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INVITED PAPER

On fractional vectorial calculus

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Abstract. This paper reviews the fractional vectorial differential operators proposed previously and introduces the fractional versions of the classic Green's, Stokes', and Ostrogradski-Gauss's integral theorems. The suitability of fractional derivatives for sciences and the Laplacian definition are also discussed.

Key words: Grünwald-Letnikov, Liouville, fractional gradient, fractional divergence, fractional curl, fractional Laplacian, Green's theorem, Stokes' theorem, Ostrogradski-Gauss's theorem.

1. Introduction

Fractional calculus (FC) was born almost the same time as the integer order calculus [1–6] and resulted from a discussion between Leibniz and Bernoulli. Recently FC abandoned a pure mathematical perspective and was recognised by the applied sciences community, namely physics and engineering [7–23], giving rise to interesting applications in many fields, such as biology [24, 25], biomedical engineering [4, 26–28], finance [29], and signal processing [30] just to mention a few.

FC is an essential tool for modelling long memory and long range processes. This non-locality in time and space can be found in many phenomena, such as the diffusion processes [31–34], anomalous porous media [35], and fractional spaces [36]. Viscoelasticity is another subject of active studies [37]. The modelling of mechanical systems, from 1-D [38] to n-D [16, 39, 40], is also an important subject that attracted the attention of researchers. This has been accomplished through the fractionalization of Hamiltonian and gradient concepts [16, 17, 39, 40]. The general case of field theories was tackled by Herrmann [41, 42].

The generalization of the vectorial operators: gradient, divergence, curl, and Laplacian was discussed in [43], starting by the space version of the fractional forward and backward Grünwald-Letnikov and Liouville derivatives [5, 44]. These operators are used to define a pair of left and right fractional gradients that lead to the corresponding divergences and curls. The fractional Laplacian is obtained from the inner product of the left and right gradients. Such operators are backward compatible in the sense that they recover the classical definitions when the order is 1.

An important aspect of our formulation concerns the use of distinct orders for different directions. These non-local tools, unlike the classical ones, look more suitable for dealing with non-homogeneous and anisotropic media.

With those operators the generalized Helmholtz decomposition theorem for fractional space and time is reviewed and decoupled by means of fractional wave equations for fields or potentials. The generalization is based on differential operators and points to the need for an integral formulation.

The importance of the classical Green's, Stokes', and Ostrogradski-Gauss's integral theorems is unquestionable. Hence, attempts to derive a fractional formulation based on two different point of views have been made, namely, fractal geometry [35, 36, 39, 45] and FC [31, 39, 46]. In this line of thought, we propose the fractional version of such theorems for rectangular domains. The starting point is the generalization of the fundamental theorem of calculus. Several researchers tried this topic [16, 46, 47], but without having the support of the notion of fractional definite integral (FDI) and, consequently, achieving limited results. However, recently this concept was tackled by means of the generalization of the Barrow formula [48]. Having defined the FDI on $\mathbb{R}$, we can introduce definite integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$ for rectangular domains. Here we use those ideas to generalize the classic Green's, Stokes', and Ostrogradski-Gauss's theorems, bearing in mind a full agreement with the fractional vectorial operators proposed in [43]. In a wider scope, this work follows the generalization of the FC the concepts and provides a straightforward methodology for the adoption of FC in applied sciences.

The manuscript is organized as follows. Section 2 defines the main fractional derivatives suitable for fractional vectorial calculus. Time and space derivatives are introduced using the Grünwald-Letnikov and Liouville formulations. Two-sided derivatives are also included due to their importance in the development of differential vectorial tools, namely the Laplacian. The tools are described in Sec. 3, where we analyze the left and right gradients, divergences, and curls, called "first generation operators", leading to a second group of "second

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generation operators" and particularly to the fractional Laplacian. This is compared with the fractional Laplacian based on the Riesz potential. The generalized Helmholtz decomposition is recalled and its solution expressed in terms of those corresponding to several fractional wave equations. In Sec. 4, the fractional definite integrals are described. They are used in Sec. 5 to extend the classic integral theorems towards the fractional case. The Green's, Stoke's, and Ostrogradski-Gauss's theorems in rectangular domains are formulated. The particular cases of uniform fractional orders is also discussed. Finally, Sec. 6 outlines the main conclusions.

1.1. Remarks

- We will assume that we are working on $\mathbb{R}^3$ and that the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ constitutes its standard orthonormal base. Therefore, each vector has the representation

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3.$$

  In particular, we define the vector

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{1}$$

- A vectorial function is represented by

$$\mathbf{f}(x_1,x_2,x_3) = f_1(x_1,x_2,x_3)\mathbf{e}_1 + f_2(x_1,x_2,x_3)\mathbf{e}_2 + f_3(x_1,x_2,x_3)\mathbf{e}_3.$$

- When necessary, we write $\overline{\alpha} = (\alpha_1,\alpha_2,\alpha_3)$ and $\overline{\omega} = (\omega_1,\omega_2,\omega_3)$ and similarly for other cases, as $\overline{s}$.
- We will use the two-sided Laplace transform (LT) of $f(x_1,x_2,x_3)$ defined on $\mathbb{R}^3$ and given by:

$$F(\overline{s}) = \mathcal{L}\left[f(x_1,x_2,x_3)\right] = \int\limits_{\mathbb{R}^3} f(x_1,x_2,x_3)e^{-(\mathbf{s}\cdot\mathbf{r})}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{2}$$

  under the usual existence conditions. The inverse LT is [49]

$$f(x_1,x_2,x_3) = \frac{1}{(i2\pi)^3}\int\limits_{a_1-i\infty}^{a_1+i\infty}\int\limits_{a_2-i\infty}^{a_2+i\infty}\int\limits_{a_3-i\infty}^{a_3+i\infty} F(\overline{s})e^{(\mathbf{s}\cdot\mathbf{r})}\mathrm{d}^3\overline{s}, \tag{3}$$

  where $\overline{a} = (a_1,a_2,a_3)$ is in the region of convergence of the transform and $i = \sqrt{-1}$.
- The Fourier transform (FT) is obtained from the LT using the substitution $\overline{s} = i\overline{\omega}$ with $\overline{\omega} \in \mathbb{R}^3$ is defined by the synthesis equation [50]

$$f(x_1,x_2,x_3) = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} F(\overline{\omega})e^{i(\overline{\omega}.\mathbf{r})}\mathrm{d}^3\overline{\omega} \tag{4}$$

  and the corresponding analysis equation

$$F(\overline{\omega}) = \mathcal{F}\left[f(x_1,x_2,x_3)\right] = \int\limits_{\mathbb{R}^3} f(x_1,x_2,x_3)e^{-i(\omega.\mathbf{r})}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \tag{5}$$

- We will work with the multivalued expressions s^α and $(-s)^\alpha$. To obtain functions from them, we fix for branchcut lines the negative real half axis for the first and the positive real half axis for the second; for both we work on the first Riemann surface.

2. Suitable fractional derivatives

2.1. Previous comments. In [51], two criteria were proposed for deciding if a given operator can be considered as a fractional derivative. According to such criteria, there are several acceptable definitions. Later, it was shown that only some of such derivatives are suitable when thinking on the generalizing classic tools [44]. In particular, they must verify the index law in order to ensure that, given a derivative, there exists its inverse, that is the anti-derivative, [48]. In the follow-up such derivatives are recalled.

2.2. About time derivatives. Let $f(t)$, $t \in \mathbb{R}$, be a function of time, having Laplace transform, $F(s)$, with a given region of convergence and introduce the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$, $n \in \mathbb{N}$, with $(a)_0 = 1$. Let us define two sets of fractional derivatives according to the arrow of time, namely the forward and backward derivatives:

Definition 2.1 (forward Grünwald-Letnikov).

$$D_f^\alpha f(t) = \lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}f(t-nh), \tag{6}$$

Definition 2.2 (backward Grünwald-Letnikov).

$$D_b^\alpha f(t) = e^{-i\pi\alpha}\lim_{h\to 0^+} h^{-\alpha}\sum_{n=0}^{\infty}\frac{(-\alpha)_n}{n!}f(t+nh). \tag{7}$$

There are several properties exhibited by (6) and (7) [51]:

- Linearity
- Additivity and commutativity of the orders (index law). If we apply (6) twice for any two orders α and β, we have

$$D_f^\alpha D_f^\beta f(t) = D_f^\beta D_f^\alpha f(t) = D_f^{\alpha+\beta} f(t). \tag{8}$$

- Neutral and inverse elements

$$D_f^\alpha D_f^{-\alpha} f(t) = D_f^0 f(t) = f(t). \tag{9}$$

  From (9) we conclude that there is always an inverse element, that is, for every order α there is always the $-\alpha$ order derivative.
- Backward compatibility ($n \in \mathbb{N}$)
  If $\alpha = n$, then:

$$D_f^n f(t) = \lim_{h\to 0}\frac{\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(t-kh)}{h^n}.$$

  We obtain this expression repeating the first order derivative.
  If $\alpha = -n$, then:

$$D_f^{-n} f(t) = \lim_{h\to 0}\sum_{k=0}^{n}\frac{(n)_k}{k!}f(t-kh)\cdot h^n,$$

  that corresponds to a n-th repeated summation [5].

- We can apply the LT to (6) and (7) to obtain

$$\mathcal{L}\left[D_f^\alpha f(t)\right] = s^\alpha \mathcal{L}\left[f(t)\right], \tag{10}$$

with $Re(s) > 0$ in the first and $Re(s) < 0$ in the second.

Let $N = \lfloor\alpha\rfloor + 1$. For functions with LT or FT, there are integral formulations for the fractional derivatives, enjoying the same set of properties.

The most general integral formulations of derivatives, in the sense of being valid for any functions defined in $\mathbb{R}$, are introduced in the folowing definitions [44, 52].

Definition 2.3 (forward regularised Liouville derivative).

$$D_f^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \cdot \left[f(t-\tau) - u(\alpha)\sum_0^{N-1} \frac{(-1)^m f^{(m)}(t)}{m!}\tau^m \right] \tau^{-\alpha-1} d\tau, \tag{11}$$

where $u(\cdot)$ is the unit step function.

Definition 2.4 (backward regularised Liouville derivative).

$$D_b^\alpha f(t) = \frac{e^{-i\alpha\pi}}{\Gamma(-\alpha)} \int_0^\infty \cdot \left[f(t+\tau) - u(\alpha)\sum_0^{N-1} \frac{f^{(m)}(t)}{m!}\tau^m \right] \tau^{-\alpha-1} d\tau. \tag{12}$$

Other Liouville derivatives can be defined on $\mathbb{R}$. One, simply called "Liouville derivative", is similar to the Riemann-Liouville [1] and the other is the "Liouville-Caputo derivative" [44, 53, 54], similar to the Caputo derivative.

2.3. On space derivatives.

For derivatives in the time domain we consider two cases with the same LT and we interpreted them as causal and anti-causal according to the region of convergence.

When dealing with derivatives in the space domain there is no need to impose causality, since we can move in all directions. Therefore, two fractional space derivatives called "left" and "right", are adopted. The left (in space) is defined by the same expression as for the forward derivative (in time) and denoted $D_{lx}^\alpha f(x)$.

Definition 2.5 (left and right Grünwald-Letnikov derivatives).

$$D_{lx}^\alpha f(x) = \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(x-nh). \tag{13}$$

The right space derivative, $D_{rx}^\alpha f(x)$, is defined as the backward, but removing the exponential factor – see (7)

$$D_{rx}^\alpha f(x) = \lim_{h\to 0^+} h^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} f(x+nh), \tag{14}$$

The lack of the exponential factor in the right derivative, when compared with the backward definition (7), has, as consequence, that the corresponding LT is different from the LT of the left derivative. In fact, if we apply the LT to (13) and (14), then we obtain

$$\mathcal{L}\left[D_{lrx}^\alpha f(x)\right] = (\pm s)^\alpha \mathcal{L}\left[f(x)\right], \tag{15}$$

with $Re(s) > 0$, in the left derivative (+ sign), and $Re(s) < 0$, in the right (– sign) case.

2.4. Two-sided derivatives.

The composition of derivatives of the same type (e.g. left) is a derivative of the same type. If the composition is mixed, say a left with a right one, then we obtain a two-sided derivative, that can be called centred derivative.

Two centred derivatives were introduced in [55–57]. Let us consider the composition of a left and a right derivatives. As shown in [55, 56], the composition $D_{lx}^\alpha D_{rx}^\beta f(x)$ leads to the GL *centred* (two-sided) fractional derivative:

$$D_{cx}^{\alpha+\beta} f(x) := \lim_{h\to 0^+} h^{-\alpha-\beta} \sum_{n=-\infty}^{+\infty} (-1)^n \cdot \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha-n+1)\Gamma(\beta+n+1)} f(x-nh). \tag{16}$$

In a general setup we would proceed as in [57] by introducing two parameters: $\gamma = \alpha + \beta$ defining the derivative order and $\theta = \alpha - \beta$ sometimes called *skewness*. In this paper it will be called *dissymmetry*, since it determines the symmetry of the binomial parameters in (16) [57]. Nonetheless, here we retained the order as $\alpha + \beta$ to enhance the two sources.

The above formula has the usual integer order derivatives as particular cases, but it introduces also others not existing before as shown in the following examples.

Examples. Some examples show the relation between the centred and the classic derivatives

- $\alpha = 1$ and $\beta = 0$

$$D_{cx}^{1+0} f(x) = \lim_{h\to 0^+} \frac{f(x) - f(x-h)}{h}.$$

This is one of the classical derivative definitions.

- $\alpha = 0$ and $\beta = 1$

$$D_{cx}^{0+1} f(x) = \lim_{h\to 0^+} \frac{f(x) - f(x+h)}{h}.$$

Aside a factor (-1) this is another classical derivative. Substituting $-h$ for h we obtain the previous expression.

- $\alpha = 1/2$ and $\beta = 1/2$

$$D_{cx}^{1/2+1/2} = \lim_{h\to 0^+} h^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n \cdot \frac{1}{\Gamma(3/2-n)\Gamma(3/2+n)} f(x-nh).$$

This represents a new derivative without having an interpretation in classical terms.

- $\alpha = 1$ and $\beta = 1$

$$D_{cx}^{1+1} f(x) = \lim_{h \to 0^+} \frac{2f(x) - f(x+h) - f(x-h)}{h^2}.$$

This derivative is the classical centred derivative of order 2. Aside a factor (-1) coincides with the classic order two derivative

$$D_{cx}^{2} f(x) = \lim_{h \to 0^+} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}.$$

We can obtain other order 2 derivatives with other combinations. For example with $\alpha = 3/2$ and $\beta = 1/2$ we obtain

$$D_{cx}^{3/2+1/2} = \lim_{h \to 0^+} h^{-1} \sum_{n=-\infty}^{+\infty} (-1)^n$$

$$\cdot \frac{1}{\Gamma(5/2 - n)\Gamma(3/2 + n)} f(x - nh).$$

This represents also a new derivative.

We must note that the LT of the centred derivatives does not exist. In fact, it would be given by $(s)^\alpha(-s)^\beta$, but the region of convergence is the empty set. However, the corresponding Fourier transform is given by [5, 55, 56]

$$\lim_{s \to ik} (s)^\alpha(-s)^\beta = |k|^{\alpha+\beta}\, e^{i\frac{\pi}{2}(\alpha-\beta)\cdot sgn(k)}.$$

We can write

$$\mathcal{F}\left[D_{cx}^{\alpha+\beta} f(x)\right] = |k|^{\alpha+\beta}\, e^{i\frac{\pi}{2}(\alpha-\beta)\cdot sgn(k)} F(ik), \tag{17}$$

where $sgn(\cdot)$ is the signum function and $F(ik)$ is the FT of $f(x)$.
The special case where $\alpha = \beta$

$$\mathcal{F}\left[D_{cx}^{2\alpha} f(x)\right] = |k|^{2\alpha}\, F(ik) \tag{18}$$

will be very important due to its relation with the FT of the Laplacian. The properties of the two-sided derivatives are discussed in detail in [55, 56]. In the following we describe the most important:

- Linearity
- Additivity and commutativity
  If $2\alpha + 2\beta > -1$, [55, 56]

$$D_{cx}^{2\alpha}\left[D_{cx}^{2\beta} f(x)\right] = D_{cx}^{2\beta}\left[D_{cx}^{2\alpha} f(x)\right] = D_{cx}^{2\alpha+2\beta} f(x) \tag{19}$$

- Neutral and inverse elements
  In particular, with $2\alpha + 2\beta = 0$, relations (19) show that, for any centred derivative of order α, with $|\alpha| < 1/2$, there is an anti-derivative, with order $-\alpha$, [55, 56] and it can be obtained by using formula (18)

$$D_{cx}^{2\alpha}\left[D_{cx}^{-2\alpha} f(x)\right] = D_{cx}^{0} f(x) = f(x). \tag{20}$$

  This implies that the derivative of order $2\alpha = 1$ does not have inverse given by (16).
- Relation with the Riesz-Feller derivative
  In the 1-D case, the Riesz and the Riesz-Feller operators are closely related with the centred derivative defined in (16). In particular such operators can be obtained for $\alpha = \beta$, and for $\alpha - \beta = \pm 1$, respectively ([55, 56] and [1], page 214).

Remark 2.1. *All the results derived using the GL derivative can be obtained with the integral formulations, due to their equivalence, at least for functions with LT [52].*

Remark 2.2. *Partial derivatives are readily obtained from the above definitions. Usually the symbol ∂_x is adopted for denoting the partial derivatives. Here, we will continue using D_x. For example*

$$D_{lx_2}^{\alpha} = \lim_{h \to 0^+} h^{-\alpha} \sum_{n=0}^{+\infty} (-1)^n \frac{(-\alpha)_n}{n!} f(x_1, x_2 - nh, x_3).$$

For the other derivatives, the notations are similar.

3. Fractional vectorial differential operators

3.1. The gradients, divergences, and curls. Let us consider a scalar field $f(x_1, x_2, x_3)$. The usual integer order gradient is a vector with components corresponding to the partial derivatives of the scalar field. The gradient points in the direction of the largest rate of function increase and its magnitude is the slope of the graph in that direction. This important and useful tool has some limitations since it requires a smooth field and uses derivatives of order one.

Here we extend its applications by considering not only fractional derivatives, but also (and more important) different orders. With this in mind, we are able to model non-homogenous and non-isotropic spaces. Essentially we can face complexity in different directions.

For any $\alpha \in \mathbb{R}$, we define *left gradient* operator by

Definition 3.1.

$$grad_l^{\overline{\alpha}}(\cdot) = \nabla_l^{\overline{\alpha}}(\cdot) := D_{lx_1}^{\alpha_1}(\cdot)\mathbf{e}_1 + D_{lx_2}^{\alpha_2}(\cdot)\mathbf{e}_2 + D_{lx_3}^{\alpha_3}(\cdot)\mathbf{e}_3. \tag{21}$$

Similarly, for any $\overline{\alpha} \in \mathbb{R}^3$, we define *right gradient* operator through

Definition 3.2.

$$grad_r^{\overline{\beta}}(\cdot) = \nabla_r^{\overline{\beta}}(\cdot) := D_{rx_1}^{\beta_1}(\cdot)\mathbf{e}_1 + D_{rx_2}^{\beta_2}(\cdot)\mathbf{e}_2 + D_{rx_3}^{\beta_3}(\cdot)\mathbf{e}_3. \tag{22}$$

The gradients act on a scalar function defined on $\mathbb{R}^3$ and generate vectors having as components the partial derivatives of the function: *the nabla is a vectorial differential operator*. Their action over other vectors, namely vectorial functions with components defined on $\mathbb{R}^3$, originates two pairs of differential vectors obtained using the *inner or scalar product* and the *cross or vectorial product*. Let $\mathbf{f}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ be a vectorial function. Calculating the inner product of the left gradient and f we get the *left divergence* of f

$$div_l^{\overline{\alpha}}(\mathbf{f}) = \nabla_l^{\overline{\alpha}} \cdot \mathbf{f} := D_{lx_1}^{\alpha_1}\mathbf{f}_1 + D_{lx_2}^{\alpha_2}\mathbf{f}_2 + D_{lx_3}^{\alpha_3}\mathbf{f}_3. \tag{23}$$

With the right nabla we obtain the *right divergence* ($div_r^{\overline{\alpha}}(\mathbf{f})$), similarly defined.

If, instead of the inner product, we use the cross product, then we derive the *left and right curls*. For the *left curl* we have

$$curl_l^{\overline{\alpha}}(\mathbf{f}) = (\nabla_l^{\overline{\alpha}} \times \mathbf{f}) := \begin{vmatrix} D_{lx_1}^{\alpha_1} & D_{lx_2}^{\alpha_2} & D_{lx_3}^{\alpha_3} \\ \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{vmatrix}. \tag{24}$$

We can write

$$\begin{aligned}(\nabla_l^{\overline{\alpha}} \times \mathbf{f}) = & \left[D_{lx_2}^{\alpha_2}\mathbf{f}_3 - D_{lx_3}^{\alpha_3}\mathbf{f}_2\right]\mathbf{e}_1 \\ & + \left[D_{lx_3}^{\alpha_3}\mathbf{f}_1 - D_{lx_1}^{\alpha_1}\mathbf{f}_3\right]\mathbf{e}_2 \\ & + \left[D_{lx_1}^{\alpha_1}\mathbf{f}_2 - D_{lx_2}^{\alpha_2}\mathbf{f}_1\right]\mathbf{e}_3.\end{aligned} \tag{25}$$

The corresponding right ($curl_r^{\overline{\alpha}}(\mathbf{f})$) is readily obtained.

The three pairs of operators defined in (21)–(24) will be called *first generation operators* [43], because they use only the left or right derivatives, that is, not both.

Remark 3.1. *With the LT such operators can be formulated easily from a transform perspective. To exemplify, the LT of the divergence is given in the LT domain by the inner product of the vector* $\mathbf{s}^{\overline{\alpha}}$ *and the transform of the vector field,*

$$\mathcal{L}\left[div_{lr}^{\overline{\alpha}}(\mathbf{f})\right] = \left((\pm\overline{\mathbf{s}})^{\overline{\alpha}} \cdot \mathcal{L}\left[\mathbf{f}\right]\right).$$

Remark 3.2. *The choice of the region of convergence* ($Re(\overline{s}) > 0$ *or* $Re(\overline{s}) < 0$) *and the sign determines the type of divergence: left or right.*

Similarly, we obtain for the LT of the curl

$$\mathcal{L}\left[curl_{lr}^{\overline{\alpha}}(\mathbf{f})\right] = \left((\pm\overline{\mathbf{s}})^{\overline{\alpha}} \times \mathcal{L}\left[\mathbf{f}\right]\right),$$

under the conditions stated in Remark 3.2.

3.2. Mixed operators. The above operators can be combined to get *second generation operators* in several distinct ways, but in agreement with the previous ideas. The most important are those resulting from combinations of left-right operators that produce two-sided derivatives.

3.2.1. Internabla operations. Divergence and curl operators were defined by inner and outer products of the nabla (left or right) and a vectorial function. If we substitute this function by the left/right nabla, then we get the following operators

1. (a) $(\nabla_l^{\overline{\alpha}} \times \nabla_r^{\overline{\alpha}})$
   Using (25) we can write

$$\begin{aligned}(\nabla_l^{\overline{\alpha}} \times \nabla_r^{\overline{\alpha}}) = & \left[D_{lx_2}^{\alpha_2}D_{rx_3}^{\alpha_3} - D_{lx_3}^{\alpha_3}D_{rx_2}^{\alpha_2}\right]\mathbf{e}_1 \\ & + \left[D_{lx_3}^{\alpha_3}D_{rx_1}^{\alpha_1} - D_{lx_1}^{\alpha_1}D_{rx_3}^{\alpha_3}\right]\mathbf{e}_2 \\ & + \left[D_{lx_1}^{\alpha_1}D_{rx_2}^{\alpha_2} - D_{lx_2}^{\alpha_2}D_{rx_1}^{\alpha_1}\right]\mathbf{e}_3\end{aligned}$$

   It is well known that in integer order calculus we have the important property: $\nabla^1 \times \nabla^1 \equiv 0$. This is not valid with the operator $(\nabla_l^{\overline{\alpha}} \times \nabla_r^{\overline{\alpha}})$ that in general is not null.

   (b) $(\nabla_l^{\overline{\alpha}} \times \nabla_l^{\overline{\alpha}})$ and $(\nabla_r^{\overline{\alpha}} \times \nabla_r^{\overline{\alpha}})$
   We have:

$$(\nabla_l^{\overline{\alpha}} \times \nabla_l^{\overline{\alpha}}) = (\nabla_r^{\overline{\alpha}} \times \nabla_r^{\overline{\alpha}}) = 0 \tag{26}$$

   that will be useful in 3.3.

2. (a) $(\nabla_l^{\overline{\alpha}} \cdot \nabla_l^{\overline{\alpha}})$,
   With the results presented in 3.1 we have

$$(\nabla_l^{\overline{\alpha}} \cdot \nabla_l^{\overline{\alpha}}) = D_{lx_1}^{2\alpha_1} + D_{lx_2}^{2\alpha_2} + D_{lx_3}^{2\alpha_3}$$

   (b) $(\nabla_r^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}})$
   This is similar to the previous case

$$(\nabla_r^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}}) = D_{rx_1}^{2\alpha_1} + D_{rx_2}^{2\alpha_2} + D_{rx_3}^{2\alpha_3}.$$

   Remark 3.3. *The two last operators are scalar one-sided. They are similar to the fractional Laplacian that we will define next.*

   (c) $(\nabla_l^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}})$,
   Attending to the results presented in Subsec. 3.1, we obtain easily:

$$(\nabla_l^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}}) = D_{lx_1}^{\alpha_1}D_{rx_1}^{\alpha_1} + D_{lx_2}^{\alpha_2}D_{rx_2}^{\alpha_2} + D_{lx_3}^{\alpha_3}D_{rx_3}^{\alpha_3}$$

   and from (16) we derive that

$$\begin{aligned}D_{lx_1}^{\alpha_1}D_{rx_1}^{\alpha_1}f(x_1,x_2,x_3) &= D_c^{2\alpha_1}f(x_1,x_2,x_3) \\ = \lim_{h\to 0} h^{-\alpha} &\sum_{n=-\infty}^{\infty}\frac{(-1)^n\,\Gamma(2\alpha_1+1)}{\Gamma(\alpha_1+n+1)\Gamma(\alpha_1-n+1)} \\ & f(x-nh,y,z).\end{aligned} \tag{27}$$

   This expression is the centred fractional derivative [5] of order $2\alpha_1$ relatively to the variable x_1. For x_2 and x_3 the expressions are similar.

Definition 3.3. *With (27) we can define the fractional Laplacian with order* $\overline{\alpha}$ *by*

$$\Delta^{\overline{\alpha}} := (\nabla_l^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}}) = D_{cx_1}^{2\alpha_1} + D_{cx_2}^{2\alpha_2} + D_{cx_3}^{2\alpha_3}. \tag{28}$$

Remark 3.4. *When* $\alpha_1 = \alpha_2 = \alpha_3 = 1$ *the centred derivative equals the classic centred derivative of order* 2*, but we do not recover the classic Laplacian, because it uses a non centred derivative and as consequence, it yields a* (-1) *factor.*

Attending to these considerations about the use of LT in Remarks 3.1 and 3.2 and relations (17) and (18) we conclude that, for this operator, the region of convergence degenerates into the imaginary axis. This means that the suitable transform is the FT.
For the FT of the Laplacian we use (18) and the properties of the two-sided derivatives [55, 56] to obtain

$$FT\left[\Delta^{\overline{\alpha}}f(x_1,x_2,x_3)\right] = \|\mathbf{k}\|^{2\overline{\alpha}}\,F(i\overline{k}), \tag{29}$$

where

$$\|\mathbf{k}\|^{2\overline{\alpha}} = |k_1|^{2\alpha_1} + |k_2|^{2\alpha_2} + |k_3|^{2\alpha_3}. \tag{30}$$

When $\alpha_i = 1/2$, $i = 1,\ldots,3$, we obtain a new Laplacian of order $\overline{\alpha} = 1/2$ that results from the application of derivatives of order $1/2$.

The above Laplacian is valid for orders verifying $\alpha_i > -1/2$, due to the conditions of existence of the two-sided derivatives, [55,56]. However, due to the relation between the centred derivative and the Riesz potential referred above, we can compute the partial derivatives in (28) for $\alpha_i \le -1/2$ with the 1-D Riesz potential, which enlarges the validity of the definition to any orders. This means that the Laplacian introduced in (28) can be computed for any real order.

It is relevant to refer here that we can define a more general Laplacian by using operators with different orders:

$$\Delta^{\overline{\alpha},\overline{\beta}} = (\nabla_l^{\overline{\alpha}} \cdot \nabla_r^{\overline{\beta}}) = D_{cx_1}^{\alpha_1+\beta_1} + D_{cx_2}^{\alpha_2+\beta_2} + D_{cx_3}^{\alpha_3+\beta_3}$$

with FT given by

$$FT\left[\Delta^{\overline{\alpha},\overline{\beta}} f(x_1,x_2,x_3)\right] = (\overline{k})^{\overline{\alpha},\overline{\beta}} F(i\overline{k}),$$

where

$$(\overline{k})^{\overline{\alpha},\overline{\beta}} = \sum_1^3 |k_m|^{\alpha_m+\beta_m} e^{i\frac{\pi}{2}(\alpha_m-\beta_m)\cdot sgn(k_m)}.$$

3.2.2. On the Laplacian. The Laplacian is an important operator in physics and engineering. However, the standard definition does not cope with domains that are neither homogeneous, nor isotropic. Researchers in applied sciences have been interested in a definition of fractional Laplacian suitable for describing such type of media. In recent studies the fractional Laplacian was implemented by means of the inverse of the Riesz potential [58–60]. However, it is not straightforward that this is the suitable option when thinking in applications where the properties change with direction. In fact, the inverse of the Riesz potential cannot be expressed as a sum of partial derivatives and is only defined using the inverse of Fourier transform of the operator [61].

The classical Laplacian is defined in 3-D by

$$\Delta f(\overline{x}) = \frac{\partial^2 f(\overline{x})}{\partial x_1^2} + \frac{\partial^2 f(\overline{x})}{\partial x_2^2} + \frac{\partial^2 f(\overline{x})}{\partial x_3^2}, \tag{31}$$

where $\overline{x} = (x_1,x_2,x_3)$. This operator is fundamental in physics, namely in electromagnetism. Applying the FT to equation (31), we obtain

$$\mathcal{F}(-\Delta f)(\overline{k}) = |\overline{k}|^2 \widehat{f}(\overline{k}) \tag{32}$$

with

$$|\overline{k}|^2 = \sum_{j=1}^3 k_j^2. \tag{33}$$

Remark 3.5. *If instead of the usual right derivative used to define the Laplacian, the centred one is adopted, then the sign − can be removed.*

In his work [58], Riesz implicitly suggested several possible definitions of fractional Laplacian [3,58], but he did not present the explicit realization of such operators. Nonetheless, Riesz deduced expressions suggesting that the potential defined by

$$(\mathbb{I}_R^{\alpha} f)(\overline{x}) := \frac{1}{\gamma_3(\alpha)} \int_{\mathbb{R}^3} f(\overline{y}) |\overline{x}-\overline{y}|^{\alpha-3} d\overline{y}, \tag{34}$$

for any positive α, could implement the inverse of the integer order Laplacian. The normalizing constant $\gamma_3(\alpha)$ is given by

$$\gamma_3(\alpha) = \frac{\pi^{\frac{3}{2}} 2^{2\alpha}\Gamma(\alpha)}{\Gamma\left(\frac{3-2\alpha}{2}\right)} = 2\frac{\pi^2\Gamma(2\alpha)}{\Gamma\left(\frac{3-2\alpha}{2}\right)\Gamma\left(\frac{1}{2}+\alpha\right)}. \tag{35}$$

In the literature several authors have proposed expressions for the Riesz Laplacian by looking for an operator satisfying the generalization of (32), that is, by verifying:

$$\mathcal{F}((-\Delta)^{\alpha} f)(\overline{k}) = |\overline{k}|^{2\alpha} \mathcal{F}(f)(\overline{k}), \tag{36}$$

with $0 < \alpha \le 1$ and

$$|\overline{k}|^{2\alpha} = \left[\sum_{j=1}^3 k_j^2\right]^{\alpha}. \tag{37}$$

This operator can be implemented by [1] means of

$$(-\Delta)^{\alpha} f(\overline{x}) := \frac{1}{\gamma_3(\alpha)} \int_{\mathbb{R}^3} \frac{(\Delta_y^l f)(\overline{x})}{|\overline{x}-\overline{y}|^{2\alpha+3}} d\overline{y}, \tag{38}$$

where $(\Delta_y^l f)(\overline{x})$ is the l-th difference of $f(\overline{x})$. For $l > 2\alpha$ the above integral is absolutely convergent. Often we have $\alpha < 1$ and l is chosen as $l = 1$ yielding [59, 60]

$$(-\Delta)^{\alpha} f(\overline{x}) := \frac{1}{\gamma_3(\alpha)} \int_{\mathbb{R}^3} \frac{f(\overline{y}) - f(\overline{x})}{|\overline{x}-\overline{y}|^{2\alpha+3}} d\overline{y}. \tag{39}$$

Operator (39) has been used to implement the fractional Laplacian [59, 60]. However, the Riesz based Laplacian does not fit into our proposal pointing to the generalization of classic vectorial operators.

3.3. Vectorial Laplacian and mixed operators. Starting from the Laplacian formulation, (28), we can define the *vectorial Laplacian*

$$lap^{\overline{\alpha}}\mathbf{f} = \Delta^{\overline{\alpha}}\mathbf{f}_1\mathbf{e}_1 + \Delta^{\overline{\alpha}}\mathbf{f}_2\mathbf{e}_2 + \Delta^{\overline{\alpha}}\mathbf{f}_3\mathbf{e}_3 \tag{40}$$

that enjoys a very important relation shared with two pairs of **mixed operators**

1. Gradient of a divergence
   We define *gradient of a divergence* as

$$\begin{aligned} &grad_r^{\overline{\alpha}}(div_l^{\overline{\alpha}}\mathbf{f}) := \nabla_r^{\overline{\alpha}}(div_l^{\overline{\alpha}}\mathbf{f}) \\ &= D_{cx_1}^{2\alpha_1}\mathbf{f}_1\mathbf{e}_1 + D_{cx_2}^{2\alpha_2}\mathbf{f}_2\mathbf{e}_2 + D_{cx_3}^{2\alpha_3}\mathbf{f}_3\mathbf{e}_3 \\ &+D_{rx_1}^{\alpha_1}D_{lx_2}^{\alpha_2}\mathbf{f}_2\mathbf{e}_1 + D_{rx_2}^{\alpha_2}D_{lx_3}^{\alpha_3}\mathbf{f}_3\mathbf{e}_2 + D_{rx_3}^{\alpha_3}D_{lx_1}^{\alpha_1}\mathbf{f}_1\mathbf{e}_3 \\ &+D_{rx_1}^{\alpha_1}D_{lx_3}^{\alpha_3}\mathbf{f}_3\mathbf{e}_1 + D_{rx_2}^{\alpha_2}D_{lx_1}^{\alpha_1}\mathbf{f}_1\mathbf{e}_2 + D_{rx_3}^{\alpha_3}D_{lx_2}^{\alpha_2}\mathbf{f}_2\mathbf{e}_3. \end{aligned} \tag{41}$$

   Remark 3.6. *We obtain $grad_l^{\overline{\alpha}}(div_r^{\overline{\alpha}}\mathbf{f})$ with the substitutions l for r and r for l.*

   Remark 3.7. *We can also define $grad_l^{\overline{\alpha}}(div_l^{\overline{\alpha}}\mathbf{f})$ and $grad_r^{\overline{\alpha}}(div_r^{\overline{\alpha}}\mathbf{f})$, but they are one-sided operators of limited interest.*

2. Divergence of a curl
   We can define several *divergence of a curl* operators as before. The most interesting are the following

$$div_l^{\overline{\alpha}}(curl_l^{\overline{\alpha}}(\mathbf{f})) = div_l^{\overline{\alpha}}(\nabla_l^{\overline{\alpha}} \times \mathbf{f}) = \nabla_l^{\overline{\alpha}} \cdot \nabla_l^{\overline{\alpha}} \times \mathbf{f} = 0, \tag{42}$$

$$div_r^{\overline{\alpha}}(curl_r^{\overline{\alpha}}(\mathbf{f})) = div_r^{\overline{\alpha}}(\nabla_r^{\overline{\alpha}} \times \mathbf{f}) = \nabla_r^{\overline{\alpha}} \cdot \nabla_r^{\overline{\alpha}} \times \mathbf{f} = 0, \tag{43}$$

   because they are identically null as in the usual integer order case.

3. Curl of a curl

We start defining *curl of a curl* by $curl_r(curl_l(\mathbf{f}))$. To obtain a closed form for this operator we use the vectorial Laplacian to get:

$$curl_r^{\overline{\alpha}}(curl_l^{\overline{\alpha}}(\mathbf{f})) = grad_l^{\overline{\alpha}}(div_r^{\overline{\alpha}}(\mathbf{f})) - lap^{\overline{\alpha}}\mathbf{f} \tag{44}$$

in agreement with the classical result. Obviously we can have alternative order for the symbols "*l*" and "*r*" obtaining a second valid relation:

$$curl_l^{\overline{\alpha}}(curl_r^{\overline{\alpha}}(\mathbf{f})) = grad_r^{\overline{\alpha}}(div_l^{\overline{\alpha}}(\mathbf{f})) - lap^{\overline{\alpha}}\mathbf{f}. \tag{45}$$

Remark 3.8. *It is possible to generalize and to combine operators with any triplets of orders:* $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \neq \overline{\beta} = (\beta_1, \beta_2, \beta_3)$ *as we did before for the Laplacian.*

Remark 3.9. *In the above expressions it was assumed that the orders of the operators are fixed. Nevertheless, we can consider changing the orders of derivation. As concerns the derivation in time, we only have to perform another derivative using a suitable order, since the proposed time derivative verifies the additivity of the orders that can assume any real value. Concerning the space derivatives, the problem may not have a such simple solution. It was conjecturated [43] that we can substitute a given field,* $\mathbf{F}$, *by* $grad_r^{\overline{\beta}} div_l^{\overline{\alpha}}\mathbf{F}$, *but that conjecture remains to be proved.*

3.4. Fractional generalized Helmholtz decomposition theorem.

The Helmholtz decomposition theorem is a classical result that allows the decomposition a given time independent field into curl-free and divergence-free components [62, 63]. This theorem was extended by Kapuścik for the time variant case and simultaneously deduced some gauge invariants of classical field theories [64]. This development was used by Nevels [65] to present a derivation of Maxwell equations. Following Kapuścik's procedure, similar conclusions for a fractional setup were obtained in [43]. Essentially the procedure consists in the decomposition of two fields in terms of time-dependent scalar and vectorial potentials. The decomposition involves the vectorial tools presented in the previous sections and the forward fractional time derivative (to impose causality).

Theorem 3.1. *Helmholtz decomposition theorem*

1. *Consider a pair of time-dependent vector fields* $\mathbf{F_1}$ *and* $\mathbf{F_2}$ *that are null at an infinitely long distance. Such pair is uniquely determined by*

$$r_1 = div_l^{\overline{\alpha}}\mathbf{F}_1, \tag{46}$$

$$r_2 = div_r^{\overline{\alpha}}\mathbf{F}_2, \tag{47}$$

and

$$\mathbf{j}_1 = curl_r^{\overline{\alpha}}\mathbf{F}_1 + a_2\frac{\partial^\gamma \mathbf{F}_2}{\partial t^\gamma}, \tag{48}$$

$$\mathbf{j}_2 = curl_l^{\overline{\alpha}}\mathbf{F}_2 + a_1\frac{\partial^\gamma \mathbf{F}_1}{\partial t^\gamma}, \tag{49}$$

where a_1 *and* a_2 *are two constants with physical meaning used to match the physical dimensions.*

2. *For the above pair of fields there are four potentials, two scalar,* ϕ_1 *and* ϕ_2, *and two vectorial,* $\mathbf{A}_1$ *and* $\mathbf{A}_2$, *that are able to represent those fields according to*

$$\mathbf{F}_1 = -grad_r^{\overline{\alpha}}\phi_1 - \frac{\partial^\gamma \mathbf{A}_1}{\partial t^\gamma} + \frac{1}{a_1} curl_l^{\overline{\alpha}}\mathbf{A}_2, \tag{50}$$

$$\mathbf{F}_2 = -grad_l^{\overline{\alpha}}\phi_2 - \frac{\partial^\gamma \mathbf{A}_2}{\partial t^\gamma} + \frac{1}{a_2} curl_r^{\overline{\alpha}}\mathbf{A}_1. \tag{51}$$

Remark 3.10. *In this generalization of the Helmholtz decomposition, the gradient and the rotational have different character: one is left and the other right-sided.*

Remark 3.11. *The scalars* r_1 *and* r_2 *and the vectors* $\mathbf{j}_1$ *and* $\mathbf{j}_1$ *have the role of inputs to the system defined by (46) to (49).*

Remark 3.12. *The two fields are interlaced. The solution of (46) to (49) for obtaining the fields consists of decoupling that relationship.*

Remark 3.13. *Another manifestation of that relationship can be obtained with the divergence operators in (48) and (49) using (42) and (43) to get*

$$div_r^{\overline{\alpha}}\mathbf{j}_1 = a_2\frac{\partial^\gamma r_2}{\partial t^\gamma}, \tag{52}$$

$$div_l^{\overline{\alpha}}\mathbf{j}_2 = a_1\frac{\partial^\gamma r_1}{\partial t^\gamma}. \tag{53}$$

Remark 3.14. *The scalar and vectorial potentials serve as intermmediate tools to obtain the fields (outputs of the system).*

3.4.1. Solution in terms of potentials.

To justify this theorem it was shown that the pair of fields can be determined from the four potentials [43]. As referred above we consider r_1, r_2, $\mathbf{j}_1$, and $\mathbf{j}_2$ as inputs to the system and $\mathbf{F}_1$ and $\mathbf{F}_2$ the outputs. The scalar and vectorial potentials are auxilliary functions. Therefore, we need relations involving such potentials serving as intermmediate tools to compute the fields from the input functions.

Such relations are obtained by applying the divergence and rotational operators for both fields and manipulating the equations with the help of the vectorial operators introduced in Sec. 3. As shown in [43] the four potentials are completely determined from four equations and once they are known we obtain the fields F_1 and F_2. Such equations assume the form

$$\begin{aligned} a_1 j_1 = {} & grad_l^{\overline{\alpha}}\left[div_r^{\overline{\alpha}}\mathbf{A}_2 - a_1 a_2\frac{\partial^\gamma \phi_2}{\partial t^\gamma}\right] \\ & - \left[lap^{\overline{\alpha}}\mathbf{A}_2 + a_1 a_2\frac{\partial^{2\gamma}\mathbf{A}_2}{\partial t^{2\gamma}}\right], \end{aligned} \tag{54}$$

$$\begin{aligned} a_2 j_2 = {} & grad_r^{\overline{\alpha}}\left[div_l^{\overline{\alpha}}\mathbf{A}_1 - a_1 a_2\frac{\partial^\gamma \phi_1}{\partial t^\gamma}\right] \\ & - \left[lap^{\overline{\alpha}}\mathbf{A}_1 + a_1 a_2\frac{\partial^{2\gamma}\mathbf{A}_1}{\partial t^{2\gamma}}\right], \end{aligned} \tag{55}$$

$$r_1 = -\frac{\partial^\gamma}{\partial t^\gamma}\left[div_l^{\overline{\alpha}}\mathbf{A}_1 - a_1 a_2 \frac{\partial^\gamma \phi_1}{\partial t^\gamma}\right] - \left[\Delta^{\overline{\alpha}}\phi_1 + a_1 a_2 \frac{\partial^{2\gamma}\phi_1}{\partial t^{2\gamma}}\right], \tag{56}$$

$$r_2 = -\frac{\partial^\gamma}{\partial t^\gamma}\left[div_r^{\overline{\alpha}}\mathbf{A}_2 - a_1 a_2 \frac{\partial^\gamma \phi_2}{\partial t^\gamma}\right] - \left[\Delta^{\overline{\alpha}}\phi_2 + a_1 a_2 \frac{\partial^{2\gamma}\phi_2}{\partial t^{2\gamma}}\right]. \tag{57}$$

This formulation suggests that for each pair $(\phi_i, \mathbf{A}_i)$, $i = 1, 2$, we expect to have:

1.
$$div_l^{\overline{\alpha}}\mathbf{A}_1 - a_1 a_2 \frac{\partial^\gamma \phi_1}{\partial t^\gamma} = 0, \tag{58}$$
$$div_r^{\overline{\alpha}}\mathbf{A}_2 - a_1 a_2 \frac{\partial^\gamma \phi_2}{\partial t^\gamma} = 0, \tag{59}$$
allowing us to obtain 4 decoupled equations.
2. Let u_1 and u_2 be two scalar potentials, and $\mathbf{B}_1$ and $\mathbf{B}_2$ be two vectorial potentials verifying (58), (59), and (61). Then, the fields (50) and (51) remain invariant under the transformations:
$$\begin{aligned} \phi_1 &\longrightarrow \phi_1 - \tfrac{\partial^\gamma u_1}{\partial t^\gamma} - \tfrac{1}{a_1 a_2} div_l^{\overline{\alpha}}\mathbf{B}_1 \\ \phi_2 &\longrightarrow \phi_2 - \tfrac{\partial^\gamma u_2}{\partial t^\gamma} - \tfrac{1}{a_1 a_2} div_r^{\overline{\alpha}}\mathbf{B}_2 \\ \mathbf{A}_1 &\longrightarrow \mathbf{A}_1 + grad_r^{\overline{\alpha}} u_1 - \tfrac{\partial^\gamma \mathbf{B}_1}{\partial t^\gamma} - \tfrac{1}{a_1} curl_l^{\overline{\alpha}}\mathbf{B}_2 \\ \mathbf{A}_2 &\longrightarrow \mathbf{A}_2 + grad_l^{\overline{\alpha}} u_2 - \tfrac{\partial^\gamma \mathbf{B}_2}{\partial t^\gamma} - \tfrac{1}{a_2} curl_r^{\overline{\alpha}}\mathbf{B}_1 \end{aligned} \tag{60}$$
3. Any other scalar or vector potential used to obtain the invariants must give a null contribution to the fields, meaning that they must verify the condition:
$$\Delta^{\overline{\alpha}}\psi + a_1 a_2 \frac{\partial^{2\gamma}\psi}{\partial t^{2\gamma}} = 0. \tag{61}$$

According to the above relations (54) to (61) we conclude that we can always choose the scalar and vectorial potentials verifying (61). This together with (60) implies that the conditions (58) and (59) are also satisfied leading to simplified decoupled equations for the potentials

$$a_1 j_1 = -\left[lap^{\overline{\alpha}}\mathbf{A}_2 + a_1 a_2 \frac{\partial^{2\gamma}\mathbf{A}_2}{\partial t^{2\gamma}}\right], \tag{62}$$

$$a_2 j_2 = -\left[lap^{\overline{\alpha}}\mathbf{A}_1 + a_1 a_2 \frac{\partial^{2\gamma}\mathbf{A}_1}{\partial t^{2\gamma}}\right], \tag{63}$$

$$r_1 = -\left[\Delta^{\overline{\alpha}}\phi_1 + a_1 a_2 \frac{\partial^{2\gamma}\phi_1}{\partial t^{2\gamma}}\right], \tag{64}$$

$$r_2 = -\left[\Delta^{\overline{\alpha}}\phi_2 + a_1 a_2 \frac{\partial^{2\gamma}\phi_2}{\partial t^{2\gamma}}\right], \tag{65}$$

establishing a set of 8 independent equations involving the Laplacian. Once the potentials are computed, the fields $\mathbf{F}_1$ and $\mathbf{F}_2$ are readily obtained using (50) and (51).

3.4.2. Decoupled solution involving the fields. We can avoid passing by intermmediate potentials if we use wave equations for the fields (see [43]). Such equations are

$$lap^{\overline{\alpha}}\mathbf{F}_1 + a_1 a_2 \frac{\partial^{2\gamma}\mathbf{F}_1}{\partial t^{2\gamma}} = -curl_l^{\overline{\alpha}}\mathbf{j}_1 + grad_r^{\overline{\alpha}} r_1 + a_2 \frac{\partial^\gamma \mathbf{j}_2}{\partial t^\gamma}, \tag{66}$$

$$lap^{\overline{\alpha}}\mathbf{F}_2 + a_1 a_2 \frac{\partial^{2\gamma}\mathbf{F}_2}{\partial t^{2\gamma}} = -curl_r^{\overline{\alpha}}\mathbf{j}_2 + grad_l^{\overline{\alpha}} r_2 + a_1 \frac{\partial^\gamma \mathbf{j}_1}{\partial t^\gamma}. \tag{67}$$

Remark 3.15. *The above theory only makes sense if it is a true generalization of the classic integer order counterpart. Therefore, when the orders in the gradients, divergences, and curls become 1 we have to obtain the classic Helmholtz decomposition theorem. From the derivative definitions given in Sec. 2 we conclude that the fractional left and right gradients, divergences, and curls recover the classic corresponding operators, since there is no difference between integer order left and right derivatives. On the other hand, these operators are continuous functions of the orders. The unique difference lies in the Laplacian definition. The one proposed here leads to an expression that has a "−" sign when compared to the classic one, due to the use of the two-sided derivative.*

Remark 3.16. *There are several approaches for obtaining the fractional Maxwell equations [22, 39, 40, 66], but without the coherence given by the previous formalism. In [43] a version of fractional Maxwell equations based on the generalized Helmholtz decomposition was proposed.*

4. On the fractional definite integrals

4.1. Previous considerations. The concept of fractional definite integrals (FDI) was introduced in [48]. Starting with a generalization of Barrow formula an FDI definition was proposed, followed by the formulation of the "fractional fundamental theorem of calculus". These developments allowed the definition of double and triple integrals on rectangular spaces. Here we will recall those results in the scope of the present work.

4.2. A generalized Barrow formula. Let $f(x)$, $x \in \mathbb{R}$, such that $f_{lr}^{(\alpha)}(x)$, $\alpha > 0$, exist – it is enough to assume that $f(x)$ has LT with a non void region of convergence. We denote the two cases of left and right anti-derivatives by $f_{lr}^{(-\alpha)}(x)$.

Definition 4.1. *The α-order fractional integral (FI) of $f(x)$ over the interval (a, b) is defined through the fractional Barrow formula*

$$I_{lr}^{\alpha} f(a,b) = \int_a^b f(x)\, dx^\alpha = f_{lr}^{(-\alpha)}(b) - f_{lr}^{(-\alpha)}(a), \tag{68}$$

where the dashed integral symbol represents the FI. Using the Liouville anti-derivative expression we can write

$$I^{\alpha}_{lr} f(a,b) = \frac{1}{\Gamma(\alpha+1)} \int\limits_0^{\infty} [f(b \pm x) - f(a \pm x)] \, dx^{\alpha}, \quad (69)$$

since $\frac{Dx^{\alpha}}{\alpha} = x^{\alpha-1}$, as $x \geq 0$. From the standard (integer order) Barrow formula $\int\limits_a^b f'(x)dx = f(b) - f(a)$ we obtain the expression

$$I^{\alpha}_{lr} f(a,b) = \frac{1}{\Gamma(\alpha+1)} \int\limits_0^{\infty} \int\limits_a^b f'(y \pm x) dy dx^{\alpha}. \quad (70)$$

Since the function $f(x)$ has LT, the integral in (69) exists. The same happens with inner integral in (70) that is uniformly convergent. Consequently, we can commute the integrations to obtain

$$I^{\alpha}_{lr} f(a,b) = \int\limits_a^b f^{(-\alpha+1)}_{lr}(x) dx. \quad (71)$$

Considering as variable the upper limit, $b = z \in \mathbb{R}$, (71) leads to a *fractional formulation of the fundamental theorem of integral calculus*. In fact, setting $f(x) = D^{\alpha}_{lr} g(x)$ we can write [48]

$$I^{\alpha}_{lr} D^{\alpha}_{lr} g(a,z) = \int\limits_a^z D^{(1)}_{lr} g(x) dx = g(z) - g(a), \quad (72)$$

and

$$D^{\alpha}_{lr}\left[I^{\alpha}_{lr} f(a,z)\right] = D^{\alpha}_{lr}\left[f^{(-\alpha)}_{lr}(z) - f^{(-\alpha)}_{lr}(a)\right] = f(z), \quad (73)$$

since, for the adopted formulations of derivatives, the derivative of a constant is zero.

4.3. Integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$. Let us assume that the function g is dependent of a variable, x_1, and a parameter, x_2, that is kept fixed, $f(x_1, x_2)$. We define the parametric integral

$$\begin{aligned} I^{\alpha}_{lr} g(a_1, b_1, x_2) &= \fint\limits_{a_1}^{b_1} g(x_1, x_2) dx_1^{\alpha_1} \\ &= g^{(-\alpha_1)}_{lr}(b_1, x_2) - g^{(-\alpha_1)}_{lr}(a_1, x_2). \end{aligned} \quad (74)$$

Similarly, fixing x_1 and having x_2 for variable, we define the parametric integral

$$\begin{aligned} I^{\alpha}_{lr} g(x_1, a_2, b_2) &= \fint\limits_{a_2}^{b_2} g(x_1, x_2) dx_2^{\alpha_2} \\ &= g^{(-\alpha_2)}_{lr}(x_1, b_2) - g^{(-\alpha_2)}_{lr}(x_1, a_2). \end{aligned} \quad (75)$$

This motivates to the following definition.

Definition 4.2. *The FI on a rectangular region $(a_1, b_1) \times (a_2, b_2)$ is given by*

$$I^{\alpha}_{lr} g(a_1, b_1, a_2, b_2) = \fint\limits_{a_1}^{b_1} \fint\limits_{a_2}^{b_2} g(x_1, x_2) dx_1^{\alpha_1} dx_2^{\alpha_2}, \quad (76)$$

where each integration is obtained by the fractional Barrow formula.

If the orders of the FD are equal (i.e., $\alpha_1 = \alpha_2 = \alpha$), then we can consider the standard expression for surface, $x_1 x_2 = S$, to get the fractional surface integral

$$I^{\alpha}_{lr} g(a_1, b_1, a_2, b_2) = \fint\limits_S g(x_1, x_2) dS^{\alpha}. \quad (77)$$

The integral in $\mathbb{R}^3$ is obtained in a similar way.

Definition 4.3. *The FI in $\mathbb{R}^3$ is defined by*

$$\begin{aligned} &I^{\alpha}_{lr} g(a_1, b_1, a_2, b_2, a_3, b_3) \\ &= \fint\limits_{a_1}^{b_1} \fint\limits_{a_2}^{b_2} \fint\limits_{a_3}^{b_3} g(x_1, x_2, x_3) dx_1^{\alpha_1} dx_2^{\alpha_2} dx_3^{\alpha_3}, \end{aligned} \quad (78)$$

where each integration is performed by means of the fractional Barrow formula.

If the derivative orders are equal (i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$), then we get

$$I^{\alpha}_{lr} g(a_1, b_1, a_2, b_2, a_3, b_3) = \fint\limits_V g(x_1, x_2, x_3) dV^{\alpha}, \quad (79)$$

where we considered the standard expression for volume, $V = x_1 x_2 x_3$.

5. Fractional classic integral theorems in rectangular domains

In this section we generalize the classic integral theorems of Green, Stokes, and Ostrogradski-Gauss by following a procedure similar to the one adopted in [46]. Furthermore, we will use the left derivative unless another option is explicitly mentioned.

5.1. The Green's theorem. Let us consider a closed path γ defining a rectangle in the horizontal plane $\Sigma = \{(x_1, x_2) : a_1 \leq b_1, a_2 \leq b_2\}$ and a vectorial function $\mathbf{f}(x_1, x_2) = f_1(x_1, x_2)\vec{e}_1 + f_2(x_1, x_2)\vec{e}_2$, where $\vec{e}_i$, $i = 1, 2$, define the usual orthogonal base in $\mathbb{R}^2$. Let α_i, $i = 1, 2$, be the order of the derivative with respect to the variable x_i. The line FI along γ is defined by:

$$\begin{aligned} I^{\overline{\alpha}}_{lr} f(\gamma) &= \fint\limits_{a_1}^{b_1} f_1(x_1, a_2) dx_1^{\alpha_1} + \fint\limits_{a_2}^{b_2} f_2(b_1, x_2) dx_2^{\alpha_2} \\ &+ \fint\limits_{b_1}^{a_1} f_1(x_1, b_2) dx_1^{\alpha_1} + \fint\limits_{b_2}^{a_2} f_2(a_1, x_2) dx_2^{\alpha_2} \\ &= \fint\limits_{a_1}^{b_1} [f_1(x_1, a_2) - f_1(x_1, b_2)] \, dx_1^{\alpha_1} \\ &+ \fint\limits_{a_2}^{b_2} [f_2(b_1, x_2) - f_2(a_1, x_2)] \, dx_2^{\alpha_2}. \end{aligned}$$

Using the fractional formulation of the fundamental theorem of integral calculus (72) and the Fubini theorem [67], we can write

$$I_{lr}^{\overline{\alpha}} f(\gamma) = \fint_{a_1}^{b_1}\fint_{a_2}^{b_2}$$
$$\cdot \left[-D_{x_2}^{\alpha_2} f_1(x_1,x_2) + D_{x_1}^{\alpha_1} f_2(x_1,x_2)\right] dx_1^{\alpha_1} dx_2^{\alpha_2}.$$

In this line of thought, introducing the notation $\fint_{\gamma} f(\gamma) d\gamma^{\overline{\alpha}} = I_{lr}^{\overline{\alpha}} f(\gamma)$, we can formulate the following theorem.

Theorem 5.1. *The fractional Green's theorem for a rectangular domain Σ is given by*

$$\fint_{\gamma} f(\gamma) d\gamma^{\overline{\alpha}} = \fint_{a_1}^{b_1}\fint_{a_2}^{b_2} \cdot \left[D_{x_1}^{\alpha_1} f_2(x_1,x_2) - D_{x_2}^{\alpha_2} f_1(x_1,x_2)\right] dx_1^{\alpha_1} dx_2^{\alpha_2}. \tag{80}$$

This theorem is valid for a region formed by juxtaposing a finite number of small rectangles. In this case the integration paths is a succession of horizontal and vertical segments. We can write $d\gamma^{\alpha} = c dx_1^{\alpha_1} + (1-c) dx_2^{\alpha_2}$, with $c = 1$ and $c = 0$ for horizontal and vertical segments, respectively.

Consider two rectangles placed side by side, $\Sigma_1 = \{(x_1,x_2) : a_1 \leq b_1, a_2 \leq b_2\}$, and $\Sigma_2 = \{(x_1,x_2) : b_1 \leq 2b_1 - a_1, a_2 \leq b_2\}$. The corresponding surrounding lines are γ_1 and γ_2. Let Σ be the union of the two rectangles and γ the surrouding line. We have:

$$\fint_{\gamma_1} f(\gamma) d\gamma^{\overline{\alpha}} + \fint_{\gamma_2} f(\gamma) d\gamma^{\overline{\alpha}}$$
$$= \fint_{a_1}^{b_1} f_1(x_1,a_2) dx_1^{\alpha_1} + \fint_{a_2}^{b_2} f_2(b_1,x_2) dx_2^{\alpha_2}$$
$$+ \fint_{b_1}^{a_1} f_1(x_1,b_2) dx_1^{\alpha_1} + \fint_{b_2}^{a_2} f_2(a_1,x_2) dx_2^{\alpha_2} + \fint_{b_1}^{\widehat{b}_1} f_1(x_1,a_2) dx_1^{\alpha_1}$$
$$+ \fint_{a_2}^{b_2} f_2(b_1,x_2) dx_2^{\alpha_2} + \fint_{\widehat{b}_1}^{b_1} f_1(x_1,b_2) dx_1^{\alpha_1} + \fint_{b_2}^{a_2} f_2(b_1,x_2) dx_2^{\alpha_2}$$
$$= \fint_{a_1}^{\widehat{b}_1} f_1(x_1,a_2) dx_1^{\alpha_1} + \fint_{b_2}^{a_2} f_2(2b_1 - a_1,x_2) dx_2^{\alpha_2}$$
$$+ \fint_{\widehat{b}_1}^{b_1} f_1(x_1,b_2) dx_1^{\alpha_1} + \fint_{a_2}^{b_2} f_2(a_1,x_2) dx_2^{\alpha_2} = \fint_{\gamma} f(\gamma) d\gamma^{\overline{\alpha}}.$$

We set $2b_1 - a_1 = \widehat{b}_1$. This procedure can be used with regions bounded by integration paths similar to that illustrated in Fig. 1.

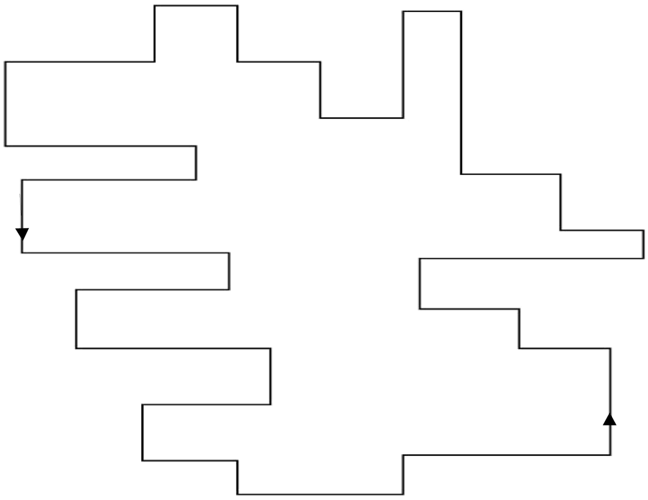

Fig. 1. Example of a rectangular domain in $\mathbb{R}^2$. The boundary is a rectangular line γ

5.2. The Stokes' theorem. In this case we adopt a strategy similar to the one followed for the Green's theorem, with the difference that now the integration path is a line in $\mathbb{R}^3$ with 6 segments parallel to the coordinate axis.

Let us consider a surface constituted by 3 contiguous faces of a parallelepiped $\Psi = \{(x_1,x_2,x_3) : a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3\}$ and a vectorial function $\mathbf{f}(x_1,x_2,x_3) = f_1(x_1,x_2,x_3)\vec{e}_1 + f_2(x_1,x_2,x_3)\vec{e}_2 + f_3(x_1,x_2,x_3)\vec{e}_3$, and α_i, $i = 1,2,3$, the derivative orders relatively to variables x_i. The 3 faces define a path, γ, with 6 segments. To compute the fractional line integral along γ we apply to each face the Green's theorem above introduced. The direction of the walk must be the same in each face. This leads to a cancellation of the contributions in edges belonging to two contiguous faces. We can write

- Upper horizontal face

$$\fint_{\gamma_3} f(\gamma) d\gamma^{\alpha} = \fint_{a_1}^{b_1}\fint_{a_2}^{b_2} \left[D_{x_1}^{\alpha_1} f_2(x_1,x_2,b_3) - D_{x_2}^{\alpha_2} f_1(x_1,x_2,b_3)\right] dx_1^{\alpha_1} dx_2^{\alpha_2}. \tag{81}$$

- Close frontal face

$$\fint_{\gamma_1} f(\gamma) d\gamma^{\alpha} = \fint_{a_2}^{b_2}\fint_{a_3}^{b_3} \left[D_{x_2}^{\alpha_2} f_3(b_1,x_2,x_3) - D_{x_3}^{\alpha_3} f_2(b_1,x_2,x_3)\right] dx_2^{\alpha_2} dx_3^{\alpha_3}. \tag{82}$$

- Right hand face

$$\fint_{\gamma_2} f(\gamma) d\gamma^{\alpha} = \fint_{a_1}^{b_1}\fint_{a_3}^{b_3} \left[D_{x_1}^{\alpha_1} f_3(x_1,b_2,x_3) - D_{x_3}^{\alpha_3} f_1(x_1,b_2,x_3)\right] dx_1^{\alpha_1} dx_3^{\alpha_3}. \tag{83}$$

Attending to the definition of the fractional curl operator introduced in Sec. 3, we have

$$\begin{aligned}(\nabla_l^{\overline{\alpha}} \times \mathbf{f}) &= \left[D^{\alpha_2}_{lx_2}\mathbf{f}_3 - D^{\alpha_3}_{lx_3}\mathbf{f}_2\right]\mathbf{e}_1 \\ &+ \left[D^{\alpha_3}_{lx_3}\mathbf{f}_1 - D^{\alpha_1}_{lx_1}\mathbf{f}_3\right]\mathbf{e}_2 + \left[D^{\alpha_1}_{lx_1}\mathbf{f}_2 - D^{\alpha_2}_{lx_2}\mathbf{f}_1\right]\mathbf{e}_3\end{aligned} \tag{84}$$

we can rewrite the relations (81)–(83) in the form

- Upper horizontal face ($x_3 = b_3$)

$$\int_{\gamma_3} f(\gamma)d\gamma^{\alpha} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \nabla_l^{\overline{\alpha}} \times \mathbf{f}\cdot\vec{e}_3\, dx_1^{\alpha_1}dx_2^{\alpha_2}. \tag{85}$$

- Close frontal face ($x_1 = b_1$)

$$\int_{\gamma_1} f(\gamma)d\gamma^{\alpha} = \int_{a_2}^{b_2}\int_{a_3}^{b_3} \nabla_l^{\overline{\alpha}} \times \mathbf{f}\cdot\vec{e}_1\, dx_2^{\alpha_2}dx_3^{\alpha_3}. \tag{86}$$

- Right hand face ($x_2 = b_2$)

$$\int_{\gamma_2} f(\gamma)d\gamma^{\alpha} = \int_{a_1}^{b_1}\int_{a_3}^{b_3} \nabla_l^{\overline{\alpha}} \times \mathbf{f}\cdot\vec{e}_2\, dx_1^{\alpha_1}dx_3^{\alpha_3}. \tag{87}$$

Adding the 3 terms we obtain the following theorem.

Theorem 5.2. *The fractional Stokes's theorem for a parallelepipedic domain, Ψ, becomes*

$$\int_{\gamma} f(\gamma)d\gamma^{\overline{\alpha}} = \sum_{i=1}^{3}\int_{a_j}^{b_j}\int_{a_k}^{b_k} \nabla_l^{\overline{\alpha}} \times \mathbf{f}\cdot\vec{e}_i\, dx_j^{\alpha_j}dx_k^{\alpha_k}, \tag{88}$$

where $j,k = 1,2,3,\ j<k,\ j,k \neq i$, and, for each $i = 1,2,3$, the variable x_i is kept constant equal to b_i.

As in the classical case, we define the flux, Φ_3, through a horizontal face by

$$\begin{aligned}\Phi_3 &= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \mathbf{f}(x_1,x_2,x_3)\cdot\vec{e}_3\, dx_1^{\alpha_1}dx_2^{\alpha_2} \\ &= \pm\int_{a_1}^{b_1}\int_{a_2}^{b_2} f_3(x_1,x_2,x_3)dx_1^{\alpha_1}dx_2^{\alpha_2},\end{aligned}$$

where the $-$ or $+$ signs apply when $x_3 = a_3$ or $x_3 = b_3$. The fluxes Φ_1 and Φ_2, through frontal and lateral faces are defined in a similar way. Therefore, the factional Stokes' states that the fractional integral of a vectorial field along a rectangular line is equal to the total flux of the curl of such field through a surface supported on such line.

5.3. The Ostrogradski-Gauss's theorem. Let us consider a parallelepipedic region Ψ previously defined and a vectorial function $\mathbf{f}(x_1,x_2,x_3) = f_1(x_1,x_2,x_3)\vec{e}_1 + f_2(x_1,x_2,x_3)\vec{e}_2 + f_3(x_1,x_2,x_3)\vec{e}_3$. Since each face is parallel to a coordinate plane, its normal is parallel to one of the unity vectors $\vec{e}_i,\ i = 1,2,3$.

According to the fractional fundamental theorem of integral calculus, summing the flux contributions of two parallel faces, we get

$$\begin{aligned}&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[f_3(x_1,x_2,b_3) - f_3(x_1,x_2,a_3)\right]dx_1^{\alpha_1}dx_2^{\alpha_2} \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} D^{\alpha_3}_{x_3}f_3(x_1,x_2,x_3)dx_1^{\alpha_1}dx_2^{\alpha_2}dx_3^{\alpha_3}.\end{aligned}$$

We can sum of the six contributions and represent symbolically the result by

$$\begin{aligned}&\oint_S (\mathbf{f}(x_1,x_2,x_3)\cdot\vec{n})dS^{\alpha} \\ &= \int_{a_2}^{b_2}\int_{a_3}^{b_3} D^{\alpha_1}_{x_1}f_1(b_1,x_2,x_3)dx_2^{\alpha_2}dx_3^{\alpha_3} \\ &+ \int_{a_1}^{b_1}\int_{a_3}^{b_3} D^{\alpha_2}_{x_2}f_2(x_1,b_2,x_3)dx_1^{\alpha_1}dx_3^{\alpha_3} \\ &+ \int_{a_1}^{b_1}\int_{a_2}^{b_2} D^{\alpha_3}_{x_3}f_3(x_1,x_2,b_3)dx_1^{\alpha_1}dx_2^{\alpha_2},\end{aligned}$$

where $j,k = 1,2,3,\ j \neq k \neq i$ and $x_i = b_i$. The normal $\vec{n}$ is equal to one of the base vectors, $\vec{e}_i,\ i = 1,2,3$, according to the integration surface.

With this result we can state the following theorem.

Theorem 5.3. *Let the left divergence operator be defined in (23). Then the fractional Ostrogradski-Gauss or Divergence theorem can be written as*

$$\begin{aligned}&\oint_S \left(\mathbf{f}(x_1,x_2,x_3)\cdot\vec{n}\right)\, dS^{\alpha} \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} div_l^{\overline{\alpha}}(\mathbf{f})\, dx_1^{\alpha_1}dx_2^{\alpha_2}dx_3^{\alpha_3}.\end{aligned} \tag{89}$$

In particular, $\mathbf{f}(x_1,x_2,x_3)$ can be set equal to the gradient of a scalar function: $\mathbf{f}(x_1,x_2,x_3) = grad_r^{\alpha}(g)$ leading to

Corollary 5.1.

$$\begin{aligned}&\oint_S \left(grad_r^{\overline{\alpha}}(g)\cdot\vec{n}\right)\, dS^{\alpha} \\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \Delta^{\overline{\alpha}}(g)\, dx_1^{\alpha_1}dx_2^{\alpha_2}dx_3^{\alpha_3}.\end{aligned} \tag{90}$$

Remark 5.1. *As referred previously, the results deduced for a parallelepiped can be generalized to the region that results from joining as many parallelepipeds as needed in such a way that all the small surfaces are parallel to one coordinated plane.*

5.4. Uniform derivative order cases. In the previous sections we assumed different orders for the three space directions. Herein, we derive formulae for the particular case of uniform order, that is, for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

In this case the FI on a rectangular region constituted by contiguous rectangles is given by (77) and is reproduced here

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2) = \fint_S f(x_1, x_2) dS^{\alpha}. \tag{91}$$

The integral in $\mathbb{R}^3$ was obtained in a similar way and the FI in $\mathbb{R}^3$ was defined in (79) and reads

$$I^{\alpha}_{lr} f(a_1, b_1, a_2, b_2, a_3, b_3) = \fint_V f(x_1, x_2, x_3) dV^{\alpha}. \tag{92}$$

A particular case of the Green's theorem occours when $\alpha_1 = \alpha_2 = \alpha$ and can be written as

$$\fint_\gamma f(\gamma) d\gamma^{\alpha} = \fint_S \left[D^{\alpha}_{x_1} f_2(x_1, x_2) - D^{\alpha}_{x_2} f_1(x_1, x_2) \right] dS^{\alpha}. \tag{93}$$

Similarly, for Stokes' theorem 5.2, if $\alpha_i = \alpha,\ i = 1, 2, 3$, then we can write from (88)

$$\fint_\gamma f(\gamma) d\gamma^{\alpha} = \sum_{i=1}^{3} \fint_{S_i} \nabla^{\alpha} \times \mathbf{f} \cdot \vec{e}_i \, dS_i^{\alpha}, \tag{94}$$

where S_i, $i = 1, 2, 3$, is a surface with normal given by $\vec{e}_i$, $i = 1, 2, 3$.

Concerning the Ostrogradski-Gauss's theorem 5.3, in the case of uniform orders we can write, from (89)

$$\oint_S \mathbf{f}(x_1, x_2, x_3) \cdot \vec{n} \, dS^{\alpha} = \fint_V div^{\alpha}(\mathbf{f}) \, dV^{\alpha}. \tag{95}$$

With $\alpha = 1$, we recover the classical results.

6. Conclusions

In this paper, a generalization of classical fractional vectorial calculus was proposed, by considering the differential and integral formulations. Concerning the first we presented fractional versions of the vectorial tools, namely, the gradient, divergence, curl, and Laplacian. For this purpose, suitable formulations of fractional derivatives were described. Time and space derivatives were introduced using the Grünwald-Letnikov and Liouville formulations. Two-sided derivatives were also presented due to their importance in the development of differential vectorial tools, namely the Laplacian. We described left and right gradient, divergence, and curl, calling them "first generation operators", since they led to the "second generation operators" and, in particular, to the fractional Laplacian. This formulation was compared with the fractional Laplacian based on the Riesz potential. The generalized Helmholtz decomposition was recalled and its solution expressed in terms of the solutions of several fractional wave equations. Before going into the integral tools the fractional definite integrals were defined. They were adopted to generalize the classic integral theorems to the fractional case. Considering rectangular domains the Green's, Stoke's, and Ostrogradski-Gauss's theorems were presented. A discussion of the particular case having uniform fractional orders was also included.

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REFERENCES

[1] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach Sc. Publ., London, 1993.
[2] S. Dugowson, *Les Différentielles Métaphysiques*, Ph.D. Thesis, Université Paris Nord, Villetaneuse, France, 1994, [In French].
[3] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
[4] R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Connecticut, 2006.
[5] M.D. Ortigueira, *Fractional calculus for scientists and engineers*, Lecture Notes in Electrical Engineering, 84, Springer, Dordrecht (2011).
[6] D. Valério, J.J. Trujillo, M. Rivero, J.A.T. Machado, and D. Baleanu, "Fractional calculus: A survey of useful formulas", *Eur. Phys. J. Special Topics* 222, 1827–1846, EDP Sciences, Springer-Verlag (2013).
[7] K.S. Cole and R.H. Cole, "Dispersion and absorption in dielectrics – I alternating current characteristics", *J. Chem. Phys.* 9, 341–352 (1941).
[8] K.S. Cole and R.H. Cole, "Dispersion and absorption in dielectrics – II direct current characteristics", *J. Chem. Phys.* 10, 98–105 (1942).
[9] S. Westerlund and L. Ekstam, "Capacitor theory", *IEEE Trans. Dielectr. Electr. Insul.* 1(5), 826–839 (1994).
[10] B.E. Conway, *Electrochemical Supercapacitors: Scientific Principles and Technological Application*, Plenum, New York, (1999).
[11] I. Schäfer and K. Krüger, "Modelling of coils using fractional derivatives", *Journal of Magnetism and Magnetic Materials* 307, 91–98 (2006).
[12] K. Biswas, S. Sen, and P. Dutta, "Realization of a constant phase element and its performance study in a differentiator circuit", *IEEE Trans. Circuits Syst. II, Exp. Briefs* 53(9), 802–806 (2006).
[13] D. Baleanu, A.K. Golmankhaneh, and M.C. Baleanu, "Fractional electromagnetic equations using fractional forms", *Int J. Theor. Phys.* 48, 3114–3123 (2009).
[14] I.S. Jesus and J.T, Machado, "Development of fractional order capacitors based on electrolyte processes", *Nonlinear Dynamics* 56, 45–55 (2009).
[15] I.S. Jesus and J.T. Machado, "Comparing integer and fractional models in some electrical systems", *Proceedings of FDA'10. The 4th IFAC Workshop Fractional Differentiation and its Applications*, Badajoz, Spain, October 18–20 (2010).
[16] V.E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Higher Education Press and Springer, Beijing (2010).
[17] C.S. Drapaca and S. Sivaloganathan, "A fractional model of continuum mechanics", *J. Elast.* 107, 105–123 (2012).

[18] M. D'Ovidio and R. Garra, *Fractional Gradient and its application to the Fractional Advection Equation*, arXiv:1305.4400v1 math-ph, May 19, (2013).
[19] A.S. Ali, A.G. Radwan, and A.M. Soliman, "Fractional order Butterworth filter: Active and passive realizations", *IEEE Journal on Emerging and Selected Topics in Circuits and Systems* 3(3), (2013).
[20] S.I.R. Arias, D.R. Muñoz, J.S. Moreno, S. Cardoso, R. Ferreira, and P.J. Freitas, "Fractional modeling of the AC large-signal frequency response in magnetoresistive current sensors", *Sensors* 13, 17516–17533 (2013).
[21] A.E. Çalik, H. Ertik, B. Öder, and H. Şirin, "A fractional calculus approach to investigate the alpha decay processes", *International Journal of Modern Physics E* 22(7), 1350049 (2013).
[22] F.A.A. El-Salam, "Fractional approach of Maxwell equations in the curved spacetime", *Journal of Taibah University for Science* 7, 173–179 (2013).
[23] V. Martyniuk and M.D. Ortigueira, "Fractional model of an electrochemical capacitor", *Signal Processing* 107, 355–360 (2015).
[24] I.S. Jesus, J.T. Machado, and J.B. Cunha, "Fractional electrical impedances in botanical elements", *Journal of Vibration and Control* 14, 1389 (2008).
[25] U. Skwara, J. Martins, P. Ghaffari, M. Aguiar, J. Boto, and N. Stollenwerk, "Applications of fractional calculus to epidemiological models", *Numerical Analysis and Applied Mathematics, ICNAAM 2012*, AIP Conf. Proc. 1479, 1339–1342 (2012).
[26] M.J. Lewis and M.A. McNarry, "Twenty-four hour variation in heart rate variability indices derived from fractional differintegration", *Clin. Physiol. Funct. Imaging* 35(1), 57–63 (2015).
[27] C. Ionescu, J.T. Machado, and R. De Keyser, "Modeling of the lung impedance using a fractional order ladder network with constant phase elements", *IEEE Transactions on Biomedical Circuits and Systems* 5(1), 83–89 (2011).
[28] D. Copot, R. De Keyser, E. Derom, M.D. Ortigueira, and C.M. Ionescu, "Reducing bias in fractional order impedance estimation for lung function evaluation", *Biomedical Signal Processing and Control* 39, 74–80 (2018).
[29] J.A.T. Machado, "And I say to myself: What a fractional world!", *Fractional Calculus and Applied Analysis* 14, 635 (2011).
[30] R. Magin, M.D. Ortigueira, I. Podlubny, and J. Trujillo, "On the Fractional Signals and Systems", *Signal Processing* 91, 350–371 (2011).
[31] M.M. Meerschaert, J. Mortensen, and S.W. Wheatcraft, "Fractional vector calculus for fractional advection–dispersion", *Physica A* 367, 181–190 (2006).
[32] M.M. Meerschaert, *Fractional Calculus, Anomalous Diffusion, and Probability*, in Fractional Dynamics: Recent Advances, edited by: J. Klafter, S. C. Lim, and R. Metzler, World Scientific Publishing Co. Pte. Ltd. (2011).
[33] R.L. Magin, C. Ingo, L. Colon-Perez, W. Triplett, and T.H. Mareci, "Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy", *Microporous and Mesoporous Materials* 178, September 15, pp. 39–43 (2013).
[34] L. Boyadjiev and Y. Luchko, "Multi-dimensional α-fractional diffusion–wave equation and some properties of its fundamental solution", *Computers & Mathematics with Applications* 73(12), 2561–2572 (2017).
[35] M. Ostoja-Starzewski, "Continuum mechanics models of fractal porous media: Integral relations and extremum principles", *Journal of Mechanics of Materials and Structures* 4(5), 901–912 (2009).
[36] G. Calcagni, "Geometry of fractional spaces", *Adv. Theor. Math. Phys.* 16, 549–644 (2012).
[37] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, Imperial College Press, (2010).
[38] M.P. Velasco and L. Vázquez, "On the fractional Newton and wave equation in one space dimension", *Appl. Math. Modelling* 38(13), 3314–3324 (2014).
[39] V.E. Tarasov, "Fractional generalization of gradient and Hamiltonian systems", *J. Phys. A: Math. Gen.* 38, 5929–5943 (2005).
[40] V.E. Tarasov, "Review of some promising fractional physical models", *International Journal of Modern Physics B* 27(9), 1330005 (2013).
[41] R. Herrmann, "Gauge invariance in fractional field theories", *Vol. 2, Physics Letters A* 372, 5515–5522 (2008).
[42] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific Publishing Co. Pte. Ltd. (2011).
[43] M.D. Ortigueira, M. Rivero, and J. Trujillo, "From a generalized Helmholtz decomposition theorem to the fractional Maxwell equations", *Commun Nonlinear Sci Numer Simulat* 22, 1036–1049 (2015).
[44] M.D. Ortigueira and J. Machado, "Which derivative? Fractal and fractional", *Fractal and Fractional* 1(1), 3 (2017).
[45] M. Ostoja-Starzewski, J. Li, H. Joumaa, and P.N. Demmie, "From fractal media to continuum mechanics", *Z. Angew. Math. Mech.* 94, 373–401 (2014).
[46] V.E. Tarasov, "Fractional vector calculus and fractional Maxwell's equations", *Annals of Physics* 323(11), 2756–2778 (2008).
[47] E.C. Grigoletto and E.C. de Oliveira, "Fractional versions of the fundamental theorem of calculus", *Applied Mathematics* 4, 23–33 (2013).
[48] M.D. Ortigueira and J. Machado, "Fractional definite integral", *Fractal and Fractional* 1(1), 2 (2017).
[49] A. Aghili and P.B. Moghaddam, "Certain theorems on two dimensional Laplace Transform and non-homogeneous parabolic partial differential equations", *Surveys in Mathematics and its Applications* 6, 165–174 (2011).
[50] M.J. Roberts, *Signals and Systems: Analysis Using Transform Methods and Matlab*, McGraw-Hill, 2003.
[51] M.D. Ortigueira and J.T. Machado, "What is a fractional derivative?", *J. Comput. Phys.* 293, 4–13 (2015).
[52] M.D. Ortigueira, R.L. Magin, J.J. Trujillo, M.P. Velasco, "A real regularized fractional derivative", *Signal Image Video P*, 6, 351–358 (2012).
[53] J. Liouville, "Memóire sur le calcul des différentielles à indices quelconques", *Journal de l'École Polytechnique* 13, 71–162 (1832). (In French)
[54] J. Liouville, "Memóire sur l'usage que l'on peut faire de la formule de Fourier, dans le calcul des différentielles à indices quelconques", *Journal für die reine und angewandte Mathematik (Journal de Crelle)* 13, 219–232 (1835). (In French)
[55] M.D. Ortigueira, "Riesz potential operators and inverses via fractional centred derivatives", *International Journal of Mathematics and Mathematical Sciences* 2006, ID 48391, pp. 1–12 (2006).

[56] M.D. Ortigueira, "Fractional central differences and derivatives", *Journal of Vibration and Control* 14(9–10), 1255–1266 (2008).

[57] M.D. Ortigueira and J.J. Trujillo, "A unified approach to fractional derivatives", *Commun Nonlinear Sci. Numer. Simulat.* 17, 5151–5157 (2012).

[58] M. Riesz, "L'integral de Riemann-Liouville et le probleme de Cauchy", *Acta Mathematica* 81(1), 1–222 (1949).

[59] W. Chen and S. Holm, "Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency dependency", *The Journal of the Acoustical Society of America* 115(4), 1424–1430 (2004).

[60] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Communications in Partial Differential Equations* 32(8), 1245–1260 (2007).

[61] M.D. Ortigueira, T.-L. Laleg-Kirati, and J.T. Machado, "Riesz potential versus fractional Laplacian", *Journal of Statistical Mechanics: Theory and Experiment* 2014(9), P09032 (2014).

[62] D.A.W. Pecknold, "On the role of the Stokes-Helmholtz decomposition in the derivation of displacement potentials in classical elasticity", *Journal of Elasticity* 1(2), 171–174 (1971).

[63] D.D. Joseph, "Helmholtz decomposition coupling rotational to irrotational flow of a viscous fluid", *Proceedings of the National Academy of Sciences of USA* 103(39), 14272–14277 (2006).

[64] E. Kapuścik, "Generalized Helmholtz theorem and gauge invariance of classical field theories", *Lett. Nuovo Cimento* 42, 263–266 (1985).

[65] R.D. Nevels, *A Simple 'Derivation' of Maxwell's Equations Relying on the new Extended Helmholtz Theorem*, IEEE Ant. Prop. Soc. Int. Symp. Newport Beach, California, pp. 822–825 (1995).

[66] E.K. Jaradat, R.S. Hijjawi, and J.M. Khalifeh, "Maxwell's equations and electromagnetic Lagrangian density in fractional form", *Journal of Mathematical Physics* 53, 033505 (2012).

[67] T. Apostol, *Mathematical Analysis*, Addison Wesley, Fifth Edition, (1981).