\xi = 1$) results in a poor forecasting performance (see Figure 3).

Starting calibration with $\xi = 1$ and gradually decreasing the ξ decreases forecast errors considerably. Nonetheless, this procedure is limited since optimal hyperparameter value (i.e. minimizing scores) is achieved around 1×10^{-4} (see Figure 3). The process of calibrating hyperparameters may be time-consuming, but as shown in Tables 1-5, with reasonably chosen triad ξ, κ_1, κ_3 the Villani specification is able to outperform not only the frequentist VAR, but also the Sims-Zha BVAR.

Table 3: Mean forecast errors for selected horizons

Model (prior type)	**Specification** ξ	κ_1	κ_3	**h=1**	**h=3**	**h=6**	**h=9**	**h=12**
Villani	0.0005	0.1	1	-0.0280	-0.0406	-0.0510	-0.0576	-0.0531
	0.0005	0.1	0.1	-0.0571	-0.0733	-0.0874	-0.0991	-0.1008
	0.01	0.1	1	-0.0295	-0.0477	-0.0435	-0.0557	-0.0501
	0.0005	0.001	1	0.0110	0.0192	0.0201	0.0196	0.0211
	0.0005	**0.1**	**2**	**-0.0038**	**0.0007**	**-0.0013**	**-0.0010**	**0.0053**
	1.2 (ZoS)	0.1	1	-0.0608	-0.0842	-0.0740	-0.0848	-0.0821
	1	0.1	1	-0.0644	-0.0764	-0.0837	-0.0911	-0.0862
	0.0001	0.1	1	-0.0287	-0.0380	-0.0518	-0.0578	-0.0555
	0.0005 (ZoS)	0.1	1	-0.0287	-0.0392	-0.0489	-0.0530	-0.0502
Sims-Zha	λ_1	λ_3	λ_5					
	0.9	1	1.2	-0.0561	-0.1060	-0.1042	-0.1045	-0.1136
	0.45	1	1.2	-0.0409	-0.0823	-0.0978	-0.0982	-0.1081
	0.9	3	1.2	-0.0473	-0.0831	-0.0922	-0.0907	-0.0962
	0.9	1	0.5	-0.0571	-0.0981	-0.0874	-0.0856	-0.0957
	0.9	0.5	1.2	-0.0640	-0.1202	-0.1182	-0.1192	-0.1184
benchmark	*frequentist VAR*			-0.0487	-0.1683	-0.2206	-0.2227	-0.2195

The best score at each horizon is bolded; ZoS stands for zeros on seasonal parameters. Overall tightness from S-Z specification $\lambda_0 = 0.2$ preserved for comparison.

In opposition to Villani approach, calibrating hyperparameters in S-Z framework is more straightforward. Firstly, an estimation of S-Z model is faster what gives a possibility to check large number of hyperparameters combinations in a short time. Secondly, results of such a grid search procedure lead to conclusion that forecast error measures are not very sensitive to the hyperparameters choice. In terms of RMSFE criterion in nowcasting ($h = 1$), we find λ_1 and λ_5 set to the value of around one (or more) a reasonable combination (see Figure 4). While hyperparameters decreasing towards zero, RMSFE rises significantly faster in the case of λ_5 than λ_1. In practice, however, a setting with lambdas lower than one is not really very misspecified. In terms of e.g. CRPS, all of the specifications perform very similarly (see Table 5). Log scores reveal greater discrepancies, although in general BVAR models with S-Z priors show weakness here, being outperformed by both, Villani specifications and benchmark model for longer horizons (see Table 2).

Table 4: MAFE values for selected horizons

Model (prior type)	Specification ξ	κ_1	κ_3	h=1	h=3	h=6	h=9	h=12
Villani	0.0005	0.1	1	0.1337	0.1333	0.1349	0.1477	0.1420
	0.0005	0.1	0.1	0.1423	0.1536	0.1550	0.1686	0.1680
	0.01	0.1	1	0.1755	0.1671	0.1545	0.1723	0.1639
	0.0005	0.001	1	0.1368	0.1273	0.1269	0.1284	0.1281
	0.0005	**0.1**	**2**	0.1409	**0.1234**	**0.1201**	**0.1267**	**0.1255**
	1.2 (ZoS)	0.1	1	0.2372	0.2177	0.2037	0.2194	0.2143
	1	0.1	1	0.2422	0.2176	0.2015	0.2112	0.2083
	0.0001	0.1	1	0.1349	0.1285	0.1354	0.1467	0.1427
	0.0005 (ZoS)	0.1	1	0.1646	0.1471	0.1499	0.1565	0.1533
	λ_1	λ_3	λ_5					
Sims-Zha	**0.9**	**1**	**1.2**	**0.1287**	0.1706	0.1593	0.1668	0.1696
	0.45	1	1.2	0.1533	0.1600	0.1566	0.1702	0.1670
	0.9	3	1.2	0.1442	0.1666	0.1547	0.1634	0.1637
	0.9	1	0.5	0.1296	0.1719	0.1481	0.1718	0.1613
	0.9	0.5	1.2	0.1292	0.1834	0.1672	0.1742	0.1739
benchmark	*frequentist VAR*			0.1604	0.2290	0.2648	0.2648	0.2614

The best score at each horizon is bolded; ZoS stands for zeros on seasonal parameters. Overall tightness from S-Z specification $\lambda_0 = 0.2$ preserved for comparison.

Table 5: CRPS values for selected horizons

Model (prior type)	Specification ξ	κ_1	κ_3	h=1	h=3	h=6	h=9	h=12
Villani	0.0005	0.1	1	0.1027	0.1062	0.1074	0.1141	0.1114
	0.0005	0.1	0.1	0.1036	0.1116	0.1144	0.1225	0.1202
	0.01	0.1	1	0.1212	0.1220	0.1156	0.1239	0.1191
	0.0005	0.001	1	0.1124	0.1129	0.1129	0.1138	0.1140
	0.0005	**0.1**	**2**	0.1091	0.1101	0.1084	**0.1124**	0.1125
	1.2 (ZoS)	0.1	1	0.1673	0.1576	0.1459	0.1559	0.1510
	1	0.1	1	0.1650	0.1543	0.1475	0.1517	0.1474
	0.0001	**0.1**	**1**	**0.1018**	**0.1039**	**0.1068**	0.1125	**0.1105**
	0.0005 (ZoS)	0.1	1	0.1235	0.1208	0.1220	0.1253	0.1253
	λ_1	λ_3	λ_5					
Sims-Zha	0.9	1	1.2	0.1041	0.1306	0.1334	0.1592	0.1590
	0.45	1	1.2	0.1151	0.1328	0.1471	0.1693	0.1724
	0.9	3	1.2	0.1092	0.1306	0.1357	0.1481	0.1491
	0.9	1	0.5	0.1098	0.1349	0.1342	0.1620	0.1626
	0.9	0.5	1.2	0.1053	0.1350	0.1364	0.1612	0.1608
benchmark	*frequentist VAR*			0.1265	0.1660	0.1831	0.1877	0.1812

The best score at each horizon is bolded; ZoS stands for zeros on seasonal parameters. Overall tightness from S-Z specification $\lambda_0 = 0.2$ preserved for comparison.

Figure 3: Log scores of Villani prior-structure model dependent on seasonal tightness hyperparameter ξ given $\kappa_1 = 0.1$, $\kappa_3 = 1$

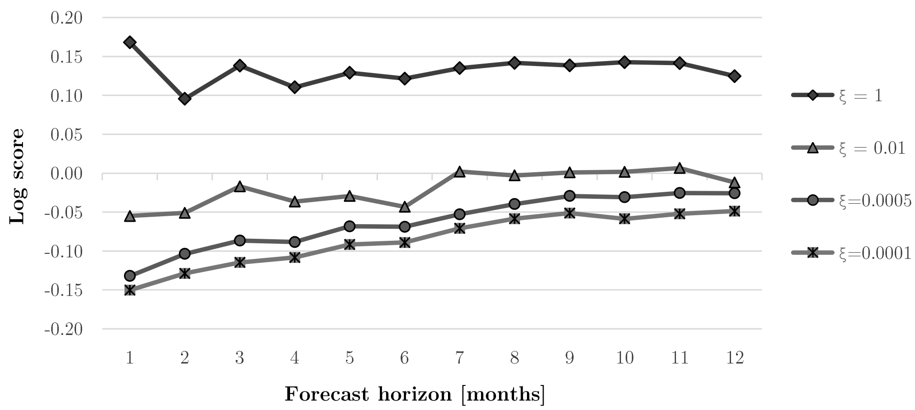
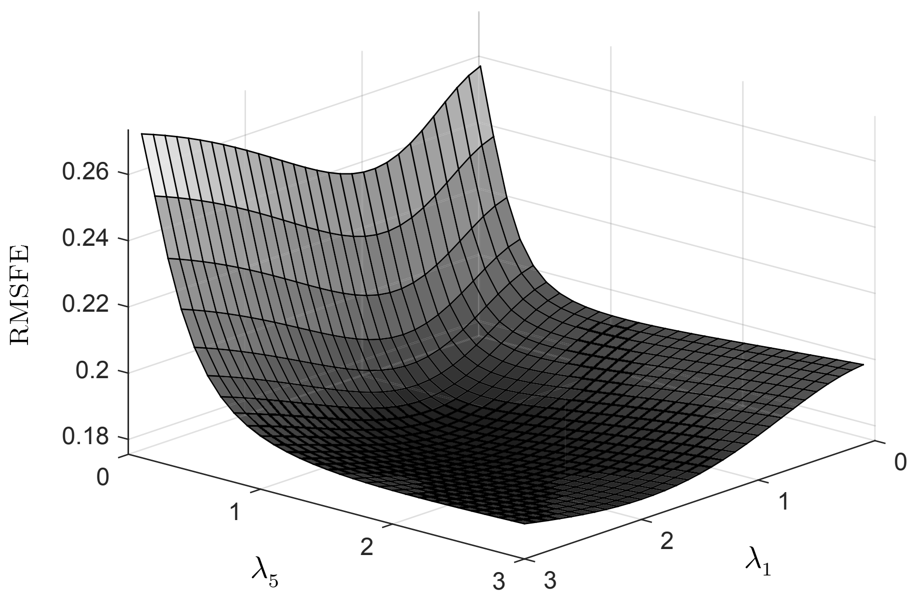


Figure 4: Root mean squared forecast error as a function of hyperparameters λ_5 and λ_1 in S-Z BVAR model given $\lambda_0 = 0.2$ and $\lambda_3 = 1$. Forecast horizon $h = 1$



In order to confirm the results in a statistical manner, the test of Amisano-Giacomini (Amisano and Giacomini, 2007) is performed on log scores and CRPS, separately. Null hypothesis states that average scores, $\bar{S}_A$ and $\bar{S}_B$, obtained in pair of models A and B are equal. It is tested against an alternative hypothesis of forecasting advantage of model B over A with the following test statistics:

$$t_{AG} = \frac{\bar{S}_A - \bar{S}_B}{\hat{\sigma}}\sqrt{n} \tag{9}$$

where $\hat{\sigma}$ is calculated as in Gneiting and Ranjan (2011) and n is a number of forecasts. Statistics t_{AG} is asymptotically standard normal under the null. All the scores investigated in this paper are negatively oriented, thus we reject the null in favour of model B if t_{AG} exceeds one-tailed critical value.

We consider the following three pairs of model specifications to be tested: (1) S-Z vs Villani, (2) benchmark vs Villani, (3) benchmark vs S-Z. Possibly best prior specifications of S-Z and Villani specifications are used. The detailed results are given in Table 6 (at significance level of 0.1).

The best model with Villani prior system performs better than S-Z BVAR (see results for pair (1) in Table 6) and benchmark (see pair (2), respectively) for all horizons except for nowcasting ($h = 1$). In the case of pair (3), we cannot reject the null of average scores equality measured by log scores (for any horizon), while CRPS from S-Z approach are significantly lower for longer horizons (see Table 6). The conclusion is, Villani outperforms the benchmark and S-Z for any horizon except for one month, while S-Z superiority to the benchmark is questionable in terms of Amisano-Giacomini test at significance level of 0.1.

Summing up, longer horizon forecasting performance of Villani BVAR is well shown with all of the point- and density-based error measures (see Figure 2). However, nowcasting performance from among: Villani, benchmark and Sims-Zha specifications, is quite indistinguishable in terms of MFE, MAFE and CRPS. Only RMSFE and log score show some clear differences in favour of Villani.

5 Final remarks

Undoubtedly, among the examined specifications BVAR model with steady-state prior structure offers the potential to produce superior pseudo out-of-sample forecasts of the Polish inflation in the examined period. However, what should be highlighted, BVAR model with Sims-Zha priors, being less complex in structure, provides comparable inflation forecasts even with quite uninformative prior assumptions on seasonal factors. It also outperforms frequentist VAR in terms of CRPS. Therefore, we regard Sims-Zha BVAR as a useful tool for forecasting inflation in Poland, too.

The type of prior distribution used in steady-state Villani approach produces predictions that are generally superior to Sims-Zha approach and to a benchmark frequentist VAR both in terms of point forecasts (RMSFE, MFE, and MAFE) and

Table 6: Results of Amisano-Giacomini tests of forecast performance equality. '1' indicates a rejection of the null hypothesis in favour of the second model in pair, while '0' implies forecasting performance equality

Models compared	Horizon	Log score	CRPS
Pair (1):	h=1	0	0
	h=3	1	1
S-Z ($\lambda_0 = 0.2$, $\lambda_1 = 0.9$, $\lambda_3 = 1$, $\lambda_5 = 1.2$)	h=6	1	1
vs Villani ($\xi = 0.0005$, $\kappa_1 = 0.1$, $\kappa_3 = 1$)	h=9	1	1
	h=12	1	1
Pair (2):	h=1	0	0
	h=3	1	1
benchmark (freqVAR)	h=6	1	1
vs Villani ($\xi = 0.0005$, $\kappa_1 = 0.1$, $\kappa_3 = 1$)	h=9	1	1
	h=12	1	1
Pair (3):	h=1	0	0
	h=3	0	1
benchmark (freqVAR)	h=6	0	1
vs S-Z ($\lambda_0 = 0.2$, $\lambda_1 = 0.9$, $\lambda_3 = 1$, $\lambda_5 = 1.2$)	h=9	0	1
	h=12	0	1

Significance level at 0.1, forecast horizon in months.

density forecasts (significant differences in log score and CRPS). BVAR models with Sims-Zha priors are second best choice being generally less sensitive to the hyperparameters choice and seasonality beliefs than Villani approach. With more than 350 parameters to be estimated in the selected five-variate VAR(12) models tight and informative priors are necessary to produce forecasts with a precision superior to VAR estimated with ML method. There are however some limits to the Minnesota-type shrinkage in both BVAR frameworks. The research supports a view that moderate values of Minnesota-type hyperparameters is the most successful in producing adequate point and density forecasts. Following this approach gains are considerably smaller in nowcasting than in forecasting for longer horizons, namely up to 12 months ahead.

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