Polynomial alignment using general transition curves

Andrzej KOBRYŃ [*] [ID]

Faculty of Civil Engineering and Environmental Sciences, Bialystok University of Technology,
ul. Wiejska 45E, 15-351 Bialystok, Poland

Abstract. This paper presents a new approach to the design methodology of road routes, in literature often referred to as the polynomial alignment. The author proposes the use of the so-called general transition curves that have been described in detail in his earlier research papers. General transition curves employ only one curvature extremum, and the whole curved transition between two extreme points of zero curvature value is described by a single equation. As a result, the curves are very useful for the creation of route geometry in accordance with the principles of polynomial alignment. The paper describes the main concept of polynomial alignment and presents equations of curves which can be used in the proposed alignment procedure. In addition, the paper gives a detailed description of design procedures.

Key words: road design; horizontal alignment; general transition curves.

1. Introduction

In general, it can be stated that a well-developed road network is one of the conditions for the development of any country or region. The road network is needed to improve connections between the most important towns as well as between the most important towns and subregional towns [1].

Various approaches are used in road route design to obtain the route that will best meet the adopted design criteria. In recent years, various design methods and optimization models have been proposed that can be useful in the comprehensive design of horizontal as well as vertical alignment. Among others, the studies [2–7] can be mentioned here.

Numerous studies present various approaches to design and optimization of the vertical alignment. Among others, the works [8–17] can be mentioned here. It is worth mentioning that the main design criterion adopted in most of these studies was the minimization of earthworks cost. Furthermore, various novel methods have been used, such as, for example, linear programming or spline techniques.

The subject of appropriate design and optimization proposals was also the horizontal alignment. Also, in case of horizontal alignment, the use of spline techniques has been proposed [18,19]. Papers [20,21] also include interesting proposals for horizontal alignment.

Other proposals for the design and optimization of road alignment have also been published recently [22–29]. The article [22] presents a design model that uses a geographical information system and allows to design highway alignment minimizing total cost as the objective function. In turn, in [23],

a geometric model is proposed including horizontal transition curves and vertical curves, and a two-stage algorithm is used for optimizing infrastructure costs. The article [24] presents a modified motion-planning based algorithm for developing new horizontal alignments with optimized costs and the proposed algorithm is integrated with the GIS database. Interesting proposals for the optimization of the horizontal alignment using straight sections, circular curves and spiral curves are also presented in the article [25], where the objective function represents the cost of a specific highway course. The article [26] deals with the problem of finding optimal corridors in which to construct a new road, taking into account the increased accuracy of cost formulation. In turn, in articles [27, 28] the authors analyze various design problems related to the highway horizontal alignment design using so-called polynomial parametric curve. Interesting proposals for the use of artificial intelligence in highway location and alignment optimization include a monograph [29].

Some papers describe the design of transport network or road alignment using genetic algorithms [30,31]. It can be noted that the methods of multi-criteria analysis may also be used in road planning and design [1, 32–34].

Various studies in the field of design or optimization of horizontal and vertical alignment concern not only the design of roads and highways. Several papers on railway alignment optimization have also been published recently, among others [35–39].

It should be noted that the above works on design and optimization of horizontal and vertical alignment of road routes analyzed not only traditional geometric elements (such as straight lines, circular arcs, and transition curves), but also various unconventional elements used in spline techniques. This includes, among others, a so-called polynomial alignment.

The polynomial alignment is an example of a total departure from the traditional road route consisting of straight lines,

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*e-mail: a.kobryn@pb.edu.pl

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circular curves and transition curves, (usually clothoids). The polynomial approach involves describing road routes by polynomial spline functions.

The road route, consisting of successive segments in the form of algebraic polynomials of the n-th degree, is completely curvilinear and characterized by a variable value of curvature. The idea of polynomial alignment has been developed primarily for easier fitting of the route to the constraints found, for example, in mountainous or urban areas. In mountainous areas fitting the route to the terrain configuration is required in order to reduce construction outlays. However, in urban areas it is often necessary to carry out a route so that the process does not interfere with existing objects. It is generally accepted that it is much easier to deal with the above problems using polynomial spline functions rather than create a route using conventional geometric elements in the form of straight lines, circular arcs and clothoids. Similar problems must also be solved when reconstructing existing road routes.

This article proposes original solutions for the polynomial approach, which are based in the use of so-called general transition curves. General transition curves have been presented by the author in his earlier works [40–42].

The following sections of the paper present the general principles of polynomial alignment and give general equations of transition curves that can be used as geometric elements forming the planned route. Further sections describe specific/suitable appropriate procedures to design routes with the use of these curves.

2. Principles of polynomial alignment

Pioneering description of the polynomial alignment was presented in the paper by Calogero [18]. This concept involves route formation by means of polynomials defined by the following equation:

$$y = y'_0 x + \frac{1}{2} y''_0 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n, \qquad (1)$$

where y'_0 and y''_0 represent respectively the first and second derivative of the start point of the i-th polynomial. They should be consistent with the corresponding derivatives at the end point of the preceding polynomial. According to Calogero's concept [18], the degree of polynomial (n) and subsequent coefficients a_i (wherein $i = 3, 4, \ldots, n$) are determined iteratively using a suitable computer program. Computational procedures ensure the control of preset minimum values of curvature radii as well as deviations from the direction points on a predetermined position by a pair of X and Y coordinates. This is done by taking certain route distance D from the direction point and assigning appropriate weights W_i to particular route points. The maximum permissible deviation of the route from any direction point is then equal to D/W_i.

The weight W_i for each direction point is determined by taking into account the importance of this point and required closer route to the point.

The theoretical basis of the polynomial alignment presented by Calogero was an impetus to take up this topic by a number of other researchers. The subject matter concerning polynomial alignment was discussed, among others, in papers [43–45], and more recently in [19].

Generally speaking, the concept of polynomial alignment involves the formation of a route using polynomial elements of the n-th degree in the form of:

$$y_i = f(x_i) = a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n. \qquad (2)$$

The polynomial route is composed of an appropriate number of polynomials whose boundaries are limited by points $\bar{x}_{k-1}$ and $\bar{x}_k$, wherein $k = 1, 2, \ldots, n$. In order to ensure a smooth transition of the route, relevant conditions of regularity at the connection points of each of the polynomials must be fulfilled. These conditions include the following:

$$f_j(\bar{x}_j) = f_{j+1}(\bar{x}_j), \qquad (3)$$

$$f'_j(\bar{x}_j) = f'_{j+1}(\bar{x}_j), \qquad (4)$$

$$f''_j(\bar{x}_j) = f''_{j+1}(\bar{x}_j). \qquad (5)$$

Conditions (3), (4) and (5) ensure the continuity of the route, including the compatibility of tangent positions and curvature values at subsequent connections of the polynomials.

As a rule, practical implementation of polynomial alignment involves preliminary determination of the so-called direction points through or close to which the route should run. Then the coefficients of subsequent polynomials constituting the polynomial route are determined. Generally, this is affected by the least squares method with respect to the route deviation from the preset direction points. These coefficients are selected so that the deviation of the designed route from the preset direction points remain within the accepted limit. It is not always necessary to draw the route exactly through the direction points. Due to the above, a distinction is made here between the so-called strict direction points (where the route is expected to pass through the points) and the so-called relative direction points (where the route should run in the defined vicinity of the points). A special case can also arise here involving the so-called direction points with tangent, which – apart from the location – also includes the tangent direction to the route.

The Introduction section contains an overview of publications that present a wide variety of approaches to designing and optimizing of road routes. All approaches are aimed at achieving the lowest possible construction costs of road routes. One of the factors that influences it directly is the adjustment of the route to specific terrain restrictions, such as topography, intensive land use, etc. While no work has been published on polynomial alignment in recent years, it should be noted that polynomial alignment is a way to achieve such adjustment. In [18, 19] and [43–45] approaches to polynomial alignment with the use of spline functions have been presented. The author of this article expresses the opinion that the general transition curves presented in his works [40–42] can also be used in the polynomial approach. This is due to the general characteristics of the gen-

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eral transition curves (which are described in Section 3), thanks to which general transition curves can be a very useful design tool in engineering practice.

In practical applications of various polynomial alignment methods, it is required to cope with the problems related to the occurrence of undesirable extremes of curvature within individual polynomials forming the route. As a result, there are difficulties with maintaining the required distance of visibility and appropriate conditions for traffic safety. However, according to the author of this paper, these problems could be avoided by implementing the idea of polynomial alignment using special geometry solutions in the form of general transition curves.

3. Polynomial solutions of general transition curves

General transition curves can be defined as geometric elements that make it possible to describe, with a single equation, a curvilinear transition between two straight line sections. In such a case, the curvature within the general transition curve increases from zero at the start point to reach a specified maximum value, and then falls again to zero at the endpoint.

The subject matter concerning general transition curves was investigated by Kobryń [40–42]. The solutions of general transition curves presented in these papers can be used, among others, in polynomial alignment. The solutions were determined on the basis of a polynomial function of the form:

$$y = \sum_{i=0}^{i=n} a_i x^i \qquad (6)$$

with $i = 4, 5, 6$ and 7.

According to [40], the first one of the solutions set of general transition curves is:

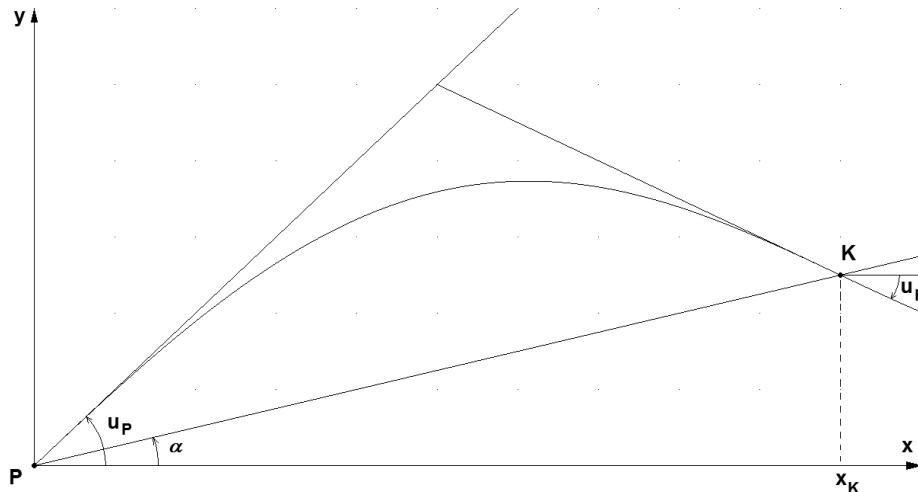$$y = x_K \left(G_0 \tan \alpha + G_1 \tan u_P + G_2 \tan u_K \right), \qquad (7)$$

where

$$G_0 = 35t^4 - 84t^5 + 70t^6 - 20t^7,$$

$$G_1 = t - 20t^4 + 45t^5 - 36t^6 + 10t^7,$$

$$G_2 = -15t^4 + 39t^5 - 34t^6 + 10t^7,$$

whereby: $t = x/x_K$, $t \in\, <0; 1>$ and (Fig. 1):

x_K — abscissa of point K in the local coordinates system of the curve,

$\tan u_P$ — tangent inclination at start point P,

$\tan u_K$ — tangent inclination at end point K,

$\tan \alpha$ — inclination of the main chord connecting points P and K.

According to [40], curve (7) has a desirable curvature distribution if:

$$\tan \alpha = G_{1/0} \tan u_P + G_{2/0} \tan u_K, \qquad (8)$$

wherein $G_{1/0} \in \langle 3/7; 4/7 \rangle$, $G_{2/0} \in \langle 3/7; 4/7 \rangle$.

The second solution proposed in paper [40] takes the form of:

$$y = x_K \left(F_1 \tan u_P + F_2 \tan u_K \right), \qquad (9)$$

where

$$F_1 = t - \frac{5}{2}t^4 + 3t^5 - t^6,$$

$$F_2 = \frac{5}{2}t^4 - 3t^5 + t^6.$$

The analysis of solution (9) carried out in [40] shows that – regardless of the mutual relationships between x_K, $\tan u_P$ and $\tan u_K$ – solution (9) has a curvature distribution that is characteristic of general transition curves.

The third solution presented in [40] has the form shown below:

$$y = x_K \left(M_0 \tan \alpha + M_1 \tan u_P + M_2 \tan u_K \right), \qquad (10)$$

Fig. 1. The general transition curve in the local coordinate system

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where

$$M_0 = 10t^3 - 15t^4 + 6t^5,$$

$$M_1 = t - 6t^3 + 8t^4 - 3t^5,$$

$$M_2 = -4t^3 + 7t^4 - 3t^5.$$

According to [38], in the case of curves (10) we are dealing with curve distribution typical of general transition curves, if:

$$\tan\alpha = M_{1/0}\tan u_P + M_{2/0}\tan u_K, \qquad (11)$$

wherein: $M_{1/0} \in \langle 2/5; 3/5\rangle$, $M_{2/0} \in \langle 2/5; 3/5\rangle$.

The last one of the solutions by Kobryń [41] takes the form:

$$y = x_K\left(N_1\tan u_P + N_2\tan u_K\right), \qquad (12)$$

where

$$N_1 = t - t^3 + \frac{1}{2}t^4,$$

$$N_2 = t^3 - \frac{1}{2}t^4.$$

According to [41], analyses of solution (12) have shown that curves (12), just like curves (9), have curvature distribution that is typical of general transition curves regardless of the mutual relationships between x_K, $\tan u_P$ and $\tan u_K$.

At this point, attention should be paid to special cases of solutions (7) and (10) that result from Eqs. (7) and (10) after taking $\tan\alpha = 0$ in them. The first case follows from (7) and has the form:

$$y = x_K\left(G_1\tan u_P + G_2\tan u_K\right). \qquad (13)$$

The desired curvature distribution within curves (13) can be obtained if:

$$-\frac{4}{3} \leq \frac{\tan u_P}{\tan u_K} \leq -\frac{3}{4}. \qquad (14)$$

While the second special case arises from (10) and has the form:

$$y = x_K\left(M_1\tan u_P + M_2\tan u_K\right). \qquad (15)$$

The equivalent of condition (14) is the following inequality for curves (15):

$$-\frac{3}{2} \leq \frac{\tan u_P}{\tan u_K} \leq -\frac{2}{3}. \qquad (16)$$

4. Proposed approach to the polynomial alignment using general transition curves

4.1. General principles. First, a properly designed polynomial route should above all meet the regularity conditions (3), (4) and (5) at the joining points between subsequent curves, as well as ensure that the acceptable radii of curvature are not exceeded in order to keep the required distance visibility.

In the case of a polynomial route created by general transition curves, condition (3) is fulfilled by locating the beginning of the subsequent curve at the end point of the preceding curve (Fig. 2). The only exception to this can be one of the two following situations:

- joining the end of the designed curve to the so-called direction point with the tangent;
- joining the end of the designed curve to a the planned und of the route with a given location or at the same time a given location and a given tangent direction (this case is described later on).
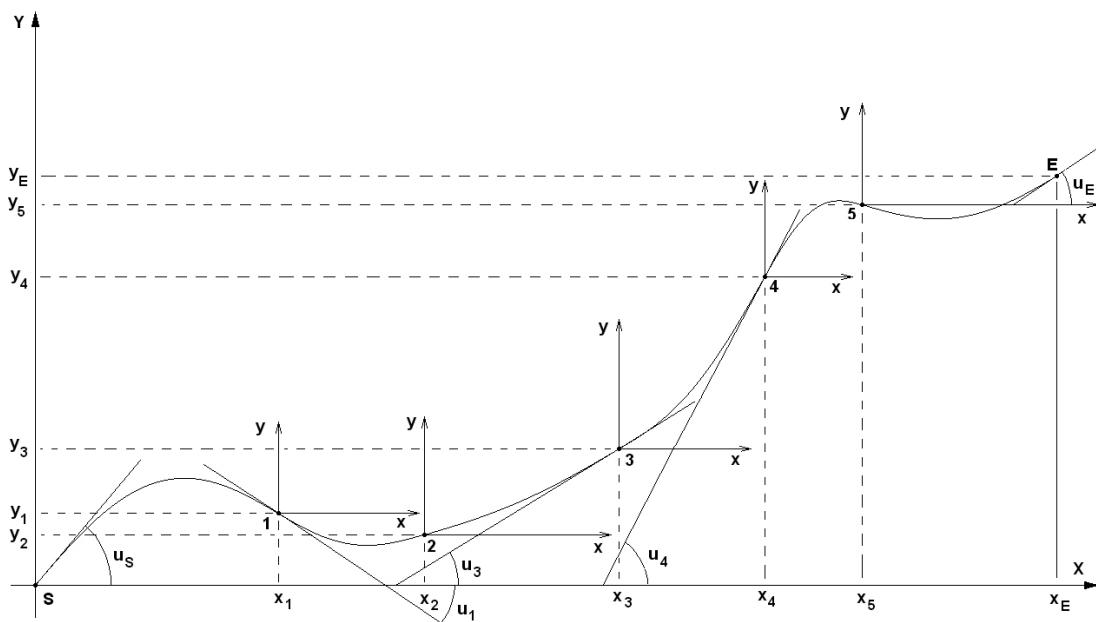

Fig. 2. Polynomial route created by general transition curves

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The coordinates of the points lying within consecutive curves are expressed in the superordinated coordinate system XY as:

$$X_i^{(j)} = X_K^{(j-1)} + x_i^{(j)}, \qquad (17)$$

$$Y_i^{(j)} = Y_K^{(j-1)} + y_i^{(j)}, \qquad (18)$$

wherein:

$X_i^{(j)}, Y_i^{(j)}$ – coordinates of the *i*-th point of the *j*-th curve in the XY superordinated coordinate system,

$x_i^{(j)}, y_i^{(j)}$ – coordinates of the *i*-th point of the *j*-th curve in its local coordinate system, i.e. $x_i^{(j)} = X_i^{(j)} - X_K^{(j-1)}$, whereas $y_i^{(j)}$ is the value expressed for $x_i^{(j)}$ by an appropriate equation of curve (7), (9), (10) or (12) wherein $X_i^{(j)} \in \langle X_K^{(j-1)}; X_K^{(j)} \rangle$,

$X_K^{(j-1)}, Y_K^{(j-1)}$ – coordinates of the end of curve K preceding (*j*−1) in the XY superordinated coordinate system.

It should be added that abscissa x_K of the end of the *j*-th curve in its local coordinate system results from values $X_K^{(j-1)}$ and $X_K^{(j)}$. It is as follows:

$$x_K = x_K^{(j)} = X_K^{(j)} - X_K^{(j-1)}. \qquad (19)$$

It should also be noted that appropriate ordinates y_K of the ends of the curves in the local coordinate systems result from the equations of individual curves. These ordinates are:

- for curves (7) and (10)

$$y_K = x_K \tan\alpha, \qquad (20)$$

- for curves (9) and (12)

$$y_K = \frac{1}{2} x_K (\tan u_P + \tan u_K). \qquad (21)$$

Another condition of regularity, i.e., condition (4) is satisfied automatically if, in the design of subsequent curves, it is assumed that

$$\tan u_P^{(j)} = \tan u_K^{(j-1)}. \qquad (22)$$

As can be seen, value $\tan u_K^{(j)}$ is essential for proper design of subsequent curves.

In the case of curves (7) and (10), it is necessary to take into account conditions (8) and (11) which give:

- for curves (7)

$$\tan u_K^{(j)} = \frac{1}{G_{2/0}} \tan\alpha^{(j)} - \frac{G_{1/0}}{G_{2/0}} \tan u_P^{(j)}. \qquad (23)$$

- for curves (10)

$$\tan u_K^{(j)} = \frac{1}{M_{2/0}} \tan\alpha^{(j)} - \frac{M_{1/0}}{M_{2/0}} \tan u_P^{(j)}. \qquad (24)$$

In further considerations, it should be taken into account that:

$$\tan\alpha^{(j)} = \frac{y_K^{(j)}}{x_K^{(j)}}, \qquad (25)$$

wherein:

$$\begin{aligned} y_K^{(j)} &= Y_K^{(j)} - Y_K^{(j-1)}, \\ x_K^{(j)} &= X_K^{(j)} - X_K^{(j-1)}. \end{aligned} \qquad (26)$$

After positioning the end of the *j*-th curve, it is necessary to determine value $\tan u_K^{(j)}$ using (23) or (24), so that it is possible to obtain a proper curvature distribution typical of general transition curves.

However, from (21) and (25), in the case of curves (9) and (12), it follows that:

$$\tan u_K^{(j)} = 2\tan\alpha^{(j)} - \tan u_P^{(j)}. \qquad (27)$$

Having assumed the location of the end of the *j*-th curve, values $y_K^{(j)}$ and $x_K^{(j)}$ determine – just like for curves (7) and (10) – the inclination of the main chord $\tan\alpha^{(j)}$ and consequently the tangent inclination at the end point of the curve.

Similarly to condition (3), condition (4) requires a distinct treatment in the situation regarding the connecting of the end of the designed curve to the so called tangent direction point or the route's end with preset tangent direction. As already mentioned, this case will be described later on.

The last one of the regularity conditions that must be met in polynomial alignment is condition (5). Using general transition curves, this condition is satisfied by definition, since the characteristic feature of these curves is zero curvature value and hence zero value of the second derivative at the extreme points. In the proposed methodology of polynomial alignment these points also constitute the joining points of consecutive elements of the route.

When using curves (7) and (10), we may have to deal with such a position of the support points of individual curves that will result in large inclinations of the main chords joining extreme points of successive curves. Due to the limited interval of permissible values $G_{1/0}$, $G_{2/0}$ for curves (7) or $M_{1/0}$, $M_{2/0}$ for curves (10), it may be more convenient to use Eqs. (13) or (15), which are special cases of solutions (7) and (10) after assuming in their $\tan\alpha = 0$. In this case, however, it is necessary to transform rectangular coordinates of individual points of the curve from the local coordinate system *xy* into the local coordinate system *x'y'*, whose axes are parallel to the respective axes of superordinated coordinate system XY (Fig. 3).

Regardless of the concavity/convexity of the arc, the transformation can be accomplished by the following equations:

$$x_i' = x_i \cos\beta - y_i \sin\beta, \qquad (28)$$

$$y_i' = x_i \sin\beta + y_i \cos\beta. \qquad (29)$$

Angle β is a directional angle fulfilling the following conditions: $\beta > 0$ if $Y_K > Y_P$ and $\beta < 0$ if $Y_K < Y_P$, where Y_P, Y_K are

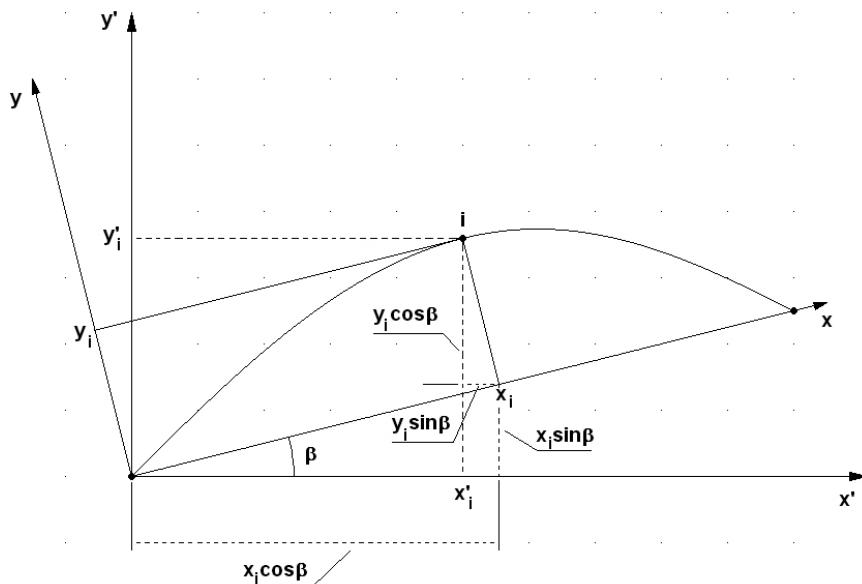Bull. Pol. Acad. Sci. Tech. Sci. 69(3) 2021, e137195

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Fig. 3. Transformation of curve coordinates from the local system xy to the local system $x'y'$

the ordinates of both the start and end of the curve in the super-ordinated coordinate system. The transformed coordinates of successive points of the curve in the superordinated coordinate system ultimately amount to:

$$X_i = X_P + x_i \cos\beta - y_i \sin\beta, \tag{30}$$

$$Y_i = Y_P + x_i \sin\beta + y_i \cos\beta, \tag{31}$$

wherein X_P, Y_P are the coordinates of the start point P of the curve in the superordinated coordinate system.

4.2. Special cases. This section analyzes the case referred to earlier, concerning the connection of the end of the designed curve with the tangent direction point or the end of the route

with a preset tangent direction. This special case requires the use of a road system consisting of two general transition curves. As shown in Fig. 4, the case involves setting the start position (P) of the first curve and the end of the second curve (Q).

Also, tangential directions $\tan u_P^{(j-1)}$ and $\tan u_K^{(j)}$ are set at these points. For simplicity's sake, in further analyzes, these points are denoted in accordance with Fig. 4, as $\tan u_P$ and $\tan u_K$. In order to achieve a smooth route transition, it is essential in this case to determine the location of point W to connect the curves in such a way as to satisfy conditions (3) and (4).

Computational procedure for curves (7) and (10)

First, the problem is analyzed with respect to curves (7) and (10). This is illustrated using curves (7). As shown in Fig. 4,
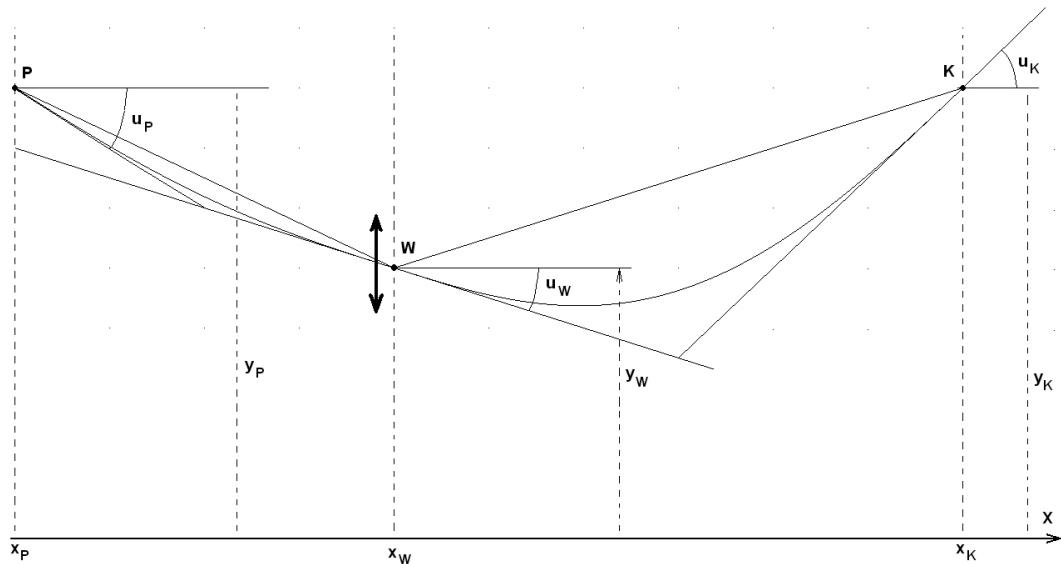

Fig. 4. A connection the end of the designed curve to the direction point with tangent or the end point of the route with a given tangent direction

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the inclination of the main chord (connecting the end points of the first curve) is denoted as $\tan \alpha_{PW}$. On the other hand, the inclination of the main chord of the second curve is denoted as $\tan \alpha_{WK}$.

According to condition (8), these inclinations can be described by the following equations:

$$\tan \alpha_{PW} = G_{1/0}^{(P)} \tan u_P + G_{2/0}^{(P)} \tan u_W \qquad (32)$$

and

$$\tan \alpha_{WK} = G_{1/0}^{(K)} \tan u_W + G_{2/0}^{(K)} \tan u_K, \qquad (33)$$

wherein $G_{1/0}^{(P)}$, $G_{2/0}^{(P)}$ and $G_{1/0}^{(K)}$, $G_{2/0}^{(K)}$ are the values selected from the permissible intervals respectively for the first and for the second curve.

Based on the above equations, value $\tan u_W$ can be expressed as follows:

$$\tan u_W = \frac{1}{G_{2/0}^{(P)}} \left(\tan \alpha_{PW} - G_{1/0}^{(P)} \tan u_P \right) \qquad (34)$$

and

$$\tan u_W = \frac{1}{G_{1/0}^{(K)}} \left(\tan \alpha_{WK} - G_{2/0}^{(K)} \tan u_K \right). \qquad (35)$$

In order to ensure a smooth route transition at point W, values $\tan u_W$ resulting from the above two equations must, of course, be the same. It is thus clear:

$$\frac{1}{G_{2/0}^{(P)}} \left(\tan \alpha_{PW} - G_{1/0}^{(P)} \tan u_P \right) =$$
$$= \frac{1}{G_{1/0}^{(K)}} \left(\tan \alpha_{WK} - G_{2/0}^{(K)} \tan u_K \right). \qquad (36)$$

Since we have:

$$\tan \alpha_{PW} = \frac{\Delta Y_{PW}}{\Delta X_{PW}} \qquad (37)$$

and

$$\tan \alpha_{WK} = \frac{\Delta Y_{WK}}{\Delta X_{WK}}. \qquad (38)$$

Hence Eq. (36) can be written in the form of:

$$\frac{1}{G_{2/0}^{(P)}} \frac{\Delta Y_{PW}}{\Delta X_{PW}} - \frac{1}{G_{1/0}^{(K)}} \frac{\Delta Y_{WK}}{\Delta X_{WK}} = \frac{G_{1/0}^{(P)}}{G_{2/0}^{(P)}} \tan u_P - \frac{G_{2/0}^{(K)}}{G_{1/0}^{(K)}} \tan u_K. \quad (39)$$

The coordinate increments found in the above equation also satisfy the following conditions:

$$\Delta Y_{WK} = \Delta Y_{PK} - \Delta Y_{PW}, \qquad (40)$$

$$\Delta X_{WK} = \Delta X_{PK} - \Delta X_{PW}. \qquad (41)$$

Considering (40) and (41) in Eq. (39) and having made the required transformations, we have:

$$\Delta Y_{PW} =$$
$$= \frac{\Delta X_{PW} \left(\Delta X_{PK} - \Delta X_{PW} \right) \left(G_{1/0}^{(P)} G_{1/0}^{(K)} \tan u_P - G_{2/0}^{(P)} G_{2/0}^{(K)} \tan u_K \right)}{\Delta X_{PK} G_{1/0}^{(K)} - \Delta X_{PW} \left(G_{1/0}^{(K)} - G_{2/0}^{(P)} \right)} +$$
$$+ \frac{\Delta X_{PW} \Delta Y_{PK} G_{2/0}^{(P)}}{\Delta X_{PK} G_{1/0}^{(K)} - \Delta X_{PW} \left(G_{1/0}^{(K)} - G_{2/0}^{(P)} \right)}. \qquad (42)$$

Using the above equation, it is possible to propose a procedure determining point W. As can be seen, it is necessary to assume an appropriate abscissa X_W of point W. As a result, on the basis of value ΔX_{PW}:

$$\Delta X_{PW} = X_W - X_P. \qquad (43)$$

Equation (42) yields the appropriate value ΔY_{PW}, and hence:

$$Y_W = Y_P + \Delta Y_{PW}. \qquad (44)$$

It should be noted that after adopting $G_{1/0}^{(P)} = G_{2/0}^{(P)} = G_{1/0}^{(K)} = G_{2/0}^{(K)} = G$, increment ΔY_{PW} in this particular case can be expressed as:

$$\Delta Y_{PW} = G \left(\tan u_P - \tan u_K \right) \frac{\Delta X_{PW}}{\Delta X_{PK}} \left(\Delta X_{PK} - \Delta X_{PW} \right) +$$
$$+ \Delta X_{PW} \frac{\Delta Y_{PK}}{\Delta X_{PK}}. \qquad (45)$$

The appropriate computational procedure for the use of curves (10) would be analogous. In the respective equations, instead of $G_{1/0}^{(P)}$, $G_{2/0}^{(P)}$, $G_{1/0}^{(K)}$ and $G_{2/0}^{(K)}$, values $M_{1/0}^{(P)}$, $M_{2/0}^{(P)}$, $M_{1/0}^{(K)}$ and $M_{2/0}^{(K)}$ should only be substituted from the permissible interval of curves (10).

Computational procedure for curves (9) and (12)

A similar procedure, leading to the determination of point W, may also be proposed for curves (9) and (12). As follows from Eq. (21), the following condition is met in the case of the curves:

$$2 \tan \alpha = \tan u_P + \tan u_K. \qquad (46)$$

Making use of the above equation, we can write:

- for the first curve (based on PW chord)

$$\tan u_W = 2 \tan \alpha_{PW} - \tan u_P; \qquad (47)$$

- for the second curve (based on WK chord)

$$\tan u_W = 2 \tan \alpha_{WK} - \tan u_K. \qquad (48)$$

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As mentioned before, in order to ensure a smooth route transition at point W, values $\tan u_W$ resulting from both curves meeting at point W must be the same. Thus, it follows from Eqs. (47) and (48):

$$\tan \alpha_{PW} - \tan \alpha_{WK} = \frac{1}{2} \left(\tan u_P - \tan u_K \right). \qquad (49)$$

Taking into account (37) and (38), the above equation can be written as:

$$\frac{\Delta Y_{PW}}{\Delta X_{PW}} - \frac{\Delta Y_{WK}}{\Delta X_{WK}} = \frac{1}{2} \left(\tan u_P - \tan u_K \right). \qquad (50)$$

Considering (40) and (41) and having made the required transformations, it follows from Eq. (50):

$$\Delta Y_{PW} = \frac{1}{2} \left(\tan u_P - \tan u_K \right) \frac{\Delta X_{PW}}{\Delta X_{PK}} \left(\Delta X_{PK} - \Delta X_{PW} \right) + \\ + \Delta X_{PW} \frac{\Delta Y_{PK}}{\Delta X_{PK}}. \qquad (51)$$

As a result, adopting abscissa X_W of point W and using (51), it is possible to determine the increment ΔY_{PW}. Hence, using (44) we have corresponding ordinate Y_W.

At this point, it is worth noting that Eq. (51) takes the form identical to Eq. (45), if it assumes $G = 1/2$.

4.3. Fitting the curves to locations of direction points. In polynomial alignment with general transition curves, it is possible to adopt a general principle that the support points (connections) of consecutive curves coincide with direction points. Basically, this would involve the so-called strict direction points or tangent direction points.

In the case of the so-called relative direction points, some divergence from the route's initial course set by these points is allowed. Of course, there are also no contraindications to locate consecutive connection points in the vicinity of the direction

points belonging to this group, if the permissible deviation of the route from a given point is taken into account. Generally, however, the fitting process of any given route segment defined by a general transition curve to the course of a pre-set relative direction points, should be carried out using the least squares method by minimizing the sum of squares of curve deviations from these points (Fig. 5).

With a set location of both the curve's start point and initial tangent direction (due to the design of the preceding curve), the problem is reduced – depending on the type of curve under consideration – to determining the inclination of $\tan \alpha$ and $\tan u_K$, or exclusively $\tan u_K$. Taking into account, for example, the differences in the ordinates of individual direction points and also in the ordinates of relevant points of the general transition curve in the local coordinates of the curve, it is possible to create the following function:

$$F_y = \sum_{i=1}^{i=q} \Delta y_i^2, \qquad (52)$$

wherein $\Delta y_i = y_i - \hat{y}_i$ and:

$\hat{y}_i$ – ordinate of the i-th direction point of abscissa X_i in the local coordinate system of the current (being fitted) curve, i.e., $\hat{y}_i = Y_i - Y_P$,

y_i – ordinate of the i-th point of the curve, corresponding to abscissa $x_i = X_i - X_P$.

Computational procedure for curves (7) and (10)

The computational procedure is exemplified using the first family of general transition curves, i.e. (7). Their fitting to the given direction points requires the designation of appropriate values of $\tan \alpha$ and $\tan u_K$. In this case, the existence of a minimum of function (52) is subjected to the following conditions:

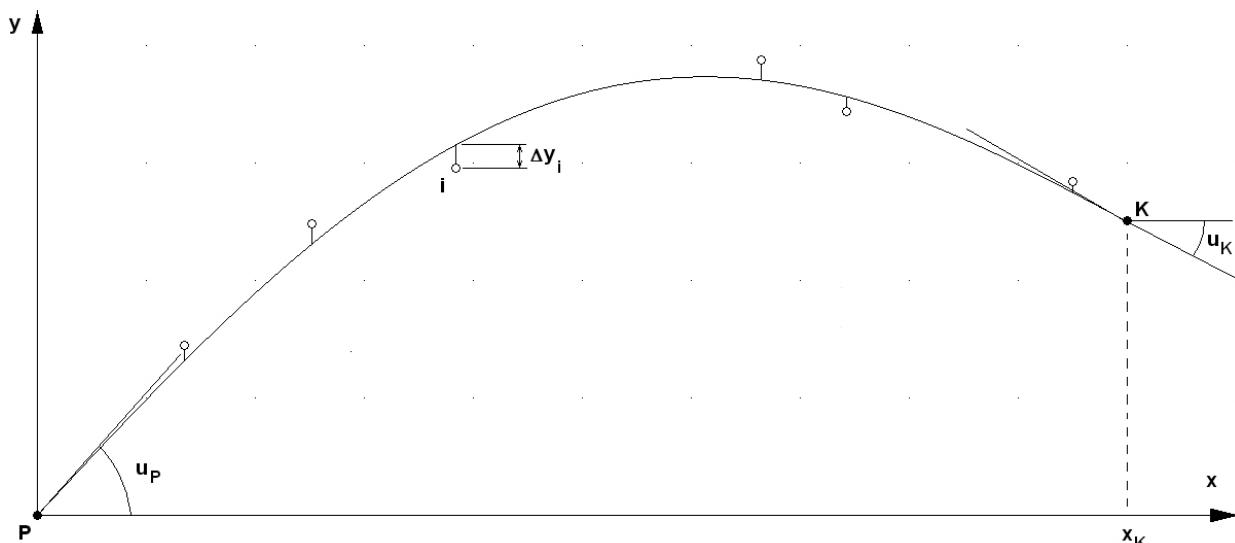$$\frac{\partial F}{\partial (\tan \alpha)} = 0 \qquad (53)$$

Fig. 5. A fitting of the general transition curve to the given direction points

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and

$$\frac{\partial F}{\partial (\tan u_K)} = 0. \tag{54}$$

The above conditions with respect to function (52) for curves (7) lead to the following system of equations:

$$\sum_i y_i G_0^{(i)} = \sum_i \hat{y}_i G_0^{(i)}, \tag{55}$$

$$\sum_i y_i G_2^{(i)} = \sum_i \hat{y}_i G_2^{(i)}. \tag{56}$$

Expressing y_i by Eq. (7) and basing on (55) and (56) results in:

$$x_K \tan\alpha \sum_i G_0^{(i)^2} + x_K \tan u_K \sum_i G_0^{(i)} G_2^{(i)} =$$
$$= \sum_i \hat{y}_i G_0^{(i)} - x_K \tan u_P \sum_i G_0^{(i)} G_1^{(i)}, \tag{57}$$

$$x_K \tan\alpha \sum_i G_0^{(i)} G_2^{(i)} + x_K \tan u_K \sum_i G_2^{(i)^2} =$$
$$= \sum_i \hat{y}_i G_2^{(i)} - x_K \tan u_P \sum_i G_1^{(i)} G_2^{(i)}. \tag{58}$$

Solving Eqs. (57) and (58) yields the following:

$$\tan\alpha = \frac{1}{x_K} \frac{\sum_i G_2^{(i)^2} \sum_i \hat{y}_i G_0^{(i)} - \sum_i G_0^{(i)} G_2^{(i)} \sum_i \hat{y}_i G_2^{(i)}}{\sum_i G_0^{(i)^2} \sum_i G_2^{(i)^2} - \left(\sum_i G_0^{(i)} G_2^{(i)}\right)^2} -$$
$$- \tan u_P \frac{\sum_i G_2^{(i)^2} \sum_i G_0^{(i)} G_1^{(i)} - \sum_i G_0^{(i)} G_2^{(i)} \sum_i G_1^{(i)} G_2^{(i)}}{\sum_i G_0^{(i)} \sum_i G_2^{(i)^2} - \left(\sum_i G_0^{(i)} G_2^{(i)}\right)^2} \tag{59}$$

and

$$\tan u_k = \frac{1}{x_K} \frac{\sum_i G_0^{(i)^2} \sum_i \hat{y}_i G_2^{(i)} - \sum_i G_0^{(i)} G_2^{(i)} \sum_i \hat{y}_i G_0^{(i)}}{\sum_i G_0^{(i)^2} \sum_i G_2^{(i)^2} - \left(\sum_i G_0^{(i)} G_2^{(i)}\right)^2} -$$
$$- \tan u_P \frac{\sum_i G_0^{(i)^2} \sum_i G_1^{(i)} G_2^{(i)} - \sum_i G_0^{(i)} G_2^{(i)} \sum_i G_0^{(i)} G_1^{(i)}}{\sum_i G_0^{(i)} \sum_i G_2^{(i)^2} - \left(\sum_i G_0^{(i)} G_2^{(i)}\right)^2}. \tag{60}$$

This procedure would apply in the case when merely the value of the abscissa of the curve's end was given. If, however, both the abscissa and ordinate of the curve's end were given, then value $\tan\alpha$ would be defined. At that point, only condition (54) should be taken into account, resulting in the equation

analogous to (56). Having employed value y_i in Eq. (7) and making appropriate transformations, we arrive at:

$$\tan u_K =$$
$$= \frac{1}{x_K} \frac{\sum_i \hat{y}_i G_2^{(i)} - x_K \tan\alpha \sum_i G_0^{(i)} G_2^{(i)} - x_K \tan u_P \sum_i G_1^{(i)} G_2^{(i)}}{\sum_i G_2^{(i)^2}}. \tag{61}$$

A similar procedure would apply also in the case of matching the route to direction points using curve (10). Appropriate equations describing $\tan\alpha$ and $\tan u_K$ would be the same as (59), (60) and (61) having replaced $G_0^{(i)}$, $G_1^{(i)}$ and $G_2^{(i)}$ with $M_0^{(i)}$, $M_1^{(i)}$ and $M_2^{(i)}$ in them.

Computational procedure for curves (9) and (12)

When fitting curves (9) or (12), condition (54) should be considered.

For curves (9), the following equation is used:

$$\sum_i y_i F_2^{(i)} = \sum_i \hat{y}_i F_2^{(i)} \tag{62}$$

and for curves (12), the equation used is:

$$\sum_i y_i N_2^{(i)} = \sum_i \hat{y}_i N_2^{(i)}. \tag{63}$$

Once expressed by Eqs. (9) and (12), and making appropriate transformations, the equations describing value $\tan u_K$ take the following form:

- for curves (9)

$$\tan u_K = \frac{\sum_i \hat{y}_i F_2^{(i)} - x_K \tan u_P \sum_i F_1^{(i)} F_2^{(i)}}{x_K \sum_i F_2^{(i)^2}}, \tag{64}$$
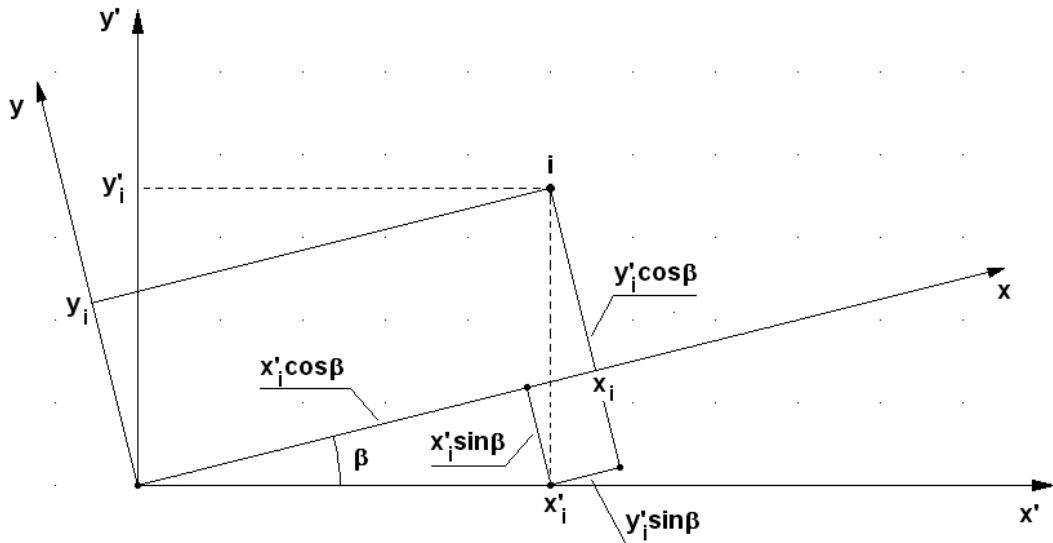
- for curves (12)

$$\tan u_K = \frac{\sum_i \hat{y}_i N_2^{(i)} - x_K \tan u_P \sum_i N_1^{(i)} N_2^{(i)}}{x_K \sum_i N_2^{(i)^2}}. \tag{65}$$

Computational procedure for curves (13) and (15)

When fitting curves (13) and (15) to the given direction points, first of all, it is required to transform the coordinates of direction points from the local coordinate system $x'y'$ (with the axes parallel to the corresponding axis of the superordinated system XY) into the local coordinate system xy of curve (Fig. 6). This can be achieved using the following equations:

$$\widehat{x}_i = x'_i \cos\beta + y'_i \sin\beta, \tag{66}$$

$$\widehat{y}_i = -x'_i \sin\beta + y'_i \cos\beta. \tag{67}$$

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Fig. 6. Transformation of curve coordinates from the local system $x'y'$ to the local system xy

Next condition (54) must be fulfilled, which analogically to the way described above, yields:

- for curves (13)

$$\tan u_K = \frac{\sum_i \widehat{y}_i G_2^{(i)} - x_K \tan u_P \sum_i G_1^{(i)} G_2^{(i)}}{x_K \sum_i G_2^{(i)2}}, \qquad (68)$$

- for curves (15)

$$\tan u_K = \frac{\sum_i \widehat{y}_i M_2^{(i)} - x_K \tan u_P \sum_i M_1^{(i)} M_2^{(i)}}{x_K \sum_i M_2^{(i)2}}. \qquad (69)$$

In the above equations values $G_1^{(i)}$, $G_2^{(i)}$ and $M_1^{(i)}$, $M_2^{(i)}$ for $t_i = \widehat{x}_i / \widehat{x}_K$ would be expressed using appropriate formulas describing G_1, G_2, M_1 and M_2.

4.4. Supplementary notes. From the point of view of the problems connected with the formation of polynomial route curvatures, a significant advantage of using general transition curves is the ease of defining their design parameters in such a way as to not exceed the maximum allowable value of curvature $1/R$. The only thing required here is the assumption of an appropriate, minimal length of abscissa x_K of the end point in the local coordinate system of the curve.

Owing to this, the curves demonstrate a high suitability for solving various design problems of in the geometrical shaping road routes. Value x_K results from the known formula describing the curvature:

$$k = \frac{|y''|}{(1 + y'^2)^{3/2}}. \qquad (70)$$

For example, value x_K for curve (7), taking into account derivatives y' and y'' function (7), can be written as:

$$x_K = $$
$$= R \frac{\left| G_0''(t_E) \tan \alpha + G_1''(t_E) \tan u_P + G_2''(t_E) \tan u_K \right|}{\left[1 + \left(G_0'(t_E) \tan \alpha + G_1'(t_E) \tan u_P + G_2'(t_E) \tan u_K \right)^2 \right]^{3/2}}, \qquad (71)$$

wherein $G_0'(t_E)$, $G_1'(t_E)$, $G_2'(t_E)$, $G_0''(t_E)$, $G_1''(t_E)$, $G_2''(t_E)$ are the values that define respectively, the first and second derivatives of coefficients G_0, G_1 and G_2 for $t = t_E$. They are as follows:

$$G_0' = 140t^3 - 420t^4 + 420t^5 - 140t^6,$$
$$G_1' = 1 - 80t^3 + 225t^4 - 216t^5 + 70t^6,$$
$$G_2' = -60t^3 + 195t^4 - 204t^5 + 70t^6,$$

and

$$G_0'' = 420t^2 - 1680t^3 + 2100t^4 - 840t^5,$$
$$G_1'' = -240t^2 + 900t^3 - 1080t^4 + 420t^5,$$
$$G_2'' = -180t^2 + 780t^3 - 1020t^4 + 420t^5.$$

The argument $t = t_E$ that occurs in Eq. (71) is a value indicating the position of curvature extremum. The value results from the equation describing the prerequisite for the existence of curvature extremum, whose general form is as follows:

$$\frac{\mathrm{d}k}{\mathrm{d}x} = \frac{y'''(1 + y'^2) - 3y'y''^2}{(1 + y'^2)^{5/2}} = 0. \qquad (72)$$

In the case of the other curves, i.e. (9), (10) and (12) the required length of abscissa x_K can be calculated the same way as described above for curves (7). Appropriate formulas expressing derivatives y', y'' and y''' for all the curves are presented in Tables 1 and 2.

Table 1
Equations describing derivatives y', y'' and y''' for curves (5) and (7)

Curves	Derivatives	Curves	Derivatives
(5)	$y' = M_0 \tan\alpha + M_1 \tan u_P + M_2 \tan u_K$ where: $M_1' = 1 - 18t^2 + 32t^3 - 15t^4,$ $M_2' = -12t^2 + 28t^3 - 15t^4,$ $M_3' = 30t^2 - 60t^3 + 30t^4$	(7)	$y' = G_0 \tan\alpha + G_1 \tan u_P + G_2 \tan u_K$ where: $G_0' = 140t^3 - 420t^4 + 420t^5 - 140t^6,$ $G_1' = 1 - 80t^3 + 225t^4 - 216t^5 + 70t^6,$ $G_2' = -60t^3 + 195t^4 - 204t^5 + 70t^6$
	$y'' = \dfrac{1}{x_K}\left(M_0'' \tan\alpha + M_1'' \tan u_P + M_2'' \tan u_K\right)$ where: $M_1'' = -36t + 96t^2 - 60t^3,$ $M_2'' = -24t + 84t^2 - 60t^3,$ $M_3'' = 60t - 180t^2 + 120t^3$		$y'' = \dfrac{1}{x_K}\left(G_0'' \tan\alpha + G_1'' \tan u_P + G_2'' \tan u_K\right)$ where: $G_0'' = 420t^2 - 1680t^3 + 2100t^4 - 840t^5,$ $G_1'' = -240t^2 + 900t^3 - 1080t^4 + 420t^5,$ $G_2'' = -180t^2 + 780t^3 - 1020t^4 + 420t^5$
	$y''' = \dfrac{1}{x_K^2}\left(M_0''' \tan\alpha + M_1''' \tan u_P + M_2''' \tan u_K\right)$ where: $M_1''' = -36 + 192t - 180t^2,$ $M_2''' = -24 + 168t - 180t^2,$ $M_3''' = 60 - 360t + 360t^2$		$y''' = \dfrac{1}{x_K^2}\left(G_0''' \tan\alpha + G_1''' \tan u_P + G_2''' \tan u_K\right)$ where: $G_0''' = 840t - 5040t^2 + 8400t^3 - 4200t^4,$ $G_1''' = -480t + 2700t^2 - 4320t^3 + 2100t^4,$ $G_2''' = -360t + 2340t^2 - 4080t^3 + 2100t^4$

Table 2
Equations describing derivatives y', y'' and y''' for curves (9) and (12)

Curves	Derivatives	Curves	Derivatives
(9)	$y' = F_1' \tan u_P + F_2' \tan u_K$ where: $F_1' = 1 - 10t^3 + 15t^4 - 6t^5,$ $F_2 = 10t^3 - 15t^4 + 6t^5$	(12)	$y' = N_1' \tan u_P + N_2' \tan u_K$ where: $N_1' = 1 - 3t^2 + 2t^3,$ $N_2' = 3t^2 - 2t^3$
	$y'' = \dfrac{1}{x_K}\left(F_1'' \tan u_P + F_2'' \tan u_K\right)$ where: $F_1'' = -30t^2 + 60t^3 - 30t^4,$ $F_2'' = 30t^2 - 60t^3 + 30t^4$		$y'' = \dfrac{1}{x_K}\left(N_1'' \tan u_P + N_2'' \tan u_K\right)$ where: $N_1'' = -6t + 6t^2,$ $N_2'' = 6t - 6t^2$
	$y''' = \dfrac{1}{x_K^2}\left(F_1''' \tan u_P + F_2''' \tan u_K\right)$ where: $F_1''' = -60t + 180t^2 - 120t^3,$ $F_2''' = 60t - 180t^2 + 120t^3$		$y''' = \dfrac{1}{x_K^2}\left(N_1''' \tan u_P + N_2''' \tan u_K\right)$ where: $N_1''' = -6 + 12t,$ $N_2''' = 6 - 12t$

In polynomial alignment with general transition curves, it is possible to adopt a general principle that the support points (connections) of consecutive curves coincide with direction points. Basically, this would involve the so-called strict direction points or tangent direction points.

5. Summary

Polynomial alignment is one possible design method of road route geometry. This subject has been extensively discussed in the literature, as evidenced by, among others, selected works included in the references/bibliography. It should be noted,

however, that this way of shaping the geometry of the road routes is not widespread in practice.

It can be assumed that the reason for this has been relatively high requirements with regard to the necessary software, which until recently has constituted some problem.

Other problems could be related to the difficulties in the design method associated with the occurrence of unacceptable extremes of curvature within individual polynomials forming the polynomial route. The result of this were problems with maintaining the required distance of visibility.

It should be noted, however, that in certain conditions, such as constructing routes in highly urbanized or mountainous areas, alignment makes it possible to find a better solution match-

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ing the specific conditions than when using traditional design methods.

Accordingly, a new approach to polynomial alignment involving the use of general transition curves has been proposed here. This approach makes it possible to avoid the problems mentioned above while keeping acceptable curvature values.

In addition, it should be noted that the methodology of applying general transitions curves in polynomial alignment is strongly algorithmic in nature allowing for a relatively simple implementation supported if appropriate software or computer systems are used for road design.

It may be added that in the design of a road route some constraints are mandatory, such as avoiding forbidden areas (for example, environmentally protected areas). Obviously, the proposed approach fully allows for such limitations to be taken into account. One might even risk the statement that it is easier to consider the alignment of the road route to such constraints with general transition curves than with traditional solutions. General transition curves are characterized by a wide range of shaping possibilities of their geometry due to their design conditions, which are given in section 3 for each family of curves.

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