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The vector space of imperceptible observation errors: a supplement to the theory of network reliability

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Abstract: A concept of the vector space of imperceptible observation errors in linear Gauss-Markov models with uncorrelated observations, initially proposed in the earlier work of the author, is presented together with some improvements and new developments. The gross errors falling into that vector space pass absolutely undetected through all possible statistical tests set in the least squares estimation and unnoticeably distort the resulting values of one or more of the model parameters. The relationship is established between the concept of imperceptible gross errors and the concept, proposed by other authors, of the gross errors which can be detected but not identified due to specific properties of a network's structure. The theory is illustrated with a simple numerical example.

Keywords: Network reliability, orthogonal projection, least squares estimation, imperceptible errors, perceptible errors, unidentifiable errors

1. Introduction

The conventional theory of reliability (Baarda, 1968), its further developments (e.g. Cross and Price, 1985; Teunissen, 1985, 1998; Ding and Coleman, 1996; Prószyński, 1997) and a new approach, being a merger of reliability and strain analysis (Vaniček et al., 2001), are all focused on the observation errors that can be detected at a certain "signal to noise" level. In other words, the error of certain magnitude and location in a particular geodetic network can be either detectable or undetectable, depending on the accuracy of observations, network configuration and the probability levels assumed in a test on outliers. In (Cen et al., 2003), the models are considered where the gross errors, while being detectable, cannot be identified due to specific properties of the model's structure. Those errors will further on be termed shortly *unidentifiable* errors.

Still more dangerous type of gross errors can be those that pass absolutely undetected through all the possible statistical tests, finally distorting the results of parameter estimation. The reasoning presented in (Caspary, 1988) encouraged the author to investigate more thoroughly that type of errors. This resulted in the concept of the vector space of imperceptible observational disturbances (Prószyński, 2000) with disturbances meaning essentially the gross errors, but also covering the random errors.

In the present paper, the theory is given on a more rigorous and complete basis. The principal part of the paper contains the results of further studies on the space of imperceptible observation errors. To link this concept to other types of observation gross errors, the models, which in their fragments, may be a seat for unidentifiable gross errors (Cen et al., 2003), are subjected to complementary analysis, based on a linear parametric model. In the analysis, the use is made of the projection operator well known both in the theory of network reliability (redundancy or reliability matrix) and the sensitivity analysis in linear regression (hat matrix – a complementary operator, see Chatterjee and Hadi, 1988). Some new resulting properties of such models are presented, expressed in terms of internal reliability indices being the diagonal elements of the redundancy matrix. On that basis the relationship is established between the concept of imperceptible gross errors and the concept of unidentifiable gross errors.

Also, the classification of observation gross errors with respect to their perceptibility, detectability and identifiability in a system is proposed.

2. Basic notations and auxiliary formulae

Let us consider a class of linear parametric models, written both in the original and the standardised form

$$\mathbf{Ax} + \mathbf{e} = \mathbf{y}; \quad \mathbf{e} \sim (\mathbf{0}, \mathbf{C}) \quad \Rightarrow \quad \mathbf{A}_*\mathbf{x} + \mathbf{e}_* = \mathbf{y}_*; \quad \mathbf{e}_* \sim (\mathbf{0}, \mathbf{I}) \tag{1a}$$

$$\mathbf{Bx} = \mathbf{0} \qquad\qquad \mathbf{Bx} = \mathbf{0} \tag{1b}$$

where

$\mathbf{y}$ – the $n \times 1$ vector of observations;
$\mathbf{A}$ – the $n \times u$ matrix of coefficients; rank $\mathbf{A} = u - d$ (d – network defect, $d \geq 0$);
$\mathbf{x}$ – the unknown $u \times 1$ vector of parameters;
$\mathbf{e}$ – the unknown $n \times 1$ vector of random errors;
$\mathbf{C}$ – the $n \times n$ covariance matrix (diagonal); rank $\mathbf{C} = n$;
$\mathbf{B}$ – the $d \times u$ matrix of coefficients, rank $\mathbf{B} = d$, rank $[\mathbf{A}^T \quad \mathbf{B}^T] = u$;

$\mathbf{A}_* = \mathbf{GA}$, $\mathbf{y}_* = \mathbf{Gy}$, $\mathbf{e}_* = \mathbf{Ge}$, where $\mathbf{G}$ is the $n \times n$ standardisation matrix (diagonal), such that $\mathbf{GG} = \mathbf{C}^{-1}$.

Equation (1) covers the models with a full rank matrix $\mathbf{A}$ (1a) and the models with minimum constraints (1a, b), the latter being frequently used in engineering surveys.

The following types of the vector spaces associated with those models, and being the subspaces of the n–dimensional observation space, will be of our interest:

- $M(\mathbf{A})$ – the observation space generated by the original model;
- $M(\mathbf{A}_*)$, $M(\mathbf{A}_*^{\perp})$ – the observation space generated by the standardised model and its orthogonal complement being the residual vector space;

where $\mathbf{A}_*^{\perp}$ $[n \times (n-u+d)]$, rank $\mathbf{A}_*^{\perp} = n-u+d$ $(d \geq 0)$. For a given $\mathbf{A}_*$, the matrix $\mathbf{A}_*^{\perp}$ is not defined uniquely (e.g. Perelmuter et al., 1994). The operators of orthogonal projection onto the spaces $M(\mathbf{A}_*)$ and $M(\mathbf{A}_*^{\perp})$ denoted by $\mathbf{P}_{(\mathbf{A}_*)}$ and $\mathbf{P}_{(\mathbf{A}_*^{\perp})}$, respectively, are linked by the relation $\mathbf{P}_{(\mathbf{A}_*)} + \mathbf{P}_{(\mathbf{A}_*^{\perp})} = \mathbf{I}$. For the model (1a, b), i.e. with $d > 0$ one has

$$\mathbf{P}_{(\mathbf{A}_*)} = \mathbf{A}_*(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}} = \mathbf{A}_*\mathbf{A}_*^{+}; \qquad \mathbf{P}_{(\mathbf{A}_*^{\perp})} = \mathbf{I} - \mathbf{A}_*(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}} \tag{2}$$

where $(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}$ is a reflexive g-inverse of $\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*$ satisfying $(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{B}_*^{\mathrm{T}} = \mathbf{0}$, and $\mathbf{A}_*^{+}$ is the pseudo-inverse of $\mathbf{A}_*$ (Rao and Mitra, 1971). For the model (1a), i.e. with $d = 0$, the regular inverse is used.

In the theory of reliability the operator $\mathbf{P}_{(\mathbf{A}_*^{\perp})}$ corresponds to the redundancy matrix and is denoted by $\mathbf{R}$. Hence, the following relationships for the least squares (*LS*) estimation, important in the theory of reliability, can be written using $\mathbf{R}$ or $\mathbf{P}_{(\mathbf{A}_*^{\perp})}$ interchangeably

$$\mathbf{v}_{*LS} = -\mathbf{R} \cdot \mathbf{y}_* \qquad\qquad \Delta\mathbf{v}_{*LS} = -\mathbf{R} \cdot \Delta\mathbf{y}_* \tag{3}$$

the latter being a disturbance/response relationship for the standardised model, where $\Delta\mathbf{v}_{*LS}$ is the $n \times 1$ vector of increments in *LS* residuals for the standardised model, $\Delta\mathbf{y}_*$ is the $n \times 1$ vector of standardised observation errors.

The projections of the vector $\Delta\mathbf{y}_*$ onto $M(\mathbf{A}_*)$ and $M(\mathbf{A}_*^{\perp})$ will be denoted by $\Delta\mathbf{y}_*^{(\mathbf{A}_*)}$ and $\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$, respectively.

3. The vector space of imperceptible observation errors and its main properties

Seeking such $\Delta\mathbf{y}_*$ that $\Delta\mathbf{y}_* \neq \mathbf{0} \;\Rightarrow\; \Delta\mathbf{v}_* = \mathbf{0}$, which may take place in either of the cases: $\mathbf{R} = \mathbf{0}$ (holds only if rank $\mathbf{A} = n$), $\mathbf{R} \cdot \Delta\mathbf{y}_* = \mathbf{0}$, one arrives at the following definition of the vector space of imperceptible observation errors for the standardised model (Prószyński, 2000)

$$U_* = \{\Delta\mathbf{y}_* : \Delta\mathbf{y}_* \in M(\mathbf{A}_*)\} \tag{4}$$

where $\dim U_* = \dim M(\mathbf{A}_*) = \dim M(\mathbf{A}) = u - d$. The definition in (4) differs only in form from that given in (Caspary, 1988).

The elements of this space can be generated by

$$\Delta\mathbf{y}_* = \mathbf{A}_* \cdot \mathbf{k} \tag{5}$$

where $\mathbf{k}$ is the $u \times 1$ vector, such that $\mathbf{k} \notin N(\mathbf{A}_*)$, $N(\mathbf{A}_*)$ being the null space of $\mathbf{A}_*$.

Rewriting (5) as $\mathbf{G} \cdot \Delta\mathbf{y} = \mathbf{G} \cdot \mathbf{Ak}$, one gets $\Delta\mathbf{y} = \mathbf{Ak}$ and may form a corresponding definition for the original model, i.e.

$$U = \{\Delta\mathbf{y} : \Delta\mathbf{y} \in M(\mathbf{A})\} \tag{6}$$

where $\dim U = \dim M(\mathbf{A}) = \dim U_*$.

On the basis of (4) and (6) one can prove that with **C** as in (1) (and even for **C** being any positive definite matrix)

$$\Delta\mathbf{y} \in U \;\Leftrightarrow\; \Delta\mathbf{y}_* \in U_* \tag{7}$$

which means that it is only the algebraic structure of the functional model (1) that determines its space of imperceptible observation errors.

As a complement to (6), we shall define the vector space of perceptible observation errors for the original model

$$W = \{\Delta\mathbf{y} : \Delta\mathbf{y} \notin M(\mathbf{A})\} \tag{8}$$

where, due to U and W being virtually disjoint (Rao and Mitra, 1971)

$$\dim W = n - \dim U = n - u + d$$

On the basis of definitions in (4), (6) and (8) and the property in (7) one may give the following definitions of imperceptible and perceptible errors, i.e.

- an imperceptible observation error is each element of such a vector $\Delta\mathbf{y}$ (and $\Delta\mathbf{y}_* = \mathbf{G}\cdot\Delta\mathbf{y}$), that $\Delta\mathbf{y}$ belongs to the space $M(\mathbf{A})$;
- a perceptible observation error is each element of such a vector $\Delta\mathbf{y}$ (and $\Delta\mathbf{y}_* = \mathbf{G}\cdot\Delta\mathbf{y}$), that $\Delta\mathbf{y}$ does not belong to the space $M(\mathbf{A})$.

The following properties can be proved:

- neither reordering of the parameters in (1) nor rescaling of each of them affects the space of imperceptible observation errors;
- in the models with redundancies such that $\{\mathbf{R}\}_{ii} > 0$ ($i = 1, 2, ..., n$), the space of imperceptible observation errors does not contain any vector with one non-zero element. This means that a single gross error, being the only non-zero element in the vector of gross errors, cannot be an imperceptible error and hence, is perceptible.

Since the LS estimation is carried out for the standardised model, we shall concentrate now on the (unique) decomposition of the vector of standardised observation errors by orthogonal projection onto the spaces $M(\mathbf{A}_*)$ and $M(\mathbf{A}_*^{\perp})$ (Kolman and Hill, 2004), as shown in Figure 1a, i.e.

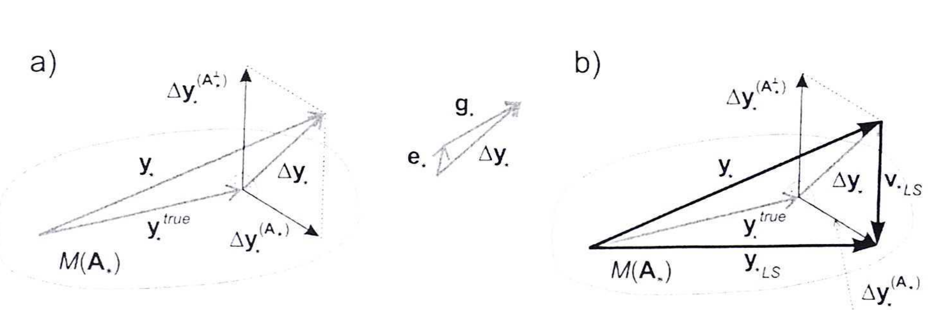

Fig. 1. Decomposition of the vector of standardised observation errors $\Delta\mathbf{y}_*$ (a) and its effect on the LS estimation results (b)

$$\Delta\mathbf{y}_* = \Delta\mathbf{y}_*^{(\mathbf{A}_*)} + \Delta\mathbf{y}_*^{(\mathbf{A}_*^\perp)} \tag{9}$$

With $\Delta\mathbf{y}_* = \mathbf{e}_* + \mathbf{g}_*$, where $\mathbf{e}_*$ is the vector of random errors and $\mathbf{g}_*$ is the vector of gross errors, one gets on the basis of (9) a detailed decomposition of $\Delta\mathbf{y}_*$

$$\Delta\mathbf{y}_* = \mathbf{e}_*^{(\mathbf{A}_*)} + \mathbf{g}_*^{(\mathbf{A}_*)} + \mathbf{e}_*^{(\mathbf{A}_*^\perp)} + \mathbf{g}_*^{(\mathbf{A}_*^\perp)} \tag{10}$$

The imperceptible and the perceptible component of $\Delta\mathbf{y}_*$ can be determined from

$$\Delta\mathbf{y}_*^{(\mathbf{A}_*)} = \mathbf{P}_{(\mathbf{A}_*)} \cdot \Delta\mathbf{y}_* \qquad \Delta\mathbf{y}_*^{(\mathbf{A}_*^\perp)} = \mathbf{P}_{(\mathbf{A}_*^\perp)} \cdot \Delta\mathbf{y}_* = (\mathbf{I} - \mathbf{P}_{(\mathbf{A}_*)}) \cdot \Delta\mathbf{y}_* \tag{11}$$

where $\mathbf{P}_{(\mathbf{A}_*)}$, $\mathbf{P}_{(\mathbf{A}_*^\perp)}$ as in (2).

4. Other properties of the vector space of imperceptible observation errors

Property 1.

The following relationship holds true

$$\dim_{\text{fr}} U = 1 - \bar{r} \tag{12}$$

where $\dim_{\text{fr}} U$ is a fractional dimension of the space of imperceptible errors

$$\dim_{\text{fr}} U = \frac{\dim U}{\dim \Phi} = \frac{u-d}{n}; \qquad \Phi - \text{observation space},$$

$\bar{r}$ is a global index of the model internal reliability; $\bar{r} = \dfrac{n-u+d}{n}$

<u>Proof</u>. Modifying slightly the expression for $\dim_{\text{fr}} U$ one gets immediately

$$\dim_{\text{fr}} U = 1 - 1 + \frac{u-d}{n} = 1 - \frac{n-u+d}{n} = 1 - \bar{r} \ \blacksquare$$

Hence, by increasing the level of the model internal reliability, one reduces the space of imperceptible observation errors, thus extending the space of perceptible observation errors. For the models without redundancies ($\bar{r} = 0$), the space of imperceptible observation errors spreads over the whole observation space.

Property 2.

The vector of *LS* residuals (see Fig. 1b) can be represented as

$$\mathbf{v}_{*LS} = -\Delta\mathbf{y}_*^{(\mathbf{A}_*^\perp)} \tag{13}$$

where $\Delta\mathbf{y}_*^{(\mathbf{A}_*^\perp)}$ is the perceptible component of the vector of observation errors.

Proof. The following can be shown:

$$\mathbf{v}_{*LS} = -\mathbf{P}_{(\mathbf{A}_*^{\perp})} \cdot \mathbf{y}_* = -\mathbf{P}_{(\mathbf{A}_*^{\perp})} (\mathbf{y}_*^{true} + \Delta\mathbf{y}_*) = -\mathbf{P}_{(\mathbf{A}_*^{\perp})} (\mathbf{y}_*^{true} + \Delta\mathbf{y}_*^{(\mathbf{A}_*)} + \Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})})$$

From the consistency of the functional model it follows that $\mathbf{y}_*^{true} \in M(\mathbf{A}_*)$, so $\mathbf{P}_{(\mathbf{A}_*^{\perp})} \cdot \mathbf{y}_*^{true} = \mathbf{0}$ and hence, we obtain the relationship as in (13). ■

Equation (13) can be considered as an algebraic interpretation of the well-known definition

$$-\mathbf{v}_{*LS} = \widehat{\Delta\mathbf{y}}_{*LS}$$

and so, one gets

$$\widehat{\Delta\mathbf{y}}_{*LS} = \Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$$

From this property, it follows that for a particular model one can generate different observation vectors resulting in identical *LS* residuals, i.e.

$$\left\{\mathbf{y}_{*id} : \mathbf{y}_{*id} = \mathbf{y}_* + \mathbf{e}_*^{(\mathbf{A}_*)} + \mathbf{g}_*^{(\mathbf{A}_*)}\right\} \tag{14}$$

where $\mathbf{y}_*$ is the vector of standardised observation values; $\mathbf{y}_{*id}$ is the observation vector yielding identical *LS* residuals as $\mathbf{y}_*$; $\mathbf{e}_*^{(\mathbf{A}_*)}$, $\mathbf{g}_*^{(\mathbf{A}_*)}$ are here the arbitrary non-zero vectors of random and gross errors belonging to the space of imperceptible observation errors.

Property 3.

The imperceptible component $\Delta\mathbf{y}_*^{(\mathbf{A}_*)}$ of the vector $\Delta\mathbf{y}_*$, where $\Delta\mathbf{y}_*^{(\mathbf{A}_*)} = \mathbf{e}_*^{(\mathbf{A}_*)}$ or $\Delta\mathbf{y}_*^{(\mathbf{A}_*)} = \mathbf{e}_*^{(\mathbf{A}_*)} + \mathbf{g}_*^{(\mathbf{A}_*)}$, does not affect the level of the system's inconsistency.

Proof. Applying an algebraic index of the system's inconsistency $q = \left\|\mathbf{P}_{(\mathbf{A}_*^{\perp})} \cdot \mathbf{y}_*\right\|_2$, one obtains on the basis of property 2, $q = \left\|\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}\right\|_2$, which does not contain $\Delta\mathbf{y}_*^{(\mathbf{A}_*)}$.■

Let us denote by q_{crit} a critical value for q, determined on a stochastic basis. The case $q \leq q_{crit}$ could be termed *insignificant inconsistency*, whereas $q > q_{crit}$ could be termed *significant inconsistency*. The former usually happens with $\Delta\mathbf{y}_* = \mathbf{e}_*$ and the latter with $\Delta\mathbf{y}_* = \mathbf{e}_* + \mathbf{g}_*$.

Property 4.

It is only the imperceptible component of the vector of observation errors that causes bias of the *LS* solution. For the models with $d > 0$ the solution is defined by

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\Delta\mathbf{y}_*^{(\mathbf{A}_*)} \tag{15}$$

or equivalently

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{I} - \mathbf{B}^{-}\mathbf{B})\mathbf{k} \tag{16}$$

where $\mathbf{B}^{-} = \mathbf{B}_0^{\mathrm{T}}(\mathbf{B}\mathbf{B}_0^{\mathrm{T}})^{-1}$, $\mathbf{B}$ is the matrix as in (1), $\mathbf{B}_0$ is the specific matrix $\mathbf{B}$ such that $\mathbf{A}\mathbf{B}_0^{\mathrm{T}} = \mathbf{0}$, and $\mathbf{k}$ is the vector such that $\Delta\mathbf{y}_*^{(\mathbf{A}_*)} = \mathbf{A}_* \cdot \mathbf{k}$ (see (5)).

Proof. With the decomposition of the vector $\mathbf{y}_*$, i.e. $\mathbf{y}_* = \mathbf{y}_*^{true} + \Delta\mathbf{y}_*^{(\mathbf{A}_*)} + \Delta\,\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$ the *LS* solution vector will take the form

$$\hat{\mathbf{x}}_{LS} = (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\mathbf{y}_*^{true} + (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\Delta\mathbf{y}_*^{(\mathbf{A}_*)} + (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$$

Since $(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\mathbf{y}_*^{true} = \mathbf{x}^{true}$ and $\mathbf{A}_*^{\mathrm{T}}\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})} = \mathbf{0}$, one obtains

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\Delta\mathbf{y}_*^{(\mathbf{A}_*)} \quad \text{(see (15))}$$

which, according to (5), can further be transformed to

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*\mathbf{k}$$

or due to $(\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_* = \mathbf{I} - \mathbf{B}^{-}\mathbf{B}$

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{I} - \mathbf{B}^{-}\mathbf{B})\mathbf{k} \quad \text{(see (16))} \ \blacksquare$$

For the models with $d = 0$, Eqs. (15) and (16) will get reduced to

$$\hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + (\mathbf{A}_*^{T}\mathbf{A}_*)^{-1}\mathbf{A}_*^{T}\Delta\mathbf{y}_*^{(\mathbf{A}_*)}, \text{ and } \hat{\mathbf{x}}_{LS} = \mathbf{x}^{true} + \mathbf{k} \tag{17}$$

respectively.

A discussion on the effects of using specific numerical options of $\mathbf{k}$ can be found in (Prószyński, 2000). In a hypothetical case, i.e. $\Delta\mathbf{y}_* = \Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$, despite the system's inconsistency be it either insignificant or significant, one gets $\mathbf{x}_{LS} = \mathbf{x}^{true}$.

Properties 3 and 4 are presented in a simplified form in Figure 2, showing the situation in a parameter space. $\Delta\hat{\mathbf{x}}_{LS}$ denotes bias of the *LS* solution (see (16) or (17)), caused by $\Delta\mathbf{y}_*^{(\mathbf{A}_*)}$. The intersection area of positional hyper-planes $h_1, h_2, ..., h_n$ (each representing an individual standardised observation equation), termed as the inconsistency zone *Iz*, is due to $\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$. Each hyper-plane runs at a distance of $\left|\Delta\mathbf{y}_{*_i}^{(\mathbf{A}_*^{\perp})}\right| \cdot \left|\mathbf{d}_i\right|^{-1}$ ($i = 1, 2, ..., n$) from the point P_{LS}, where $\Delta\mathbf{y}_{*_i}^{(\mathbf{A}_*^{\perp})}$ is the i-th element of the vector $\Delta\mathbf{y}_*^{(\mathbf{A}_*^{\perp})}$ and $\mathbf{d}_i$ a vector normal to the i-th hyper-plane.

The component $\mathbf{g}_*^{(\mathbf{A}_*)}$ only shifts the inconsistency zone *Iz*, maintaining the orientations and mutual positions of the hyper-planes (see Fig. 2). One can suppose that for any method of linear estimation (be it either orthogonal or quasi-orthogonal projection) the location of the solution point within the inconsistency zone is in both cases, i.e. $\mathbf{g}_*^{(\mathbf{A}_*)} = \mathbf{0}$, $\mathbf{g}_*^{(\mathbf{A}_*)} \neq \mathbf{0}$, identical. Hence, the corresponding bias of the solution will be equal to that in the *LS* estimation.

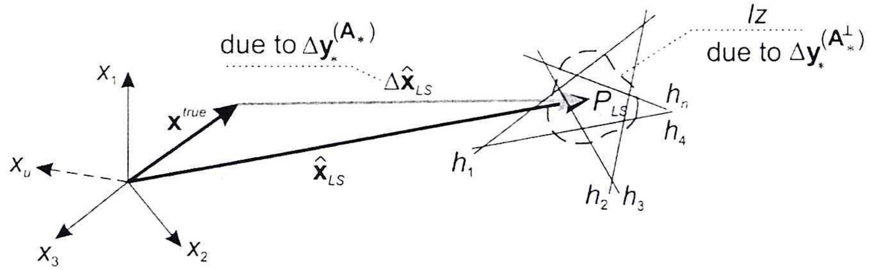

Fig. 2. The effects of the imperceptible ($\Delta\mathbf{y}_*^{(\mathbf{A}_*)}$) and perceptible ($\Delta\mathbf{y}_*^{(\mathbf{A}_*^\perp)}$) component of the error vector $\Delta\mathbf{y}_*$ upon the results of the LS estimation; $h_1, h_2, ..., h_n$ symbolize positional hyper-planes and Iz – their intersection area

Since, in general $\hat{\mathbf{x}}_{(\circ)}[\mathbf{g}_*^{(\mathbf{A}_*)} = \mathbf{0}] \neq \hat{\mathbf{x}}_{LS}[\mathbf{g}_*^{(\mathbf{A}_*)} = \mathbf{0}]$, where $(\circ)$ denotes an arbitrary linear estimation, the vector $\Delta\hat{\mathbf{x}}_{(\circ)}$ will be parallel to $\Delta\hat{\mathbf{x}}_{LS}$.

The above reasoning can be an argumentation for the following proposition:

Proposition. The imperceptible component $\mathbf{g}_*^{(\mathbf{A}_*)}$ of the error vector $\mathbf{g}_*$, resulting in a bias of the solution as in Property 4, i.e.

$$\Delta\hat{\mathbf{x}}_{LS} = \hat{\mathbf{x}}_{LS}[\mathbf{g}_*^{(\mathbf{A}_*)} \neq \mathbf{0}] - \hat{\mathbf{x}}_{LS}[\mathbf{g}_*^{(\mathbf{A}_*)} = \mathbf{0}] = (\mathbf{A}_*^{\mathrm{T}}\mathbf{A}_*)_{\mathbf{B}}^{-}\mathbf{A}_*^{\mathrm{T}}\mathbf{g}_*^{(\mathbf{A}_*)} \tag{18}$$

will cause an identical bias of the solution vector in any other method of linear estimation, i.e.

$$\Delta\hat{\mathbf{x}}_{(\circ)} = \hat{\mathbf{x}}_{(\circ)}[\mathbf{g}_*^{(\mathbf{A}_*)} \neq \mathbf{0}] - \hat{\mathbf{x}}_{(\circ)}[\mathbf{g}_*^{(\mathbf{A}_*)} = \mathbf{0}] = \Delta\hat{\mathbf{x}}_{LS} \tag{19}$$

with $\Delta\hat{\mathbf{x}}_{(\circ)}$ being parallel to $\Delta\hat{\mathbf{x}}_{LS}$.

5. Complementary analysis of the concept of unidentifiable gross errors

In (Cen et al., 2003) a method is presented of finding at a pre-adjustment stage, the observations in which the detectable gross errors are unidentifiable due to a specific network's structure. Here, there will be added some more properties of such a structure, the properties being the results of the analysis based on a parametric model as in (1) and the use of redundancy matrix $\mathbf{R}$.

Property A.

The unidentifiable gross errors belong to the class of perceptible errors for a particular model.

Explanation. The existence of an unidentifiable gross error (or errors) in a network is signalled by equal values of standardised LS residuals for several observations, exceeding the critical values. To cause non-zero increments in the values of LS residuals, this error (or errors), although of unidentified location, must be perceptible in the system.

Property B.

The necessary structural condition for the existence in a network with uncorrelated observations, of the i-th and the j-th observable such, that a gross error occurring in the i-th or in the j-th observation cannot be identified, is as follows:

$$\mathbf{R}_{\bullet j} = c_{ij} \cdot \mathbf{R}_{\bullet i} \quad \text{with} \quad c_{ij} = \sqrt{\frac{\{\mathbf{R}\}_{jj}}{\{\mathbf{R}\}_{ii}}} \quad \text{or} \quad c_{ij} = -\sqrt{\frac{\{\mathbf{R}\}_{jj}}{\{\mathbf{R}\}_{ii}}} \tag{20}$$

or equivalently

$$\left|\{\mathbf{R}\}_{ij}\right| = \sqrt{\{\mathbf{R}\}_{ii} \cdot \{\mathbf{R}\}_{jj}} \tag{21}$$

where $\mathbf{R}_{\bullet i}$, $\mathbf{R}_{\bullet j}$, $\{\mathbf{R}\}_{ii}$, $\{\mathbf{R}\}_{jj}$, $\{\mathbf{R}\}_{ij}$ are respectively, the columns and the elements of the reliability matrix R, corresponding to the i-th and the j-th observable.

Proof. The identification of gross error g being either in the i-th or in the j-th observation, i.e. $g(i)$ or $g(j)$, will not be possible if

$$\left|w_{i,g(i)}\right| = \left|w_{j,g(i)}\right| \quad \text{and} \quad \left|w_{i,g(j)}\right| = \left|w_{j,g(j)}\right| \tag{22}$$

where $w_{i,g(i)}, w_{j,g(i)}, w_{i,g(j)}, w_{j,g(j)}$ are the elements of the vectors $\mathbf{w}_{g(i)}, \mathbf{w}_{g(j)}$ of the standardised *LS* residuals, for the i-th and the j-th observation.

Premultiplying both sides of (3), first by $\mathbf{G}^{-1}$ (where $\mathbf{G}$ as in (1)) and then by the diagonal matrix $\mathbf{H} = \text{diag}\,, (\sigma_{\hat{v}_1}^{-1}, \sigma_{\hat{v}_2}^{-1}, \ldots, \sigma_{\hat{v}_n}^{-1})$, where $\sigma_{\hat{v}_i} = \sigma_{y_i}\sqrt{\{\mathbf{R}\}_{ii}}$, one shall get the following formula for the vector of standardised *LS* residuals

$$\mathbf{w} = -\mathbf{H}\mathbf{G}^{-1}\mathbf{R}\mathbf{G} \cdot \mathbf{y} = \mathbf{Q} \cdot \mathbf{y} \tag{23}$$

This formula will be used with the decomposition of the observation vector, applied earlier in this paper, i.e.

$$\mathbf{y} = \mathbf{y}^{true} + \mathbf{e} + \mathbf{g}$$

where $\mathbf{e} = [e_1, e_2, \ldots, e_n]^T$; $\mathbf{g} = [0 \ldots 0 \quad g(i) \quad 0 \ldots 0]^T$ or $\mathbf{g} = [0 \ldots 0 \quad g(j) \quad 0 \ldots 0]^T$.

Since $\mathbf{Q} \cdot \mathbf{y}^{true} = \mathbf{0}$, $\mathbf{Q} \cdot \mathbf{e} \neq \mathbf{0}$ (it is assumed that $\mathbf{e}$ is a vector of perceptible errors) and $\mathbf{Q} \cdot \mathbf{g} \neq \mathbf{0}$ (having one non-zero element, $\mathbf{g}$ is a vector of perceptible errors), one obtains from (23)

$$\mathbf{w} = \mathbf{Q}(\mathbf{e} + \mathbf{g})$$

and hence

$$w_i = \mathbf{Q}_{i\bullet}(\mathbf{e} + \mathbf{g}), \qquad w_j = \mathbf{Q}_{j\bullet}(\mathbf{e} + \mathbf{g})$$

where $\mathbf{Q}_{i\bullet}$, $\mathbf{Q}_{j\bullet}$ are the i-th and the j-th row of the matrix $\mathbf{Q}$.

To satisfy (22) one needs

$$\mathbf{Q}_{j\bullet} = \mathbf{Q}_{i\bullet} \quad \text{or} \quad \mathbf{Q}_{j\bullet} = -\,\mathbf{Q}_{i\bullet}. \tag{24}$$

Since the elements of $\mathbf{Q}_{i\bullet}$ and $\mathbf{Q}_{j\bullet}$ can be written as

$$\{\mathbf{Q}_{i\bullet}\}_k = \frac{1}{\sigma_{y_k}} \frac{\{\mathbf{R}\}_{ik}}{\sqrt{\{\mathbf{R}\}_{ii}}}; \qquad \{\mathbf{Q}_{j\bullet}\}_k = \frac{1}{\sigma_{y_k}} \frac{\{\mathbf{R}\}_{jk}}{\sqrt{\{\mathbf{R}\}_{jj}}}, \quad k = 1, 2, \ldots, n$$

one obtains after simple operations the condition as in (20), which holds also for rows due to the symmetry of $\mathbf{R}$. Equating the elements $\{\mathbf{Q}_{i\bullet}\}_k$, $\{\mathbf{Q}_{j\bullet}\}_k$ for $k = i$ and $k = j$, one gets the condition as in (21). The equivalency of (20) and (21) can be proved by showing that the relationship in (21) implies the one in (20). ■

The above proof holds for any vector $\mathbf{g}$, except for $\mathbf{g} \in M(\mathbf{A})$, which yields $\mathbf{Qg} = \mathbf{0}$. Therefore, with the condition as in (20) or (21) being satisfied, one gets for $\mathbf{g} = [0 \ldots 0 \;\; g(i) \;\; 0 \ldots 0 \;\; g(j) \;\; 0 \ldots 0]^{\mathrm{T}}$, such that $\mathbf{g} \notin M(\mathbf{A})$, the equality

$$\left| w_{i,g(i),g(j)} \right| = \left| w_{j,g(i),g(j)} \right|$$

i.e. the gross errors in the i-th and the j-th observation, such that they do not form a vector of imperceptible errors, are unidentifiable.

Property B extended upon a greater number of observables in a network, e.g. $y_1, y_2, \ldots, y_s$ would be expressed by the conditions as in (20) or (21), with the range for indices i and j being $i = 1, 2, \ldots, s-1$; $j = 2, 3, \ldots, s$; $j > i$.

Hence, one may introduce the concept of a Region of Unidentifiable Errors (*RUE*), defined by

$$RUE = \{y_1, y_2, \ldots, y_s : \left|\{\mathbf{R}\}_{ij}\right| = \sqrt{\{\mathbf{R}\}_{ii} \cdot \{\mathbf{R}\}_{jj}};\; i = 1, 2, \ldots, s-1;\; j = 2, 3, \ldots, s;\; j > i \tag{25}$$

or equivalently, with the use of (20).

A single levelling loop, which as a whole constitutes one *RUE*, is a specific case. In such a structure the gross errors in any combination, that do not form a vector of imperceptible errors, are unidentifiable.

Property C.

In a network with the observables y_i, y_j such that $\{\mathbf{R}\}_{ii} \cdot \{\mathbf{R}\}_{jj} > 0.25$, the region $\{y_1, y_2\}$ cannot be a *RUE*.

Proof. Since $\left|\{\mathbf{R}\}_{ij}\right|_{\max} = 0.5$ (Prószyński, 1997), it follows that with $\{\mathbf{R}\}_{ii}$, $\{\mathbf{R}\}_{jj}$, such that $\{\mathbf{R}\}_{ii} \cdot \{\mathbf{R}\}_{jj} > 0.25$, Eq. (21) cannot be satisfied, which contradicts the existence of $RUE = \{y_i, y_j\}$. This applies to any pair of indices i, j. ■

Property D.

In a network with internal reliability such that $\{\mathbf{R}\}_{ii} > 0.5$ ($i = 1, 2, \ldots, n$) there cannot exist any *RUE*. This results immediately from Property C.

From Properties C and D it follows that the existence of *RUE* in a network depends entirely on the level of its internal reliability. So, such regions are practically unavoidable in levelling networks, which are the structures of low internal reliability.

It should be noted that the definition of *RUE* introduced in this paper does not cover the network structures (Cen et al., 2003) where only some of the detectable gross errors are unidentifiable. This may happen in a network without a *RUE*, when after removing an outlier with gross error being detectable and identifiable, the resulting structure discloses the *RUE*, containing the detectable but unidentifiable gross errors.

6. The proposed reliability-oriented classification of gross errors

Figure 3 shows the proposed classification of observation gross errors, formed on the basis of the original (i.e. not standardised) model.

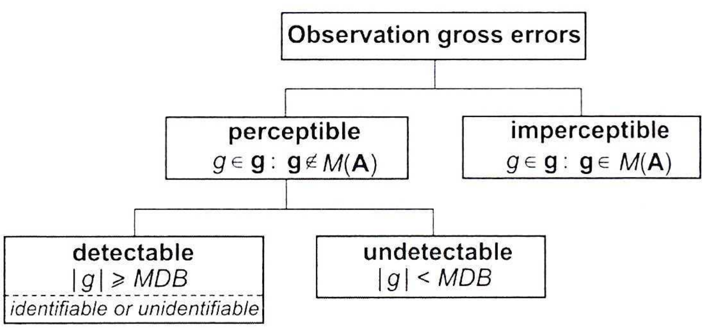

Fig. 3. The reliability-oriented classification of gross errors; $\mathbf{g}$ – vector of observation gross errors; g – gross error in the individual observation, being the element of the vector $\mathbf{g}$; *MDB* – minimal detectable bias for this observation

The minimal detectable bias (*MDB*) developed by (Baarda, 1968) for the case of single gross error, was later extended by (Teunissen, 1985, 1998) to cover the case of multiple gross errors.

The classification has two hierarchic levels, with the following criteria:

- orientation of the vector of observation gross errors in relation to the observation space generated by the model;
- relation between the magnitude of the error and the minimal detectable bias (*MDB*).

Here there are some explanatory comments to the scheme in Figure 3:

- each element g of the vector $\mathbf{g}$, where $\mathbf{g}$ belongs to the observation space generated by the model, is an imperceptible gross error. Conversely, each element g of the vector $\mathbf{g}$, where $\mathbf{g}$ does not belong to this space (i.e. it has non-zero component orthogonal to this space) is a perceptible gross error;

- by definition, the concept of perceptibility of errors is not connected with parameter estimation in a model, as is the case with the concept of detectability. However, both the concepts depend on algebraic properties of the model and hence, in this respect they are interrelated. This relation is reflected in the values of reliability indices $\{\mathbf{R}\}_{ii}$ and the *MDB* values computed on their basis. And so, for a single gross error being an imperceptible error only when it resides in the observation (outlier) with $\{\mathbf{R}\}_{ii} = 0$, the *MDB* becomes infinite, which means that such an error is absolutely undetectable. Similarly, in the multiple outlier case the *MDB* for the candidate outlier carrying an imperceptible error becomes infinite, thus indicating that the error is absolutely undetectable. For a single gross error being a perceptible error, i.e. the one which resides in the observation with $\{\mathbf{R}\}_{ii} > 0$, the *MDB* assumes finite value and the error can be either detectable or undetectable, depending on its magnitude;
- the detectable errors can be either identifiable or unidentifiable. The gross errors that lie in *RUE*, and do not form a vector belonging to the space of imperceptible errors, are unidentifiable. The unidentifiability of this type, as independent of the magnitude of gross errors, can be attributed to undetectable errors as well. This was not shown in the classification table, since in the outlier-detection strategies the undetectable gross errors are beyond our reach.

7. Numerical example

To demonstrate the properties of imperceptible gross errors and show their relation to unidentifiable gross errors discussed in (Cen et al., 2003), we shall use the same levelling network, shown in Figure 4. Its option with one observation being added to improve the network's internal reliability shall also be used.

Table 1 shows the observation values with random errors only ($\mathbf{h}$), their a priori standard deviations (σ_h), the indices of internal reliability ($\{\mathbf{R}\}_{ii}$) and the *LS* standardised residuals ($\mathbf{w}$).

The inspection of the matrix $\mathbf{R}$ (19×19) shows that the columns $\mathbf{R}_{\bullet 13}$ and $\mathbf{R}_{\bullet 14}$ are parallel vectors with $c = 0.80$ and $\{\mathbf{R}\}_{13,13} = 0.47$, $\{\mathbf{R}\}_{14,14} = 0.30$, $\{\mathbf{R}\}_{13,14} = 0.38$. One can check that both the equivalent conditions as in (20) and (21), are satisfied. One may then conclude that in the original levelling scheme there is $RUE = \{h_{13}, h_{14}\}$.

The following examples of the vectors of gross errors to be introduced into the scheme shall be considered:

$$\mathbf{g}_I = \mathbf{A} \cdot \mathbf{k} = [1\ {-6}\ 8\ 10\ {-7}\ 4\ {-3}\ {-5}\ 1\ 7\ {-1}\ 9\ {-5}\ 3\ {-1}\ 6\ 1\ {-2}\ 3]^{\mathrm{T}}$$

with $\mathbf{k} = [\underset{(A)}{1}\ \ \underset{(B)}{2}\ \ \underset{(C)}{-3}\ \ \underset{(D)}{0}\ \ \underset{(E)}{-1}\ \ \underset{(F)}{5}\ \ \underset{(G)}{4}\ \ \underset{(H)}{-6}\ \ \underset{(I)}{-2}\ \ \underset{(J)}{3}\ \ \underset{(K)}{-4}]^{\mathrm{T}}$

$$\mathbf{g}_{II} = g \cdot \mathbf{A}_{\bullet C} = [0 \ldots 0\ \underset{(13)}{g}\ \underset{(14)}{-g}\ 0 \ldots 0]^{\mathrm{T}} \text{ or represented like } \mathbf{g}_I$$

$$\mathbf{g}_{II} = \mathbf{A} \cdot \mathbf{k} \quad \text{with} \quad \mathbf{k} = [0 \;\ldots\; 0 \; \underset{(C)}{g} \; 0 \;\ldots\; 0]^{\mathrm{T}}$$

$$\mathbf{g}_{III} = [0 \;\ldots\; 0 \; \underset{(13)}{g} \; 0 \;\ldots\; 0]^{\mathrm{T}}; \qquad \mathbf{g}_{IV} = [0 \;\ldots\; 0 \; \underset{(13)}{g} \;\; \underset{(14)}{g} \; 0 \;\ldots\; 0]^{\mathrm{T}}$$

where g is a gross error; $\mathbf{A}_{\bullet C}$ is the column of $\mathbf{A}$ corresponding to the node C of the network; $(\,\cdot\,)$ is the observation or node number.

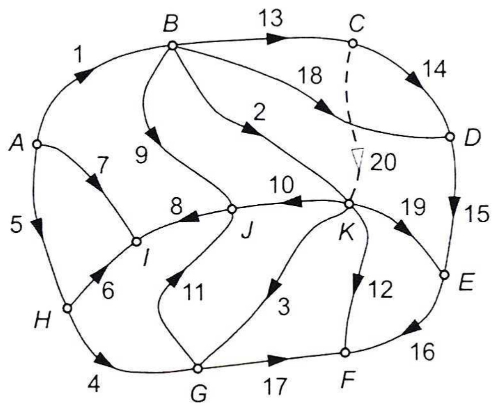

Fig. 4. The levelling network for testing the properties of gross errors (Cen et al., 2003); additional observation *C-K* is marked with a dashed line

The vectors $\mathbf{g}_I$, $\mathbf{g}_{II}$, being the combinations of the columns of matrix $\mathbf{A}$, belong to the space of imperceptible errors. Added to the vector of observation values $\mathbf{h}$, they do not change the vector of *LS* standardised residuals $\mathbf{w}$, but cause respective biases $\Delta\mathbf{H}_I$ and $\Delta\mathbf{H}_{II}$ in the solution, obtained with a reference condition $H_A = 0$, i.e.

$$\Delta\mathbf{H}_I = [\underset{(A)}{0} \;\; \underset{(B)}{1} \;\; \underset{(C)}{-4} \;\; \underset{(D)}{-1} \;\; \underset{(E)}{-2} \;\; \underset{(F)}{4} \;\; \underset{(G)}{3} \;\; \underset{(H)}{-7} \;\; \underset{(I)}{-3} \;\; \underset{(J)}{2} \;\; \underset{(K)}{-5}]^{\mathrm{T}}; \qquad \Delta\mathbf{H}_{II} = [0 \;\ldots\; 0 \; \underset{(C)}{g} \; 0 \;\ldots\; 0]^{\mathrm{T}}$$

One can see that the vector $\mathbf{g}_I$ affects all the parameters (except for the fixed H_A), whereas $\mathbf{g}_{II}$ distorts H_C only.

The vector $\mathbf{g}_{III}$ containing a single gross error, located in the observation h_{13} with $\{\mathbf{R}\}_{13,13} > 0$, does not belong to the space of imperceptible errors, and hence is perceptible. Its standardised form $\mathbf{g}_{III*}$ can be decomposed into an imperceptible component and a perceptible component. Both the components computed for $g = 2.8$ mm and thus, $g_* = 2.8\text{ mm}/0.91\text{ mm} = 3.08$, are shown in Table 1 under the symbols $\mathbf{g}_{III*}(\mathbf{A}^*)$ and $\mathbf{g}_{III*}(\mathbf{A}_*^{\perp})$, respectively. The decomposition shows that the single error $g_* = 3.08$ in the observation h_{13} is split into its imperceptible part (1.63) and the perceptible part (1.45). The latter is a result of the operation $\{\mathbf{R}\}_{13,13} \cdot g_*$, i.e. $0.47 \cdot 3.08 = 1.45$. One can notice that the perceptibility of this error is proportional to the value of reliability index for the observation h_{13}. In other words, the greater the reliability index for a particular observation, the better the perceptibility of a gross error residing in this observation. With $\{\mathbf{R}\}_{13,13} > 0.5$ the perceptible part of the error would exceed the imperceptible part. This should be taken into consideration in the design of networks. As shown in Table 1, the corresponding elements of the components $\mathbf{g}_{III*}(\mathbf{A}_*)$ and $\mathbf{g}_{III*}(\mathbf{A}_*^{\perp})$, except for the elements discussed above, have opposite signs and cancel each other.

Table 1. The results of testing the original levelling scheme

No obs.	$\mathbf{h}$ [mm]	σ_h [mm]	$\{\mathbf{R}\}_{ii}$	$\mathbf{w}$	$\mathbf{g}_{III*}(\mathbf{A}_*)$	$\mathbf{g}_{III*}(\mathbf{A}_*^{\perp})$	$\mathbf{w}_{III}$	$\mathbf{g}_{IV*}(\mathbf{A}_*)$	$\mathbf{g}_{IV*}(\mathbf{A}_*^{\perp})$	$\mathbf{w}_{IV}$
1	171303.4	1.15	0.62	–1.25	–0.09	0.09	–1.36	–0.17	0.17	–1.47
2	7514.2	0.90	0.65	–0.92	0.23	–0.23	–0.63	0.46	–0.46	–0.34
3	38636.2	0.83	0.62	0.38	–0.04	0.04	0.33	–0.08	0.08	0.28
4	117336.0	0.96	0.50	0.37	0.06	–0.06	0.45	0.12	–0.12	0.54
5	100116.5	0.63	0.39	–1.26	0.03	–0.03	–1.21	0.06	–0.06	–1.17
6	309270.0	0.66	0.45	–1.50	–0.01	0.01	–1.52	–0.02	0.02	–1.53
7	409384.6	0.59	0.37	2.05	0.02	–0.02	2.08	0.03	–0.03	2.11
8	81197.9	0.78	0.39	–0.72	–0.01	0.01	–0.73	–0.02	0.02	–0.75
9	156885.3	0.67	0.43	0.52	0.17	–0.17	0.78	0.35	–0.35	1.05
10	149373.1	0.75	0.57	–1.95	–0.12	0.12	–2.11	–0.24	0.24	–2.27
11	110735.1	0.71	0.50	0.87	–0.08	0.08	0.76	–0.16	0.16	0.65
12	134363.2	0.54	0.35	0.91	0.10	–0.10	1.08	0.21	–0.21	1.26
13	56263.7	0.91	0.47	–0.56	**1.63**	**1.45**	**–2.68**	**0.18**	**2.91**	**–4.79**
14	42425.2	0.73	0.30	–0.56	–1.16	1.16	**–2.68**	**1.52**	**2.32**	**–4.79**
15	25345.1	0.58	0.22	–0.66	–0.34	0.34	–1.38	–0.68	0.68	–2.11
16	17844.2	0.74	0.44	–0.85	–0.24	0.24	–1.20	–0.47	0.47	–1.55
17	95727.2	0.84	0.56	–0.25	0.11	–0.11	–0.11	0.21	–0.21	0.03
18	98688.4	0.79	0.51	–0.13	0.80	–0.80	0.99	1.60	–1.60	2.11
19	116519.9	0.92	0.64	–0.26	0.25	–0.25	0.05	0.50	–0.50	0.37

The observation h_{13} carrying this error belongs to $RUE = \{h_{13}, h_{14}\}$ and hence the error is unidentifiable. This is reflected in identical values of LS standardised residuals for h_{13} and h_{14}, as shown in the column $\mathbf{w}_{III}$.

The vector $\mathbf{g}_{IV}$ does not belong to the space of imperceptible errors ($\mathbf{g}_{IV} \neq g \cdot \mathbf{A}_{\cdot C}$), and hence is the vector of perceptible errors. The imperceptible and perceptible components of its standardised form $\mathbf{g}_{IV*}$ for $g = 2.8$ mm, i.e.

$$\mathbf{g}_{IV*} = [0 \; \dots \; 0 \;\; \underset{(13)}{3.08} \;\; \underset{(14)}{3.84} \;\; 0 \; \dots \; 0]^{\mathrm{T}}$$

where $g_{*(13)}$ as in $\mathbf{g}_{III}$ and $g_{*(14)} = 2.8$ mm/0.73 mm $= 3.84$, are shown in the columns $\mathbf{g}_{IV*}(\mathbf{A}_*)$ and $\mathbf{g}_{IV*}(\mathbf{A}_*^{\perp})$, respectively. Here again, each of the errors $g_{*(13)}$ and $g_{*(14)}$ is split into the imperceptible part and the perceptible part. The perceptible parts of these errors are the results of the following operations:

for $g_{*(13)}$: $\{\mathbf{R}\}_{13,13} \cdot g_{*(13)} + \{\mathbf{R}\}_{14,13} \cdot g_{*(14)} = 0.47 \cdot 3.08 + 0.38 \cdot 3.84 = 2.91$
for $g_{*(14)}$: $\{\mathbf{R}\}_{13,14} \cdot g_{*(13)} + \{\mathbf{R}\}_{14,14} \cdot g_{*(14)} = 0.38 \cdot 3.08 + 0.30 \cdot 3.84 = 2.32$

One can see that the error $g_{*(13)}$ residing in the observation with greater reliability index has better perceptibility. As in the case of the vector $\mathbf{g}_{III*}$ other corresponding elements of the components $\mathbf{g}_{IV*}(\mathbf{A}_*)$ and $\mathbf{g}_{IV*}(\mathbf{A}_*^{\perp})$ have opposite signs and cancel each other.

As both the errors reside in $RUE = \{h_{13}, h_{14}\}$, they are unidentifiable (see the column $\mathbf{w}_{IV}$).

For the scheme with the added observation h_{20} (σ_h = 0.80 mm) the columns $\mathbf{R}_{\bullet 13}$, $\mathbf{R}_{\bullet 14}$ of the matrix $\mathbf{R}$ (20 × 20) are not parallel vectors, which means that the condition as in (20) is not satisfied, and therefore the $RUE = \{h_{13}, h_{14}\}$ we had in the original scheme, does not exist in the new scheme. This is confirmed by $\{\mathbf{R}\}_{13,14} \neq \sqrt{\{\mathbf{R}\}_{13,13} \cdot \{\mathbf{R}\}_{14,14}}$, where $\{\mathbf{R}\}_{13,13}=0.61$, $\{\mathbf{R}\}_{14,14}=0.43$, $\{\mathbf{R}\}_{13,14}=0.24$ (see (21)).

Also, using two options of robust estimation – the Danish method (Krarup et al., 1980) and the method of growing rigour (Kamiński and Wiśniewski, 1992) – checks were made on Proposition given in the section on the properties of the vector space of imperceptible errors. The results verified (19) and indicated also that the vectors $\Delta\hat{\mathbf{H}}_{(\circ)}$ and $\Delta\hat{\mathbf{H}}_{LS}$ are parallel.

8. Conclusions

A recommended reliability level ($\{\mathbf{R}\}_{ii} > 0.5$, $i = 1, 2, \ldots, n$; see (Prószyński, 1994)) results in limiting the vector space of imperceptible gross errors ($\dim_{\mathrm{fr}} U < 0.5$) in a model and hence, in reducing the consequences of these errors in the form of an uncontrolled parameter shift. The greater the value $\{\mathbf{R}\}_{ii}$ for a particular observation, the better the perceptibility of a single gross error residing in this observation. With $\{\mathbf{R}\}_{ii} > 0.5$, the perceptible part of a single gross error surpasses the imperceptible part. This reliability level also eliminates the existence of RUE in a network and hence, the occurrence of unidentifiable gross errors.

Especially in traditional geodetic networks of small reliability level (i.e. levelling networks), the space of imperceptible errors is a significant element of practical importance. In such networks one frequently encounters levelling lines, that form RUE. It should be noted that the unidentifiable gross errors are a smaller threat for the quality of the network than the imperceptible gross errors. The latter, in contrary to unidentifiable gross errors, do not yield any signal about their existence in the observations. Since on basis of a pre-analysis we may locate RUE in a network, we can find the actual outlier by carrying out a limited number of additional measurements. The effects of imperceptible gross errors, as the undetectable errors, will be hidden in the distorted values of estimated parameters.

Both the concept of imperceptible errors and the concept of unidentifiable errors are associated with the properties of the network's structure, and therefore they are useful in the design of geodetic networks with respect to reliability. In specific situations, for instance, we may check whether the vector of systematic errors of presumed or hypothetical pattern falls into the space of imperceptible errors and if not, evaluate its imperceptible component. This may give us some indications on how to reshape the model and possibly, modify the measurement process, in order to minimise the effects of these errors.

The analysis of the vector space of imperceptible errors can be also practically employed in investigating the properties of the outlier detection methods using numerically simulated observations, i.e. when the true errors are known. Prior to actual computation one may then easily check whether the simulated vectors of true errors are perceptible or not in a model under question. In such a pre-analysis we may also search for *RUE*, that is a potential seat for unidentifiable gross errors.

The problem of the vector space of imperceptible observation errors in linearised models was only touched in (Prószyński, 2000) and it deserves a separate and more detailed treatment.

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Przestrzeń wektorowa niedostrzegalnych zaburzeń obserwacyjnych: Uzupełnienie teorii niezawodności sieci

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Streszczenie

Pojęcie wektorowej przestrzeni niedostrzegalnych błędów obserwacyjnych w modelach liniowych Gaussa-Markowa z obserwacjami nieskorelowanymi, zaproponowane we wcześniejszej pracy autora, przedstawione jest z pewnymi udoskonaleniami i nowymi dokonaniami w tym obszarze. Błędy grube trafiające do tej przestrzeni są zupełnie niewykrywalne w jakichkolwiek możliwych testach statystycznych wykonywanych w procesie estymacji metodą najmniejszych kwadratów i niezauważenie zniekształcają wynikowe wartości jednego bądź więcej parametrów modelu. Pokazana jest zależność między pojęciem błędów grubych niedostrzegalnych a proponowanym przez innych autorów pojęciem błędów grubych wykrywalnych ale nieidentyfikowalnych na skutek specyficznych własności struktury sieci. Teoria ilustrowana jest prostym przykładem liczbowym.