

# On $k$ -cyclic $SHn$ -algebra

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**Abstract.** In this work we consider a new class of algebra called  $k$ -cyclic  $SHn$ -algebra  $(A, T)$  where  $A$  is an  $SHn$ -algebra and  $T$  is a lattice endomorphism such that  $T^k(x) = x$ , for all  $x$ ,  $k$  is a positive integer. The main goal of this paper is to show a Priestley duality theorem for  $k$ -cyclic  $SHn$ -algebra.

**Key words:** De Morgan algebra, Łukasiewicz algebra, Heyting algebra, Lattices and duality.

## 1. Introduction

As an application to the study of switching circuits, Gr.C. Moisil introduced in [1] the symmetric Boolean algebra, that is, Boolean algebras with an automorphism of period two and later, in [2], the cyclic Boolean algebras in which the automorphism of period two is replaced by an automorphism of period  $k$ . The symmetric Boolean algebras were also studied in A. Monteiro [3, 4]. M. Abad [5] has also investigated the action of a cyclic operation on Łukasiewicz algebra, and L. Iturrioz [6] gave complete description of the variety of involutive Heyting algebra. On the other hand, it has been shown that Post and Łukasiewicz algebra are both Heyting algebra with operators. In both Łukasiewicz and Post algebra a symmetry can be expressed in terms of the primitive operations. This led to the study of more general algebra, called Symmetrical Heyting Algebras of order  $n$  (or  $SHn$ -algebra) (see [7–10]).

A  $k$ -cyclic  $SHn$ -algebra is a system  $(A, T)$ , such that  $A$  is an  $SHn$ -algebra and  $T$  is an endomorphism of  $A$  such that  $T^k(x) = x$  for every  $x \in A$ . We usually use the same notation for a structure as for its universe. The class of  $k$ -cyclic  $SHn$ -algebra forms a variety. In this paper we give a Priestley-style duality for  $k$ -cyclic  $SHn$ -algebra.

## 2. Preliminaries

In this section, in order to simplify reading, we summarize the fundamental concepts we use.

Recall that V. Sofronie-Stokkermas [11] introduce the category **SHn** of  $SHn$ -algebra and  $SHn$ -homomorphisms, where an  $SHn$ -algebra is an algebra  $\langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1, S_1, \dots, S_{n-1} \rangle$  such that  $\langle A, \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$  is a symmetric Heyting algebra (see [12]) and  $S_1, \dots, S_{n-1}$  are unary operators defined on  $A$  fulfilling the following equalities:

- (S1)  $S_i(a \wedge b) = S_i a \wedge S_i b$ ,
- (S2)  $S_i(a \Rightarrow b) = (\bigwedge_{k=i}^n S_k(a) \Rightarrow S_k(b))$ ,
- (S3)  $S_i(S_j(a)) = S_j(a)$ , for every  $i, j = 1, \dots, n-1$ ,
- (S4)  $S_1(a) \vee a = a$ ,

- (S5)  $S_i(\sim a) = \sim S_{n-i}(a)$ , for  $i = 1, \dots, n-1$ ,
- (S6)  $S_1(a) \vee \neg S_1(a) = 1$ , with  $\neg a = a \Rightarrow 0$ .

In addition, this author extended Esakia duality [13, 14] to the category **SHnSp** whose objects are  $SHn$ -spaces and whose morphisms are  $SHn$ -functions. Specifically, an  $SHn$ -space is a system  $(X, \leq, \tau, s_1, \dots, s_{n-1}, g)$  such that  $(X, \leq, \tau)$  is an Esakia space and for every  $x, y \in X$  the following conditions are satisfied:

- (1) if  $x \leq y$  then  $g(y) \leq g(x)$ ,
- (2)  $g(s_i(x)) = s_{n-i}(g(x))$ ,
- (3)  $g(g(x)) = x$ ,
- (4)  $s_j(s_i(x)) = s_j(x)$ ,
- (5)  $s_1(x) \leq x$ ,
- (6)  $x \leq s_{n-1}(x)$ ,
- (7)  $s_i(x) \leq s_j(x)$  for every  $i \leq j$ ,
- (8) if  $x \leq y$  then  $s_i(x) = s_i(y)$ ,
- (9) for all  $i = 1, \dots, n-1$ ,  $x \leq s_i(x)$  or  $s_{i+1}(x) \leq x$ .

and an  $SHn$ -function from an  $SHn$ -space  $(X, \leq, \tau, s_1, \dots, s_{n-1}, g)$  into another  $(X', \leq', \tau', s'_1, \dots, s'_{n-1}, g')$  is an Esakia morphism (continuous bounded morphism)  $f : X \rightarrow X'$  such that preserve the operations  $s_1, \dots, s_{n-1}, g$ . Besides, he proved that **SHn** is dually equivalent to **SHnSp**.

## 3. Priestley duality for $k$ -cyclic $SHn$ -algebra

In this section we study  $SHn$ -algebra endowed with a lattice endomorphism, more precisely:

**Definition 1.** An  $k$ -cyclic  $SHn$ -algebra is a pair  $(A, T)$  where  $A$  is an  $SHn$ -algebra and  $T$  is a lattice endomorphism which satisfies the following properties:

- (K)  $T^k(x) = x$ , for every  $x \in A$ .

By **kSHn** we will denote the category of  $k$ -cyclic  $SHn$ -algebra and their corresponding homomorphisms.

We indicate a Priestley-style duality for  $k$ -cyclic  $SHn$ -algebra.

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**Definition 2.** A  $k$ -space is a pair  $(X, t)$  where  $X$  is an  $SHn$ -space and the following conditions are satisfied:

- (k1)  $t : X \rightarrow X$  is an homeomorphism,
- (k2)  $t : X \rightarrow X$  is an order-isomorphism,
- (k3)  $t^k(x) = x$ , for every  $x \in X$ .

Let  $\mathbf{kSp}$  be the category whose objects are  $k$ -spaces and whose morphisms are  $SHn$ -functions such that preserve the operation  $t$ .

For every  $k$ -cyclic space  $(X, t)$ , let  $E(X)$  be the set of clopen order-filters of  $X$ . On  $E(X)$  the following operations can be defined:

- (a)  $\vee : E(X) \times E(X) \rightarrow E(X)$  is defined by  $U \vee V := U \cup V$ .
- (b)  $\wedge : E(X) \times E(X) \rightarrow E(X)$  is defined by  $U \wedge V := U \cap V$ .
- (c)  $\Rightarrow : E(X) \times E(X) \rightarrow E(X)$  is defined by  $U \Rightarrow V := \{x \in X \mid x \leq y \text{ and } y \in U \text{ implies } y \in V\}$ .
- (d)  $\sim : E(X) \times E(X) \rightarrow E(X)$  is defined for every  $U \in E(X)$  by  $\sim U := X \setminus g^{-1}(U)$ .
- (e) For every  $i = 1, \dots, n-1$ ,  $S_i : E(X) \rightarrow E(X)$  is defined for every  $U \in E(X)$  by  $S_i(U) := s^{-1}(U)$ .
- (f)  $T : E(X) \rightarrow E(X)$  is defined for every  $U \in E(X)$  by  $T(U) := t^{-1}(U)$ .

For every morphism of  $k$ -cyclic space  $\varphi : X \rightarrow X'$  let  $E(\varphi) : E(X') \rightarrow E(X)$  be defined by  $E(\varphi)(U) = \varphi^{-1}(U)$  for every  $U \in E(X')$ .

**Proposition 1.** The functor  $E : \mathbf{kSp} \rightarrow \mathbf{kSHn}$  is well-defined, i.e. the following holds:

- (i) The algebra  $(E(X), T)$  is an  $k$ -cyclic  $SHn$ -algebra.
- (ii) If  $\varphi : X \rightarrow X'$  is a morphism of  $k$ -spaces, then  $E(\varphi) : E(X') \rightarrow E(X)$  is a morphism of  $k$ -cyclic  $SHn$ -algebra.

**Proof.** We shall only prove that for all  $U \in E(X)$ , (t4) is satisfied. Indeed, the following conditions are equivalents:

- (1)  $x \in T^k(U)$ ,
- (2)  $t^k(x) \in U$ ,
- (3)  $x \in U$ .

For every  $k$ -cyclic  $SHn$ -algebra  $(A, T)$  let  $D(A) = (PF(A), \leq, \tau, s_1, \dots, s_{n-1}, g, t)$  where  $(PF(A), \leq, \tau, s_1, \dots, s_{n-1}, g)$  is the Priestley space of the  $SHn$ -algebra  $A$  and  $t : D(A) \rightarrow D(A)$  is defined for every  $F \in D(A)$  by  $t(F) = T^{-1}(F)$ .

For every morphism of  $k$ -cyclic  $SHn$ -algebra  $f : A \rightarrow A'$  let  $D(f) : D(A') \rightarrow D(A)$  be defined by  $D(f)(F) = f^{-1}(F)$  for every  $F \in D(A')$ .

**Proposition 2.** The functor  $D : \mathbf{kSHn} \rightarrow \mathbf{kSp}$  is well-defined, i.e. the following holds:

- (i) For every  $k$ -cyclic  $SHn$ -algebra  $(A, T)$ ,  $(D(A), t)$  is an  $k$ -space.

- (ii) If  $f : A \rightarrow A'$  is a morphism of  $SHn$ -algebra, then  $D(f) : D(A') \rightarrow D(A)$  is a morphism of  $k$ -spaces.

**Proof.** We can only prove that for all  $F \in D(A)$ , (K) is satisfied. Indeed, the following conditions are equivalents:

- (1)  $x \in t^k(F)$ ,
- (2)  $T^k(x) \in F$ ,
- (3)  $x \in F$ .

From Proposition 1 and 2 and taking into account the results indicad in [11] we have.

**Theorem 1.** The categories  $\mathbf{kSHn}$  and  $\mathbf{kSp}$  are dually equivalent.

## REFERENCES

- [1] Gr. C. Moisil, "Algebra schemelor cu elemente ventil", *Revista Universitatii C. I. Parhon si a Politehnicii Bucuresti Seria St. Nat.* 4–5, 9–42 (1954).
- [2] Gr. C. Moisil, *Essais sur les Logiques Non Chrysippiennes*, Academiei Bucurest, Bucurest, 1972.
- [3] A. Monteiro, *Algebras de Boole Involutivas*, Universidad Nacional del Sur, Bahía Blanca, 1969.
- [4] A. Monteiro, *Algèbres de Boole Cycliques*, *Revue Roumaine de Math. Pures et Appliquées* 23, 71–76 (1978).
- [5] M. Abad, "Estructuras cíclica y monádica de un álgebra de Łukasiewicz  $n$ -valente", *Notas de Lógica Matemática* 36, CD-ROM (1988).
- [6] L. Iturrioz, "Algèbres de Heyting trivalentes involutives", *Notas de Lógica Matemática* 31, CD-ROM (1974).
- [7] L. Iturrioz, "Łukasiewicz and symmetrical Heyting algebra", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 23, 131–136 (1977).
- [8] L. Iturrioz, *Modal Operators on Symmetrical Heyting Algebra*, Universal Algebra and Applications, Banach Center Publications 9, ed. T. Traczyk, pp. 289–303, PWN-Polish Scientific Publishers, Warsaw, 1982.
- [9] L. Iturrioz, "Symmetrical Heyting algebra with operators", *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 29, 33–70 (1983).
- [10] L. Iturrioz and V. Sofronie-Stokkermans, " $SHn$ -algebra (abbreviation of symmetrical Heyting algebra of order  $n$ )", in: *Atlas of Many-Valued Structures, COST Action 15*, eds. L. Iturrioz, E. Orłowska, E. Turunen, pp. 1235–9599, Tampere Univ. of Technology, Tampere, 2000.
- [11] V. Sofronie-Stokkermans, "Priestley duality for  $SHn$ -algebras and applications to the study of Kripke-style models for  $SHn$ -logics", *Multiple Valued Logic, Int. J.* 5 (4), 281–305 (1999).
- [12] A. Monteiro, "Sur les algèbres de Heyting symétriques", *Portugaliae Mathematica* 39, 1–237 (1980).
- [13] L.L. Esakia, "Topological Kripke models", *Soviet Math. Dokl.* 15, 147–151 (1974).
- [14] L.L. Esakia, "The problem of dualism in the intuitionistic logic and Brouwerian lattices", *V Int. Congress Logic, Methodology and Philosophy of Science* 1, 7–8 (1975).