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Address of the Editor: Institute of Automatic Control, Silesian University of Technology
Akademicka 16, 44–100 Gliwice, Poland. Tel: +(48–32) 237–23–98, fax +(48–32) 237–21–27

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Dual hesitant pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making

GUIWU WEI and MAO LU

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the Hamacher aggregation operators with dual Pythagorean hesitant fuzzy information. Then, motivated by the ideal of Hamacher operation, we have developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Key words: multiple attribute decision making (MADM); dual Pythagorean hesitant fuzzy values; dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator; dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHHG) operator; power aggregation operators.

1. Introduction

Atanassov [1,2] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set [3] whose basic component is only a membership function. Xu [4] developed the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu [5] developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator and gave an application of the IFHG

The authors are with School of Business, Sichuan Normal University, Chengdu, 610101, P.R. China. The corresponding author is Guiwu Wei, e-mail: weiguiwu@163.com

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operator to multiple attribute group decision making with intuitionistic fuzzy information. Xu and Yager [6] investigated the dynamic intuitionistic fuzzy multiple attribute decision making problems and developed some aggregation operators such as the dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. Wei [7] proposed some dynamic geometric aggregation operators such as the dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and uncertain dynamic intuitionistic fuzzy weighted geometric (UDIFWG) operator to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. Wei [8] proposed two new aggregation operators: induced intuitionistic fuzzy ordered weighted geometric (I-IFOWG) operator and induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IIFOWG) operator. Wei and Zhao [9] developed two new aggregation operators: induced intuitionistic fuzzy correlated averaging (I-IFCA) operator and induced intuitionistic fuzzy correlated geometric (I-IFCG) operator. Yu et al. [10] proposed some intuitionistic fuzzy aggregation operators such as the intuitionistic fuzzy prioritized weighted average (IFPWA) operator, the intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator. Xu [11] developed a series of operators for aggregating intuitionistic fuzzy numbers and established various properties of these power aggregation operators. Xu and Chen [12] proposed an aggregation technique called the intuitionistic fuzzy Bonferroni mean for aggregating intuitionistic fuzzy information. Xu and Xia [13] studied the induced generalized aggregation operators under intuitionistic fuzzy environments. The intuitionistic fuzzy set has received more and more attention since its appearance[14-28].

More recently, Pythagorean fuzzy set (PFS) [29-30] has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu[31]provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Peng and Yang [32] proposed the division and subtraction operations for PFNs, and also developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multicriteria group decision making problem with PFNs. Afterwards, Beliakov and James [33] focused on how the notion of "averaging" should be treated in the case of PFNs and how to ensure that the averaging aggregation functions produce outputs consistent with the case of ordinary fuzzy numbers. Reformat and Yager [34] applied the PFNs in handling the collaborative-based recommender system.

In this paper, we introduce dual hesitant Pythagorean fuzzy set (DHPFS), which is a new extension of PFS and dual hesitant fuzzy set(DHFs) [35]. It's clear that the DHPFSs consist of two parts, that is, the membership degrees function and the non-membership degrees function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of degrees in this situation. For example, in a multiple attribute decision-making problem, some decision makers consider as possible values for the membership degree of into the set a few different values 0.4, 0.5, and 0.6, and for the non-membership degrees 0.1, 0.2 and 0.3 replacing just one number or a tuple. So, the certainty and uncertainty on the possible values are somehow limited, respectively, which can reflect the original information given by the decision makers as much as possible. Utilizing DHPFSs can take much more information into account, the more values we obtain from the decision makers, the greater epistemic certainty we have, and thus, compared to the existing sets, DHPFS can be regarded as a more comprehensive set, which supports a more flexible approach when the decision makers provide their judgments.

Hamacher t-conorm and t-norm, which are the generalization of algebraic and Einstein t-conorm and t-norm [36], are more general and more flexible. There is important significance to research aggregation operators based on Hamacher operations and their application to MADM problems. However, all the above approaches are unsuitable to aggregate these dual hesitant Pythagorean fuzzy numbers on the basis of the Hamacher operations [37]. Thus, based on the Hamacher operations, how to aggregate these dual hesitant Pythagorean fuzzy numbers is an interesting topic. To solve this issue, in this paper, we shall develop some dual hesitant Pythagorean fuzzy Hamacher aggregation operators on the basis of the traditional Hamacher operations [37-42]. In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to Pythagorean fuzzy set, dual hesitant Pythagorean fuzzy set and their operational laws. In Section 3, we shall propose some dual hesitant Pythagorean fuzzy Hamacher aggregation operators. In Section 4, we shall propose some dual hesitant Pythagorean fuzzy Hamacher power aggregation operators. In Section 5, based on these operators, we shall propose some models for multiple attribute decision making problems with dual hesitant Pythagorean fuzzy information. In Section 6, we present a numerical example for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information in order to illustrate the method proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

2.1. Pythagorean fuzzy set

The basic concepts of PFSs [29-30] are briefly reviewed in this section. Afterwards, novel score and accuracy functions for PFNs are proposed. Furthermore, a new comparison method for PFNs is developed.

Definition 1 [29-30] *Let X be a fix set. A PFS is an object having the form*

$$P=\{\langle x,(\mu_P(x),\nu_P(x))\rangle\,|x\in X\} \tag{1}$$

where $\mu_P: X\to[0,1]$ the function defines the degree of membership and the function $\nu_P: X\to[0,1]$ defines the degree of non-membership of the element $x\in X$ to P, respectively, and, for every $x\in X$, it holds that

$$(\mu_p(x))^2+(\nu_p(x))^2\leqslant 1. \tag{2}$$

Definition 2 [31] *Let $\widetilde{a}_1=(\mu_1,\nu_1)$, $\widetilde{a}_2=(\mu_2,\nu_2)$, and $\widetilde{a}=(\mu,\nu)$ be three Pythagorean fuzzy numbers, and some basic operations on them are defined as follows:*

$$(1)\ \widetilde{a}_1\oplus\widetilde{a}_2=\left(\sqrt{(\mu_1)^2+(\mu_2)^2-(\mu_1)^2(\mu_2)^2},\nu_1\nu_2\right);$$

$$(2)\ \widetilde{a}_1\otimes\widetilde{a}_2=\left(\mu_1\mu_2,\sqrt{(\nu_1)^2+(\nu_2)^2-(\nu_1)^2(\nu_2)^2}\right);$$

$$(3)\ \lambda\widetilde{a}=\left(\sqrt{1-(1-\mu^2)^\lambda},\nu^\lambda\right),\lambda>0;$$

$$(4)\ (\widetilde{a})^\lambda=\left(\mu^\lambda\sqrt{1-(1-\nu^2)^\lambda}\right),\lambda>0;$$

$$(5)\ \widetilde{a}^c=(\nu,\mu).$$

2.2. Dual hesitant Pythagorean fuzzy set

In this section, we introduce dual hesitant Pythagorean fuzzy set (DHPFS), which is a new extension of PFS and dual hesitant fuzzy set [35]. It is clear that the DHPFSs consist of two parts, that is, the membership hesitancy function and the non-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and we have to handle two kinds of hesitancy in this situation.

Definition 3 *Let X be a fixed set, then a dual hesitant Pythagorean fuzzy set (DHPFS) on X is described as:*

$$D=(\langle x,h_P(x),g_P(x)\rangle\,|x\in X) \tag{3}$$

in which $h_p(x)$ and $g_p(x)$ are two sets of some values in $[0,1]$, denoting the possible membership degrees and non-membership degrees of the element $x\in X$ to the set D respectively, with the conditions:

$$\gamma^2+\eta^2\leqslant 1$$

where $\gamma\in h_P(x)$, $\eta\in g_P(x)$, for all $x\in X$. For convenience, the pair $d(x)=(h_P(x),g_P(x))$ is called a dual hesitant Pythagorean fuzzy number (DHPFN) denoted by $d=(h,g)$, with the conditions: $\gamma\in h$, $\eta\in g$, $0\leqslant\gamma$, $\eta\leqslant 1$, $0\leqslant\gamma^2+\eta^2\leqslant 1$.

To compare the DHPFNs, in the following, we shall give the following comparison laws:

Definition 4 *Let $d = (h, g)$ be a DHPFNs, $s(d) = \frac{1}{2}\left(1 + \frac{1}{\#h}\sum_{\gamma\in h}\gamma^2 - \frac{1}{\#g}\sum_{\eta\in g}\eta^2\right)$ the score function of d, and $p(d) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma^2 + \frac{1}{\#g}\sum_{\eta\in g}\eta^2$ the accuracy function of d, where $\#h$ and $\#g$ are the numbers of the elements in h and g respectively, then, let $d_i = (h_i, g_i)$ $(i = 1, 2)$ be any two DHPFNs, we have the following comparison laws:*

- *If $s(d_1) > s(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$;*
- *If $s(d_1) = s(d_2)$, then*

  (1) If $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$;

  (2) If $p(d_1) > p(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$.

Example 1 Let $d_1 = \{\{0.3, 0.4\}, \{0.6\}\}$, $d_2 = \{\{0.4, 0.5\}, \{0.3, 0.4)\}$ by Definition 4, we can get:

$$s(d_1) = \frac{1}{2}\left(1 + \frac{1}{2}(0.3^2 + 0.4^2) - 0.6^2\right) = 0.3825$$

$$s(d_2) = \frac{1}{2}\left(1 + \frac{1}{2}(0.4^2 + 0.5^2) - \frac{1}{2}(0.3^2 + 0.4^2)\right) = 0.5400$$

Thus, $s(d_2) > s(d_1)$, so $d_2 \succ d_1$. Then, we define some **new** operations on the DHPFNs d, d_1 and d_2:

(1) $d^\lambda = \cup_{\gamma\in h,\eta\in g}\left\{\{\gamma^\lambda\}, \left\{\sqrt{1-(1-\eta^2)^\lambda}\right\}\right\}, \lambda > 0;$

(2) $\lambda d = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\sqrt{1-(1-\gamma^2)^\lambda}\right\}, \{\eta^\lambda\}\right\}, \lambda > 0;$

(3) $d_1 \oplus d_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\left\{\left\{\sqrt{(\gamma_1)^2 + (\gamma_2)^2 - (\gamma_1)^2(\gamma_2)^2}\right\}, \{\eta_1\eta_2\}\right\};$

(4) $d_1 \otimes d_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\left\{\{\gamma_1\gamma_2\}, \left\{\sqrt{(\eta_1)^2 + (\eta_2)^2 - (\eta_1)^2(\eta_2)^2}\right\}\right\}.$

2.3. Hamacher operations of dual hesitant Pythagorean fuzzy set

Based on the traditional Hamacher operations [36], in the following, we shall define the Hamacher operations on the DHPFNs d, d_1 and d_2.

(1) $d^\lambda = \cup_{\gamma\in h,\eta\in g}\left\{\left\{\frac{\sqrt{\gamma}(\gamma_1)^\lambda}{\sqrt{\left(1+(\gamma-1)\left(1-(\gamma_1)^2\right)\right)^\lambda + (\gamma-1)(\gamma_1)^{2\lambda}}}\right\},\right.$

$$\left\{\sqrt{\frac{\left(1+(\gamma-1)(\eta_1)^2\right)^\lambda-\left(1-(\eta_1)^2\right)^\lambda}{\left(1+(\gamma-1)(\eta_1)^2\right)^\lambda+(\gamma-1)\left(1-(\eta_1)^2\right)^\lambda}}\right\}\Bigg\}, \ \lambda>0;$$

$$(2)\quad \lambda d=\cup_{\gamma\in h,\eta\in g}\left\{\left\{\sqrt{\frac{\left(1+(\gamma-1)(\gamma_1)^2\right)^\lambda-\left(1-(\gamma_1)^2\right)^\lambda}{\left(1+(\gamma-1)(\gamma_1)^2\right)^\lambda+(\gamma-1)\left(1-(\gamma_1)^2\right)^\lambda}}\right\},\right.$$

$$\left.\left\{\frac{\sqrt{\gamma}(\eta_1)^\lambda}{\sqrt{\left(1+(\gamma-1)\left(1-(\eta_1)^2\right)\right)^\lambda+(\gamma-1)(\eta_1)^{2\lambda}}}\right\}\right\}, \lambda>0$$

$$(3)\ d_1\oplus d_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\left\{\left\{\sqrt{\frac{(\gamma_1)^2+(\gamma_2)^2-(\gamma_1)^2(\gamma_2)^2-(1-\gamma)(\gamma_1)^2(\gamma_2)^2}{1-(1-\gamma)(\gamma_1)^2(\gamma_2)^2}}\right\},\right.$$

$$\left.\left\{\frac{\eta_1\eta_2}{\sqrt{\gamma+(1-\gamma)\left((\eta_1)^2+(\eta_2)^2-(\eta_1)^2(\eta_2)^2\right)}}\right\}\right\}, \lambda>0$$

$$(4)\ d_1\otimes d_2=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2}\left\{\left\{\frac{\gamma_1\gamma_2}{\sqrt{\gamma+(1-\gamma)\left((\gamma_1)^2+(\gamma_2)^2-(\gamma_1)^2(\gamma_2)^2\right)}}\right\},\right.$$

$$\left.\left\{\sqrt{\frac{(\eta_1)^2+(\eta_2)^2-(\eta_1)^2(\eta_2)^2-(1-\gamma)(\eta_1)^2(\eta_2)^2}{1-(1-\gamma)(\eta_1)^2(\eta_2)^2}}\right\}\right\}, \lambda>0$$

3. Dual hesitant Pythagorean fuzzy Hamacher aggregation operators

3.1. Dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operators

In the following, we shall propose some dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operator based on the Hamacher operations of DHPFNs.

Definition 5 *Let $\widetilde{d}_j\,(j=1,2,\cdots,n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher weighted average (DHPFHWA) operator as follows:*

$$DHPFHWA_\omega\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\overset{n}{\underset{j=1}{\oplus}}\left(\omega_j\widetilde{d}_j\right) \tag{4}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\widetilde{d}_j\,(j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$.

Based on the operations of the dual hesitant Pythagorean fuzzy values described and mathematical induction method, we can drive the Theorem 1.

Theorem 1 *Let $\widetilde{d}_j\,(j=1,2,\cdots,n)$ be a collection of DHPFNs, then their aggregated value by using the DHPFHWA operator is also a DHPFN, and*

$$\begin{aligned}
&DHPFHWA_\omega\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)\\
&=\overset{n}{\underset{j=1}{\oplus}}\left(\omega_j\widetilde{d}_j\right)\\
&=\cup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)(\gamma_j)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(\gamma_j)^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j}}}\right\},\right.\\
&\qquad\left.\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{n}(\eta_j)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-(\eta_j)^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}(\eta_j)^{2\omega_j}}}\right\}\right\}
\end{aligned}\tag{5}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ *be the weight vector of* $\widetilde{d}_j\,(j=1,2,\cdots,n)$*, and* $\omega_j>0$, $\sum_{j=1}^{n}\omega_j=1$, $\gamma>0$.

Now, we can discuss some special cases of the DHPFHWA operator with respect to the parameter γ.

- When $\gamma=1$, DHPFHWA operator reduces to the dual hesitant Pythagorean fuzzy weighted average (DHPFWA) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFWA}_\omega\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)\\
&=\overset{n}{\underset{j=1}{\oplus}}\left(\omega_j\widetilde{d}_j\right)\\
&=\cup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j}}\right\},\left\{\prod_{j=1}^{n}(\eta_j)^{\omega_j}\right\}\right\}
\end{aligned}\tag{6}$$

- When $\gamma = 2$, DHPFHWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein weighted average (DHPFEWA) operator as follows:

$$
\begin{aligned}
&\mathrm{DHPFEWA}_{\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) \\
&= \overset{n}{\underset{j=1}{\oplus}} \left(\omega_j \widetilde{d}_j\right) \\
&= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma_j)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma_j)^2\right)^{\omega_j} + \prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j}}} \right\}, \right. \\
&\qquad \left. \left\{ \frac{\sqrt{2}\prod_{j=1}^{n}(\eta_j)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(2-(\eta_j)^2\right)^{\omega_j} + \prod_{j=1}^{n}(\eta_j)^{2\omega_j}}} \right\} \right\}
\end{aligned} \tag{7}
$$

Definition 6 *Let $\widetilde{d}_j \, (j = 1, 2, \cdots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher ordered weighted average (DHPFHOWA) operator as follows:*

$$
\begin{aligned}
&DHPFHOWA_{w}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\oplus}} \left(w_j \widetilde{d}_{\sigma(j)}\right) \\
&= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j} - \prod_{j=1}^{n}\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n}\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}}} \right\}, \right. \\
&\qquad \left. \left\{ \frac{\sqrt{\gamma}\prod_{j=1}^{n}\left(\gamma_{\sigma(j)}\right)^{w_j}}{\sqrt{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n}\left(\gamma_{\sigma(j)}\right)^{2w_j}}} \right\} \right\}
\end{aligned} \tag{8}
$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $1, 2, \cdots, n$, such that $\widetilde{d}_{\sigma(j-1)} \geqslant \widetilde{d}_{\sigma(j)}$ for all $j = 2, \cdots, n$, and $w = (w_1, w_2, \cdots, w_n)^T$ is the aggregation-associated weight vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, $\gamma > 0$.

Now, we can discuss some special cases of the DHPFHOWA operator with respect to the parameter γ.

- When $\gamma = 1$, DHPFHOWA operator reduces to the dual hesitant Pythagorean fuzzy ordered weighted average (DHFOWA) operator as follows:

$$\begin{aligned}\mathrm{DHPFOWA}_{\mathrm{w}}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) &= \overset{n}{\underset{j=1}{\oplus}}\left(w_j\widetilde{d}_{\sigma(j)}\right)\\ &= \cup_{\gamma_{\sigma(j)}\in h_{\sigma(j)},\eta_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}}\right\},\right.\\ &\left.\left\{\frac{\sqrt{2}\prod_{j=1}^{n}\gamma_j{}^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(2-(\gamma_j)^2\right)^{\omega_j}+\prod_{j=1}^{n}(\gamma_j)^{2\omega_j}}}\right\}\right\}\end{aligned} \tag{9}$$

- When $\gamma = 2$, DHPFHOWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein ordered weighted average (DHPFEOWA) operator as follows:

$$\begin{aligned}\mathrm{DHPFEOWA}_{\mathrm{w}}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) &= \overset{n}{\underset{j=1}{\oplus}}\left(w_j\widetilde{d}_{\sigma(j)}\right)\\ &= \cup_{\gamma_{\sigma(j)}\in h_{\sigma(j)},\eta_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}-\prod_{j=1}^{n}\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}+\prod_{j=1}^{n}\left(1-\left(\gamma_{\sigma(j)}\right)^2\right)^{w_j}}}\right\},\right.\\ &\left.\left\{\frac{\sqrt{2}\prod_{j=1}^{n}\left(\eta_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(2-\left(\eta_{\sigma(j)}\right)^2\right)^{\omega_j}+\prod_{j=1}^{n}\left(\eta_{\sigma(j)}\right)^{2\omega_j}}}\right\}\right\}\end{aligned} \tag{10}$$

From Definitions 5 and 6, we know that the DHPFHWA operator weights the dual hesitant Pythagorean fuzzy argument itself, while the DHPFHOWA operator weights the ordered positions of the dual hesitant Pythagorean fuzzy arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the DHPFHWA and DHPFHOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator.

Definition 7 *A dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator is defined as follows:*

$$DHPFHHA_{w,\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\overset{n}{\underset{j=1}{\oplus}}\left(w_j\dot{\widetilde{d}}_{\sigma(j)}\right)$$

$$=\cup_{\dot{\gamma}_{\sigma(j)}\in h_{\sigma(j)},\dot{\eta}_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{\omega_j}}}\right\},\right.$$

$$\left.\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{n}\left(\dot{\eta}_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\left(\dot{\eta}_{\sigma(j)}\right)^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(\dot{\eta}_{\sigma(j)}\right)^{2\omega_j}}}\right\}\right\} \tag{11}$$

where $w=(w_1,w_2,\cdots,w_n)$ is the associated weighting vector, with $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$, and $\dot{h}_{\sigma(j)}$ is the j-th largest element of the dual hesitant Pythagorean fuzzy arguments $\dot{\widetilde{d}}\left(\dot{\widetilde{d}}=n\omega_j\widetilde{d}_j,\ j=1,2,\cdots,n\right)$, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ is the weighting vector of dual hesitant Pythagorean fuzzy arguments $\widetilde{d}_j\,(j=1,2,\cdots,n)$, with $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$, and n is the balancing coefficient, $\gamma>0$. Especially, if $w=\left(1/n,1/n,\cdots,1/n\right)^T$, then DHPFHA is reduced to the dual hesitant Pythagorean fuzzy weighted average (DHPFWA) operator; if , then DHPFHA is reduced to the dual hesitant Pythagorean fuzzy ordered weighted average (DHPFOWA) operator.

From Definition 7, we know that:

(1) The DHPFHHA operator first weights the given arguments, and then reorders the weighted arguments in descending order and weights these ordered arguments by the DHPFHHA weights, and finally aggregates all the weighted arguments into a collective one.

(2) The DHPFHHA operator generalizes both the DHPFHWA and DHPFHOWA operators, and reflects the importance degrees of both the given arguments and their ordered positions.

Now, we can discuss some special cases of the DHPFHHA operator with respect to the parameter γ.

- When $\gamma = 1$, DHPFHHA operator reduces to the hesitant Pythagorean fuzzy hybrid average (DHPFHA)operator as follows:

$$\begin{aligned} &\mathrm{DHPFHA}_{w,\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\oplus}}\left(w_j\dot{\widetilde{d}}_{\sigma(j)}\right) \\ &= \cup_{\dot{\gamma}_{\sigma(j)}\in h_{\sigma(j)},\dot{\eta}_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{w_j}}\right\},\left\{\prod_{j=1}^{n}\left(\dot{\eta}_{\sigma(j)}\right)^{w_j}\right\}\right\} \end{aligned} \tag{12}$$

- When $\gamma = 2$, DHPFHHA operator reduces to the dual hesitant Pythagorean fuzzy Einstein hybrid average (HPFEHA) operator as follows:

$$\begin{aligned} &\mathrm{DHPFEHA}_{w,\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\oplus}}\left(w_j\dot{\widetilde{d}}_{\sigma(j)}\right) \\ &= \cup_{\dot{\gamma}_{\sigma(j)}\in h_{\sigma(j)},\dot{\eta}_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{w_j}-\prod_{j=1}^{n}\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{w_j}+\prod_{j=1}^{n}\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{w_j}}}\right\},\right. \\ &\left.\left\{\frac{\sqrt{2}\prod_{j=1}^{n}\left(\dot{\eta}_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(2-\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{\omega_j}+\prod_{j=1}^{n}\left(\dot{\eta}_{\sigma(j)}\right)^{2\omega_j}}}\right\}\right\} \end{aligned} \tag{13}$$

3.2. Dual hesitant Pythagorean fuzzy Hamacher Geometric Aggregation Operators

Based on the dual hesitant Pythagorean fuzzy Hamacher arithmetic aggregation operators and the geometric mean, here we define some dual hesitant Pythagorean fuzzy Hamacher geometric aggregation operators:

Definition 8 *Let $\widetilde{d}_j\,(j = 1,2,\cdots,n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher weighted geometric (DHPFHWG) operator as follows:*

$$DHPFHWG_{\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}}\left(\widetilde{d}_j\right)^{\omega_j} \tag{14}$$

where $\omega = (\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\widetilde{d}_j\,(j = 1,2,\cdots,n)$, and $\omega_j > 0$, $\sum_{j=1}^{n}\omega_j = 1$, $\gamma > 0$.

Based on the operations of the dual hesitant Pythagorean fuzzy values described and mathematical induction methods, we can drive the Theorem 2.

Theorem 2 *Let $\widetilde{d}_j\,(j=1,2,\cdots,n)$ be a collection of DHPFNs, then their aggregated value by using the DHPFHWG operator is also a DHPFN, and*

$$\begin{aligned}
&DHPFHWG_{\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)\\
&=\mathop{\otimes}_{j=1}^{n}\left(\widetilde{d}_j\right)^{\omega_j}\\
&=\cup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left\{\frac{\sqrt{\gamma}\prod\limits_{j=1}^{n}\gamma_j{}^{\omega_j}}{\sqrt{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)\left(1-(\gamma_j)^2\right)\right)^{\omega_j}+(\gamma-1)\prod\limits_{j=1}^{n}(\gamma_j)^{2\omega_j}}}\right\},\right.\\
&\qquad\left.\left\{\sqrt{\frac{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\eta_j)^2\right)^{\omega_j}-\prod\limits_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\omega_j}}{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\eta_j)^2\right)^{\omega_j}+(\gamma-1)\prod\limits_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\omega_j}}}\right\}\right\}
\end{aligned}\tag{15}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\widetilde{d}_j\,(j=1,2,\cdots,n)$, and $\omega_j>0$, $\sum\limits_{j=1}^{n}\omega_j=1$, $\gamma>0$.

Now, we can discuss some special cases of the DHPFHWG operator with respect to the parameter γ.

- When $\gamma=1$, DHPFHWG operator reduces to the dual hesitant Pythagorean fuzzy weighted geometric (DHPFWG) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFWG}_{\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\mathop{\otimes}_{j=1}^{n}\left(\widetilde{d}_j\right)^{\omega_j}\\
&=\cup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left\{\prod_{j=1}^{n}(\gamma_j)^{\omega_j}\right\},\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\omega_j}}\right\}\right\}
\end{aligned}\tag{16}$$

- When $\gamma = 2$, DHPFHWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein weighted geometric (DHPFEWG) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFEWG}_\omega\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\widetilde{d}_j\right)^{\omega_j} \\
&= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^{n} {\gamma_j}^{\omega_j}}{\sqrt{\prod_{j=1}^{n} \left(2 - (\gamma_j)^2\right)^{\omega_j} + \prod_{j=1}^{n} (\gamma_j)^{2\omega_j}}} \right\}, \right. \\
&\left. \left\{ \sqrt{\frac{\prod_{j=1}^{n} \left(1 + (\eta_j)^2\right)^{\omega_j} - \prod_{j=1}^{n} \left(1 - (\eta_j)^2\right)^{\omega_j}}{\prod_{j=1}^{n} \left(1 + (\eta_j)^2\right)^{\omega_j} + \prod_{j=1}^{n} \left(1 - (\eta_j)^2\right)^{\omega_j}}} \right\} \right\}
\end{aligned} \tag{17}$$

Definition 9 *Let $\widetilde{d}_j \,(j = 1, 2, \cdots, n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher ordered weighted geometric (DH-PFHOWG?operator as follows:*

$$\begin{aligned}
&DHPFHOWG_w\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\widetilde{d}_{\sigma(j)}\right)^{w_j} \\
&= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n} \left(1 + (\gamma - 1)\left(1 - \left(\gamma_{\sigma(j)}\right)^2\right)\right)^{\omega_j} + (\gamma - 1) \prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{2\omega_j}}} \right\}, \right. \\
&\left. \left\{ \sqrt{\frac{\prod_{j=1}^{n} \left(1 + (\gamma - 1)\left(\eta_{\sigma(j)}\right)^2\right)^{w_j} - \prod_{j=1}^{n} \left(1 - \left(\eta_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n} \left(1 + (\gamma - 1)\left(\eta_{\sigma(j)}\right)^2\right)^{w_j} + (\gamma - 1) \prod_{j=1}^{n} \left(1 - \left(\eta_{\sigma(j)}\right)^2\right)^{w_j}}} \right\} \right\}
\end{aligned} \tag{18}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\widetilde{d}_{\sigma(j-1)} \geqslant \widetilde{d}_{\sigma(j)}$ for all $j = 2, \cdots, n$, and $w = (w_1, w_2, \cdots, w_n)^T$ is the aggregation-associated weight vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, $\gamma > 0$.

Now, we can discuss some special cases of the DHPFHOWG operator with respect to the parameter γ.

- When $\gamma = 1$, DHPFHOWG operator reduces to the dual hesitant Pythagorean fuzzy ordered weighted geometric (DHPFOWG) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFOWG}_w\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\widetilde{d}_{\sigma(j)}\right)^{w_j} \\
&= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{\left\{\prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{w_j}\right\}, \left\{\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\eta_{\sigma(j)}\right)^2\right)^{w_j}}\right\}\right\}
\end{aligned} \tag{19}$$

- When $\gamma = 2$, DHPFHOWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein ordered weighted geometric (DHPFEOWG) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFEOWG}_w\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\widetilde{d}_{\sigma(j)}\right)^{w_j} \\
&= \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} \left\{\left\{\frac{\sqrt{2} \prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n} \left(2 - \left(\gamma_{\sigma(j)}\right)^2\right)^{\omega_j} + \prod_{j=1}^{n} \left(\gamma_{\sigma(j)}\right)^{2\omega_j}}}\right\},\right. \\
&\qquad \left.\left\{\sqrt{\frac{\prod_{j=1}^{n} \left(1 + \left(\eta_{\sigma(j)}\right)^2\right)^{w_j} - \prod_{j=1}^{n} \left(1 - \left(\eta_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n} \left(1 + \left(\eta_{\sigma(j)}\right)^2\right)^{w_j} + \prod_{j=1}^{n} \left(1 - \left(\eta_{\sigma(j)}\right)^2\right)^{w_j}}}\right\}\right\}
\end{aligned} \tag{20}$$

From Definitions 8 and 9, we know that the DHPFHWG operator weights the dual hesitant Pythagorean fuzzy argument itself, while the DHPFHOWG operator weights the ordered positions of the dual hesitant Pythagorean fuzzy arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the DHPFHWG and DHPFHOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHHG) operator.

Definition 10 *A dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHHG) operator is defined as follows:*

$$
\begin{aligned}
&DHPFHHG_{w,\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\overset{n}{\underset{j=1}{\otimes}}\left(\dot{\widetilde{d}}_{\sigma(j)}\right)^{w_j}\\
&=\cup_{\dot{\gamma}_{\sigma(j)}\in h_{\sigma(j)},\dot{\eta}_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{n}\left(\dot{\gamma}_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-\left(\dot{\gamma}_{\sigma(j)}\right)^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(\dot{\gamma}_{\sigma(j)}\right)^{2\omega_j}}}\right\},\right.\\
&\left.\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(1-\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{\omega_j}}}\right\}\right\}
\end{aligned}
\tag{21}
$$

where $w=(w_1,w_2,\cdots,w_n)$ *is the associated weighting vector, with* $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$, *and* $\dot{h}_{\sigma(j)}$ *is the j-th largest element of the dual hesitant Pythagorean fuzzy arguments* $\dot{\widetilde{d}}\left(\dot{\widetilde{d}}=\left(\widetilde{d}_j\right)^{n\omega_j},\ j=1,2,\cdots,n\right)$, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ *is the weighting vector of dual hesitant Pythagorean fuzzy arguments* $\widetilde{d}_j\,(j=1,2,\cdots,n)$, *with* $\omega_j\in[0,1]$, $\sum_{j=1}^{n}\omega_j=1$, *and n is the balancing coefficient,* $\gamma>0$. *Especially, if* $w=\left(1/n,1/n,\cdots,1/n\right)^T$, *then DHPFHHG is reduced to the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator; if* $\omega=\left(1/n,1/n,\cdots,1/n\right)$, *then DHPFHHG is reduced to the dual hesitant Pythagorean fuzzy ordered weighted geometric (DHPFHOWG) operator.*

From Definition 10, we know that:

- When $\gamma=1$, DHPFHHG operator reduces to the dual hesitant Pythagorean fuzzy hybrid geometric (DHPFHG) operator as follows:

$$
\begin{aligned}
&\mathrm{DHPFHG}_{w,\omega}\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\overset{n}{\underset{j=1}{\otimes}}\left(\dot{\widetilde{d}}_{\sigma(j)}\right)^{w_j}\\
&=\cup_{\dot{\gamma}_{\sigma(j)}\in h_{\sigma(j)},\dot{\eta}_{\sigma(j)}\in g_{\sigma(j)}}\left\{\left\{\prod_{j=1}^{n}\left(\dot{\gamma}_{\sigma(j)}\right)^{w_j}\right\},\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{w_j}}\right\}\right\}
\end{aligned}
\tag{22}
$$

- When $\gamma = 2$, DHPFHHG operator reduces to the dual hesitant Pythagorean fuzzy Einstein hybrid geometric (DHPFEHG) operator as follows:

$$\begin{aligned} &\mathrm{DHPFEHG}_{w,\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \overset{n}{\underset{j=1}{\otimes}} \left(\dot{\widetilde{d}}_{\sigma(j)}\right)^{w_j} \\ &= \cup_{\dot{\gamma}_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in g_{\sigma(j)}} \left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^{n} \left(\dot{\gamma}_{\sigma(j)}\right)^{\omega_j}}{\sqrt{\prod_{j=1}^{n} \left(2 - \left(\dot{\gamma}_{\sigma(j)}\right)^2\right)^{\omega_j} + \prod_{j=1}^{n} \left(\dot{\gamma}_{\sigma(j)}\right)^{2\omega_j}}} \right\}, \right. \\ &\left. \left\{ \sqrt{\frac{\prod_{j=1}^{n} \left(1 + \left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{w_j} - \prod_{j=1}^{n} \left(1 - \left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{w_j}}{\prod_{j=1}^{n} \left(1 + \left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{w_j} + \prod_{j=1}^{n} \left(1 - \left(\dot{\eta}_{\sigma(j)}\right)^2\right)^{w_j}}} \right\} \right\} \end{aligned} \tag{23}$$

4. Dual hesitant Pythagorean fuzzy Hamacher power operators

4.1. Dual hesitant Pythagorean fuzzy Hamacher power Hamacher power weighted average (DHPFHPWA) operator

Yager [43] developed a nonlinear weighted average aggregation operator called power average (PA) operator, which can be defined as follows:

$$PA\left(a_1, a_2, \cdots, a_n\right) = \frac{\sum_{i=1}^{n} \left(1 + T\left(a_i\right)\right) a_i}{\sum_{i=1}^{n} \left(1 + T\left(a_i\right)\right)} \tag{24}$$

where $T\left(a_i\right) = \sum_{\substack{j=1 \\ j \neq i}}^{n} Sup\left(a_i, a_j\right)$, and $Sup\left(a, b\right)$ is the support for a from b, which satisfies the following three properties:

(1) $Sup\left(a, b\right) \in [0, 1]$;

(2) $Sup\left(a, b\right) = Sup\left(b, a\right)$;

(2) $Sup\left(a, b\right) \geqslant Sup\left(x, y\right)$.

In this section, we shall propose the dual hesitant Pythagorean fuzzy Hamacher power weighted average (DHPFHPWA) operator based on the power average [43] operators and Hamacher operations [36].

Definition 11 *Let $\widetilde{d}_j\,(j = 1,2,\cdots,n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher power weighted average (DHPFHPWA) operator as follows:*

$$DHPFHPWA\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right) = \bigoplus_{j=1}^{n}\left(\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)\widetilde{d}_j}{\sum\limits_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}\right) = \cup_{\gamma_j\in h_j,\eta_j\in g_j}$$

$$\left\{\left\{\sqrt{\frac{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}-\prod\limits_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+(\gamma-1)\prod\limits_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\},\right.$$

$$\left.\left\{\frac{\sqrt{\gamma}\prod\limits_{j=1}^{n}(\eta_j)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\sqrt{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)\left(1-(\eta_j)^2\right)\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+(\gamma-1)\prod\limits_{j=1}^{n}(\eta_j)^{\frac{2\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\}\right\} \tag{25}$$

where $\omega = (\omega_1,\omega_2,\cdots,\omega_n)^T$ be the weight vector of $\widetilde{d}_j\,(j = 1,2,\cdots,n)$, $\gamma > 0$, and

$$T\left(\widetilde{d}_j\right) = \sum_{\substack{i=1\\ i\neq j}}^{n}\omega_i Sup\left(\widetilde{d}_j,\widetilde{d}_i\right) \tag{26}$$

and $Sup\left(\widetilde{d}_j,\widetilde{d}_i\right)$ is the support for $\widetilde{d}_j$ from $\widetilde{d}_i$, with the conditions:

- *$Sup\left(\widetilde{d}_i,\widetilde{d}_j\right)\in[0,1]$;*
- *$Sup\left(\widetilde{d}_i,\widetilde{d}_j\right) = Sup\left(\widetilde{d}_i,\widetilde{d}_j\right)$;*
- *$Sup\left(\widetilde{d}_i,\widetilde{d}_j\right) \geqslant Sup\left(\widetilde{d}_s,\widetilde{d}_t\right)$, if $dis\left(\widetilde{d}_i,\widetilde{d}_j\right) \geqslant dis\left(\widetilde{d}_s,\widetilde{d}_t\right)$, where dis is a distance measure.*

Now, we can discuss some special cases of the DHPFHWA operator with respect to the parameter γ.

- When $\gamma = 1$, DHPFHPWA operator reduces to the dual hesitant Pythagorean fuzzy power weighted average (DHPFPWA) operator as follows:

$$
\begin{aligned}
&\mathrm{DHPFPWAA}_{\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) \\
&= \mathop{\oplus}_{j=1}^{n}\left(\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)\widetilde{d}_j}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}\right) \\
&= \cup_{\gamma_j\in h_j,\eta_j\in g_j}\left\{\left\{\sqrt{1-\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)\Big/\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}\right\},\right. \\
&\qquad\left.\left\{\prod_{j=1}^{n}(\eta_j)^{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)\Big/\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}\right\}\right\}
\end{aligned}
\tag{27}
$$

- When $\gamma = 2$, DHPFHPWA operator reduces to the dual hesitant Pythagorean fuzzy Einstein power weighted average (DHPFEPWA) operator as follows:

$$
\begin{aligned}
&\mathrm{DHPFEPWA}_{\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \mathop{\oplus}_{j=1}^{n}\left(\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)\widetilde{d}_j}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}\right) = \cup_{\gamma_j\in h_j,\eta_j\in g_j} \\
&\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}-\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\prod_{j=1}^{n}\left(1+(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+\prod_{j=1}^{n}\left(1-(\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\},\right. \\
&\left.\left\{\frac{\sqrt{2}\prod_{j=1}^{n}(\eta_j)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\sqrt{\prod_{j=1}^{n}\left(2-(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+\prod_{j=1}^{n}(\eta_j)^{\frac{2\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\Sigma_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\}\right\}
\end{aligned}
\tag{28}
$$

4.2. Dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator

Xu and Yager [44] developed power geometric (PG) operator on the basis of PA operator [43] and geometric mean [45-46], which can be defined as follows:

$$PG(a_1,a_2,\cdots,a_n)=\prod_{i=1}^{n}(a_i)^{(1+T(a_i))\Big/\sum\limits_{i=1}^{n}(1+T(a_i))} \tag{29}$$

where $T(a_i)=\sum_{\substack{j=1\\ j\neq i}}^{n} Sup(a_i,a_j)$, and $Sup(a,b)$ is the support for a from b, which satisfies the following three properties:

(1) $Sup(a,b)\in[0,1]$;

(2) $Sup(a,b)=Sup(b,a)$;

(3) $Sup(a,b)\geqslant Sup(x,y)$, if $|a-b|<|x-y|$.

In this section, we shall propose the dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator based on the power geometric [44] operators and Hamacher operations [36].

Definition 12 *Let $\widetilde{d}_j\,(j=1,2,\cdots,n)$ be a collection of DHPFNs, then we define the dual hesitant Pythagorean fuzzy Hamacher power weighted geometric (DHPFHPWG) operator as follows:*

$$DHPFHPWG\left(\widetilde{d}_1,\widetilde{d}_2,\cdots,\widetilde{d}_n\right)=\overset{n}{\underset{j=1}{\oplus}}\left(\widetilde{d}_j\right)=\cup_{\gamma_j\in h_j,\eta_j\in g_j}$$

$$\left\{\left\{\frac{\sqrt{\gamma}\prod\limits_{j=1}^{n}(\gamma_j)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\sqrt{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)\left(1-(\gamma_j)^2\right)\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+(\gamma-1)\prod\limits_{j=1}^{n}(\gamma_j)^{\frac{2\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\},\right.$$

$$\left.\left\{\sqrt{\frac{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}-\prod\limits_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\prod\limits_{j=1}^{n}\left(1+(\gamma-1)(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+(\gamma-1)\prod\limits_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\frac{2\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\}\right\} \tag{30}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *be the weight vector of* $\widetilde{d}_j$ $(j = 1, 2, \cdots, n)$, $\gamma > 0$, *and*

$$T\left(\widetilde{d}_j\right) = \sum_{\substack{i=1\\ i\neq j}}^{n} \omega_i Sup\left(\widetilde{d}_j, \widetilde{d}_i\right) \tag{31}$$

and $Sup\left(\widetilde{d}_j, \widetilde{d}_i\right)$ *is the support for* $\widetilde{d}_j$ *from* $\widetilde{d}_i$, *with the conditions:*

(1) $Sup\left(\widetilde{d}_i, \widetilde{d}_j\right) \in [0, 1]$;

(2) $Sup\left(\widetilde{d}_i, \widetilde{d}_j\right) = Sup\left(\widetilde{d}_i, \widetilde{d}_j\right)$;

(3) $Sup\left(\widetilde{d}_i, \widetilde{d}_j\right) \geqslant Sup\left(\widetilde{d}_s, \widetilde{d}_t\right)$, *if* $dis\left(\widetilde{d}_i, \widetilde{d}_j\right) \geqslant dis\left(\widetilde{d}_s, \widetilde{d}_t\right)$ *where dis is a distance measure.*

Now, we can discuss some special cases of the DHPFHPWG operator with respect to the parameter γ.

- When $\gamma = 1$, DHPFHPWG operator reduces to the dual hesitant Pythagorean fuzzy power weighted geometric (DHPFPWG) operator as follows:

$$\begin{aligned}
&\mathrm{DHPFPWGG}_{\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \bigotimes_{j=1}^{n} \left(\widetilde{d}_j\right)^{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right) \Big/ \sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)} \\
&= \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left\{ \left\{ \prod_{j=1}^{n} (\gamma_j)^{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right) \Big/ \sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)} \right\}, \right. \\
&\left. \left\{ \sqrt{1 - \prod_{j=1}^{n} \left(1 - (\eta_j)^2\right)^{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right) \Big/ \sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}} \right\} \right\}
\end{aligned} \tag{32}$$

- When $\gamma = 2$, DHPFHPWG operator reduces to the dual hesitant Pythagorean fuzzy Einstein power weighted geometric (DHPFEPWG) operator as follows:

$$\mathrm{DHPFEPWG}_{\omega}\left(\widetilde{d}_1, \widetilde{d}_2, \cdots, \widetilde{d}_n\right) = \bigoplus_{j=1}^{n} \left(\widetilde{d}_j\right) = \cup_{\gamma_j \in h_j, \eta_j \in g_j}$$

$$\left\{ \left\{ \frac{\sqrt{2} \prod_{j=1}^{n} (\gamma_j)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\sqrt{\prod_{j=1}^{n} \left(2 - (\gamma_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}} + \prod_{j=1}^{n} (\gamma_j)^{\frac{2\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n} \omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}} \right\}, \right.$$

$$\left\{\left.\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}-\prod_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}{\prod_{j=1}^{n}\left(1+(\eta)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}+\prod_{j=1}^{n}\left(1-(\eta_j)^2\right)^{\frac{\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\widetilde{d}_j\right)\right)}}}}\right\}\right\} \tag{33}$$

5. An approach to multiple attribute decision making with dual hesitant Pythagorean fuzzy information

In this section, we shall utilize the dual hesitant aggregation operators to multiple attribute decision making with dual hesitant Pythagorean fuzzy information. Let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be the state of nature. If the decision makers provide several values for the alternative A_i under the state of nature G_j with anonymity, these values can be considered as a dual hesitant Pythagorean fuzzy element $\widetilde{d}_{ij} = (h_{ij}, g_{ij})$. In the case where two decision makers provide the same value, then the value emerges only once in $\widetilde{d}_{ij}$. Suppose that the decision matrix $\widetilde{D} = \left(\widetilde{d}_{ij}\right)_{m\times n}$ is the dual hesitant Pythagorean fuzzy decision matrix, where $\widetilde{d}_{ij}\,(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ are in the form of DHPFNs.

In the following, we apply the DHPFHWA (or DHPFHWG) operator to the MADM problems for potential evaluation of emerging technology commercialization with dual hesitant Pythagorean fuzzy information.

Step 1 We utilize the decision information given in matrix $\widetilde{D}$, and the DHPFHWA operator

$$\begin{aligned}
\widetilde{d}_i &= \text{DHPFHWA}\left(\widetilde{d}_{i1}, \widetilde{d}_{i2}, \cdots, \widetilde{d}_{in}\right)\\
&= \bigoplus_{j=1}^{n}\left(\omega_j \widetilde{d}_{ij}\right)\\
&= \cup_{\gamma_{ij}\in h_{ij}, \eta_{ij}\in h_{ij}}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)(\gamma_{ij})^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-(\gamma_{ij})^2\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(\gamma_{ij})^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}\left(1-(\gamma_{ij})^2\right)^{\omega_j}}}\right\},\right.\\
&\qquad\left.\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{n}(\eta_{ij})^{\omega_j}}{\sqrt{\prod_{j=1}^{n}\left(1+(\gamma-1)\left(1-(\eta_{ij})^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}(\eta_{ij})^{2\omega_j}}}\right\}\right\}
\end{aligned} \tag{34}$$

Or the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator:

$$
\begin{aligned}
\widetilde{d}_i &= \mathrm{DHPFHWG}\left(\widetilde{d}_{i1}, \widetilde{d}_{i2}, \cdots, \widetilde{d}_{in}\right) \\
&= \bigotimes_{j=1}^{n} \left(\widetilde{d}_{ij}\right)^{\omega_j} \\
&= \cup_{\gamma_{ij}\in h_{ij}, \eta_{ij}\in h_{ij}} \left\{ \left\{ \frac{\sqrt{\gamma} \prod_{j=1}^{n} (\gamma_{ij})^{\omega_j}}{\sqrt{\prod_{j=1}^{n} \left(1+(\gamma-1)\left(1-(\gamma_{ij})^2\right)\right)^{\omega_j} + (\gamma-1) \prod_{j=1}^{n} (\gamma_{ij})^{2\omega_j}}} \right\}, \right. \qquad (35) \\
&\qquad \left. \left\{ \sqrt{\frac{\prod_{j=1}^{n} \left(1+(\gamma-1)(\eta_{ij})^2\right)^{\omega_j} - \prod_{j=1}^{n} \left(1-(\eta_{ij})^2\right)^{\omega_j}}{\prod_{j=1}^{n} \left(1+(\gamma-1)(\eta_{ij})^2\right)^{\omega_j} + (\gamma-1) \prod_{j=1}^{n} \left(1-(\eta_{ij})^2\right)^{\omega_j}}} \right\} \right\}
\end{aligned}
$$

to derive the overall preference values $\widetilde{d}_i\,(i=1,2,\cdots,m)$ of the alternative A_i.

Step 2 Calculate the scores $S\left(\widetilde{d}_i\right)$ $(i=1,2,\cdots,m)$ of the overall dual hesitant Pythagorean fuzzy preference values $\widetilde{d}_i\,(i=1,2,\cdots,m)$. If there is no difference between two scores $S\left(\widetilde{d}_i\right)$ and $S\left(\widetilde{d}_j\right)$, then we need to calculate the accuracy degrees $S\left(\widetilde{p}_i\right)$ and $S\left(\widetilde{p}_i\right)$ of the collective overall preference values $\widetilde{d}_i$ and $\widetilde{d}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $p\left(\widetilde{d}_i\right)$ and $p\left(\widetilde{d}_i\right)$.

Step 3 Rank all the alternatives $A_i\,(i=1,2,\cdots,m)$ and select the best one(s) in accordance with the scores $S\left(\widetilde{d}_i\right)$ $(i=1,2,\cdots,m)$.

Step 3 End.

6. Numerical example

Thus, in this section we shall present a numerical example for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information in order to illustrate the method proposed in this paper. Let us suppose there is a problem to deal with the supplier selection in supply chain management which is classical multiple attribute decision making problems. There are five prospect suppliers $A_i\,(i=1,2,3,4,5)$ for four attributes $G_j\,(j=1,2,3,4)$. The four attributes include product quality (G_1), service (G_2), delivery (G_3) and price (G_4), respectively. In order to avoid influence each other, the decision makers are required to evaluate the five suppliers $A_i\,(i=1,2,3,4,5)$ under the above four attributes in anonymity and the decision matrix $\widetilde{D}=\left(\widetilde{d}_{ij}\right)_{5\times 4}$ is presented in Tab. 1, where $\widetilde{d}_{ij}\,(i=1,2,3,4,5,\, j=1,2,3,4)$ are in the form of DHPFNs.

Table 1: Dual hesitant Pythagorean fuzzy decision matrix

	G_1	G_2	G_3	G_4
A1	{{0.3,0.4},{0.6}}	{{0.4,0.5},{0.3,0.4}}	{{0.2,0.3},{0.7}}	{{0.4,0.5},{0.5}}
A2	{{0.6},{0.4}}	{{0.2,0.4,0.5},{0.4}}	{{0.2},{0.6,0.7,0.8}}	{{0.5},{0.4,0.5}}
A3	{{0.5,0.7},{0.2}}	{{0.2},{0.7,0.8}}	{{0.2,0.3,0.4},{0.6}}	{{0.5,0.6,0.7},{0.3}}
A4	{{0.7},{0.3}}	{{0.6,0.7,0.8},{0.2}}	{{0.1,0.2},{0.3}}	{{0.1},{0.6,0.7,0.8}}
A5	{{0.6,0.7},{0.2}}	{{0.2,0.3,0.4},{0.5}}	{{0.4,0.5},{0.2}}	{{0.2,0.3,0.4},{0.5}}

The information about the attribute weights is known as follows: $\omega = (0.20, 0.15, 0.35, 0.30)$. In the following, we utilize the approach developed for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information.

Step 1 We utilize the decision information given in matrix $\widetilde{D}$, and the DHPFHWA operator to obtain the overall preference values $\widetilde{d}_i$ of the supplier in supply chain management $A_i\,(i=1,2,3,4,5)$. Take alternative A_i for an example (here, we take $\gamma=3$), we have

$$
\begin{aligned}
&\widetilde{d}_1 = \mathrm{DHPFHWA}_{\omega}\left(\widetilde{d}_{11},\widetilde{d}_{12},\widetilde{d}_{13},\widetilde{d}_{14}\right)\\
&= \mathop{\oplus}_{j=1}^{4}\left(\omega_j\widetilde{d}_{1j}\right)\\
&= \cup_{\gamma_{1j}\in h_{1j},\eta_{1j}\in h_{1j}}\left\{\left\{\sqrt{\frac{\prod_{j=1}^{4}\left(1+(\gamma-1)(\gamma_{1j})^2\right)^{\omega_j}-\prod_{j=1}^{4}\left(1-(\gamma_{1j})^2\right)^{\omega_j}}{\prod_{j=1}^{4}\left(1+(\gamma-1)(\gamma_{1j})^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{4}\left(1-(\gamma_{1j})^2\right)^{\omega_j}}}\right\},\right.\\
&\qquad\left.\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{4}(\eta_{1j})^{\omega_j}}{\sqrt{\prod_{j=1}^{4}\left(1+(\gamma-1)\left(1-(\eta_{1j})^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{4}(\eta_{1j})^{2\omega_j}}}\right\}\right\}\\
&= \{\{\{0.3,0.4\},\{0.5\}\},\{\{0.4,0.5\},\{0.3,0.4\}\},\{\{0.2,0.3\},\{0.5\}\},\{\{0.4,0.5\},\{0.5\}\}\}\\
&= \{\{0.3005,0.3146,0.3281,0.3328,0.3333,0.3412,0.3457,0.3461,0.3582,0.3586,\\
&0.3630,0.3703,0.3707,0.3749,0.3866,0.3979\},\{0.3097,0.3416\}\}
\end{aligned}
$$

Step 2 Calculate the scores $s\left(\widetilde{d_i}\right)$ $(i = 1,2,3,4,5)$ of the overall dual hesitant Pythagorean fuzzy preference values $\widetilde{d_i}$ $(i = 1,2,3,4,5)$:

$$s\left(\widetilde{d_1}\right) = 0.3828, s\left(\widetilde{d_2}\right) = 0.4552, s\left(\widetilde{d_3}\right) = 0.5008$$
$$s\left(\widetilde{d_4}\right) = 0.4774, s\left(\widetilde{d_5}\right) = 0.6171$$

Step 3 Rank all the suppliers $A_i\,(i = 1,2,3,4,5)$ in accordance with the scores $s\left(\widetilde{d_i}\right)$ $(i = 1,2,3,4,5)$ of the overall dual hesitant Pythagorean fuzzy numbers: $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$, and thus the most desirable supplier is A_5.

Based on the DHPFHWG operator, then, in order to select the most desirable supplier, we can develop an approach to multiple attribute decision making problems with dual hesitant Pythagorean fuzzy information, which can be described as following:

Step 1' Aggregate all dual hesitant Pythagorean fuzzy value $\widetilde{h}_{ij}\,(j = 1,2,3,4)$ by using the dual hesitant Pythagorean fuzzy weighted geometric (DHPFHWG) operator to derive the overall dual hesitant Pythagorean fuzzy values $\widetilde{d_i}\,(i = 1,2,\cdots,5)$ of the supplier A_i. Take supplier A_1 for an example (here, we take $\gamma = 3$), we have

$$\begin{aligned}
\widetilde{d_1} &= \mathrm{DHPFHWG}_\omega\left(\widetilde{d}_{11}, \widetilde{d}_{12}, \widetilde{d}_{13}, \widetilde{d}_{14}\right)\\
&= \underset{j=1}{\overset{4}{\otimes}} \left(\widetilde{d}_{1j}\right)^{\omega_j}\\
&= \cup_{\gamma_{1j}\in h_{1j}, \eta_{1j}\in h_{1j}} \left\{\left\{\frac{\sqrt{\gamma}\prod_{j=1}^{4}(\gamma_{1j})^{\omega_j}}{\sqrt{\prod_{j=1}^{4}\left(1+(\gamma-1)\left(1-(\gamma_{1j})^2\right)\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{4}(\gamma_{1j})^{2\omega_j}}}\right\},\right.\\
&\qquad \left.\left\{\sqrt{\frac{\prod_{j=1}^{4}\left(1+(\gamma-1)(\eta_{1j})^2\right)^{\omega_j}-\prod_{j=1}^{4}\left(1-(\eta_{1j})^2\right)^{\omega_j}}{\prod_{j=1}^{4}\left(1+(\gamma-1)(\eta_{1j})^2\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{4}\left(1-(\eta_{1j})^2\right)^{\omega_j}}}\right\}\right\}\\
&= \{\{\{0.3,0.4\},\{0.5\}\},\{\{0.4,0.5\},\{0.3,0.4\}\},\{\{0.2,0.3\},\{0.5)\},\{\{0.4,0.5\},\{0.5\}\}\}\\
&= \{\{0.2794, 0.2863, 0.2934, 0.3006, 0.3053, 0.3128, 0.3204, 0.3275, 0.3282, 0.3354,\\
&\quad 0.3435, 0.3518, 0.3572, 0.3657, 0.3743, 0.3832\}, \{0.5913, 0.6031\}\}
\end{aligned}$$

Step 2' Calculate the scores $s\left(\widetilde{d_i}\right)$ $(i=1,2,3,4,5)$ of the overall dual hesitant Pythagorean fuzzy values $\widetilde{d_i}\,(i=1,2,3,4,5)$ of the supplier A_i:

$$s\left(\widetilde{d_1}\right)=0.4862, s\left(\widetilde{d_2}\right)=0.4084, s\left(\widetilde{d_3}\right)=0.4076$$
$$s\left(\widetilde{d_4}\right)=0.4941, s\left(\widetilde{d_5}\right)=0.5502$$

Step 3' Rank all the suppliers in supply chain management $A_i\,(i=1,2,3,4,5)$ in accordance with the scores $s\left(\widetilde{d_i}\right)$ $(i=1,2,3,4,5)$ of the overall dual hesitant Pythagorean fuzzy values $\widetilde{d_i}\,(i=1,2,\cdots,5)$ by using definition 5: $A_5 \succ A_4 \succ A_1 \succ A_2 \succ A_3$ and thus the most desirable supplier in supply chain management is A_5.

From the above analysis, it is easily seen that although the overall rating values of the alternatives are slightly different by using two operators respectively. However, the most desirable supplier in supply chain management is A_5.

7. Conclusion

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the Hamacher aggregation operators with dual Pythagorean hesitant fuzzy information. Then, motivated by the ideal of Hamacher operation, we have developed some Hamacher aggregation operators for aggregating dual hesitant Pythagorean fuzzy information: dual hesitant Pythagorean fuzzy Hamacher weighted average (DHPFHWA) operator, dual hesitant Pythagorean fuzzy Hamacher weighted geometric (DHPFHWG) operator, dual hesitant Pythagorean fuzzy Hamacher ordered weighted average (DHPFHOWA) operator, dual hesitant Pythagorean fuzzy Hamacher ordered weighted geometric (DHPFHOWG) operator, dual hesitant Pythagorean fuzzy Hamacher hybrid average (DHPFHHA) operator and dual hesitant Pythagorean fuzzy Hamacher hybrid geometric (DHPFHHG) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the dual hesitant Pythagorean fuzzy multiple attribute decision making problems. Finally, a practical example for supplier selection in supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, we shall continue working in the extension and application of the developed operators to other domains and uncertain environments [47-66].

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An interval observer design for uncertain nonlinear systems based on the T-S fuzzy model

YAMINA MENASRIA, HICHEM BOURAS and NASREDDINE DEBBACHE

A new approach to build an interval observer for nonlinear uncertain systems is presented in this paper. Nonlinear systems modeled in the Takagi-Sugeno (T-S) form are studied. A T-S proportional observer is first issued by pole-placement and LMI tools. Secondly, time-varying change of coordinates for each dynamic state estimation error is used to design an interval observer. The system state bounds are then directly deduced.

Key words: T-S model, T-S proportional observer, interval observer, time-varying systems.

1. Introduction

The problem of state vector estimation is very challenging in control and diagnosis thories for nonlinear systems. It have recently received considerable interest among scientists in various fields and its solution remains expected in many applications. The design of the classical state estimator (observer) is not possible due to presence of uncertainty (parametric or/and signal). A called interval observer, was introduced by [9] to estimate state bounds of biological systems that are subject to parameter uncertainties. Later the framework of interval observers was used and extended for many biological processes [3, 21, 15].

Actually, there exist many interval observers proposed for linear systems in continuous and discret times [13, 20, 12, 7]. For nonlinear systems, several observer were also proposed in [14, 19, 16, 18, 6, 8, 5, 24]. By applying similarity transformation, a Hurwitz matrix can be transformed to a Hurwitz and Metzler (cooperative) one. The transformation matrix is constant and real is considered in [18] and it is a solution of the Sylvester equation. In [13, 12] the transformation is time-varying.

In this work, we propose the design of an interval observer for nonlinear systems based on the Takagi-Sugeno model with the time-varying approach [12]. The T-S fuzzy model proposed by [22] has been shown to be an universal approximator of nonlinear dynamic systems. It's a piecewise interpolation of several linear or nonlinear models

The authors are with Department of Electronic, Badji Mokhtar University, Po. Box 12, 23000, Annaba, Algeria. The corresponding author is Y. Menasria, e-mail: yamina.menasria@annaba-univ.org

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through membership functions. The fuzzy proportional observer cited in [11, 23] is designed but diverges in presence of disturbances. When disturbances are with known distribution, fuzzy unknown input observer introduced by [2] can easily be applied. Fuzzy sliding mode observer studied by [1] works if uncertainties are of known structure. In the case of unknown disturbances but bounded within known bounds, the fuzzy interval observer is a solution.

In the following, an interval observer is designed the T-S systems. Fuzzy interval observer, which is quite an important issue has not been investigated yet. This motivates us to carry out the present work. The design procedure consists in computing proportional observer gains as well as changes of coordinates by multiple time-varying transformations. The main contributions of this paper can be summarized as follows: (i) the fuzzy proportional observer gain matrices are obtained by pole-placement and LMI tools, (ii) time-varying transformation is applied for all local linear models and (iii) sufficient conditions for designing interval observers for T-S systems are given. The rest of the paper is outlined as follows. In Sect. 2, problem formulation and some necessary definitions are given. In Sect. 3, based on time-varying transformation, sufficient conditions for the existence of fuzzy interval observers are established. An example is provided to illustrate the efficiency of the proposed method in Sect. 4. Conclusions are given in Sect. 5.

2. Problem formulation and preliminaries

Consider the nonlinear system in the T-S model form:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))C_i x(t) \end{cases} \tag{1}$$

where M is the number of local models function, $x \in \Re^n$ is the state vector, $u \in \Re^p$ is the input and $y \in \Re^q$ the output. Matrices A_i, B_i C_i are constant and the premisse variable $\xi(t)$ can be the control $u(t)$ and/or the state vector $x(t)$.

The membership functions satisfy the following convexity constraints:

$$\begin{cases} \sum\limits_{i=1}^{M} \mu_i(\xi(t)) = 1 \\ 0 \leqslant \mu_i((\xi(t)) \leqslant 1 \end{cases} \tag{2}$$
$$\forall i = 1, 2 \ldots M$$

The T-S proportional observer is an interpolation of linear proportional observers initiated by [10]. It is given by the following equation:

$$\begin{cases} \dot{\hat{x}}(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))(A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t)) \\ \hat{y}(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))C_i\hat{x}(t) \end{cases} \tag{3}$$

The dynamic error state estimation is then:

$$\dot{e}(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(A_i - L_iC_i)e(t) \tag{4}$$

The pair (A_i, C_i) is detectable for all $i = 1 \ldots M$. So, there exist constant matrices $L_i \in \Re^{n\times q}$ such that $A_i - L_iC$ are Hurwitz for all $i = 1 \ldots M$. For the sake of simplicity we choose $C_i = C$ for all $i = 1 \ldots M$ and L_i gains are obtained by pole-placement. Global stability is ensured by LMI tools [4].

3. Nonlinear interval observer design

Consider the nonlinear uncertain system:

$$\begin{cases} \dot{x}(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))(A_ix(t) + B_iu(t) + \omega_{1i}(t)) \\ y(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))C_ix(t) + \omega_2(t) \end{cases} \tag{5}$$

where $\omega_{1i}(t)$, for $i = 1 \ldots M$ and $\omega(t)$ are unknown Lipschitz functions with known bounds and the initial condition $x(t_0) = x_0$ is assumed to be bounded by two known bounds:

$$\omega_{1i}^-(t) \leqslant \omega_{1i}(t) \leqslant \omega_{1i}^+(t)$$

for all $i = 1 \ldots M$ and

$$\begin{cases} \omega_2^-(t) \leqslant \omega_2(t) \leqslant \omega_2^+(t) \\ x_0^- \leqslant x_0 \leqslant x_0^+ \end{cases}$$

The dynamic error state estimation is:

$$\dot{e}(t) = \sum_{i=1}^{M} \mu_i(\xi(t))((A_i - L_iC)e(t)) + \sum_{i=1}^{M} \mu_i(\xi(t))(\omega_{1i}(t) - L_i\omega_2(t)) \tag{6}$$

Remark 1 For $i = 1...M$, if the corresponding $(A_i - L_iC)$ Jordan matrices [17] are not cooperatives (off diagonal entries negative), the system state estimation error (6) can be transformed into cooperative one combining M linear time-varying change of coordinates.

Theorem 3 *The following system:*

$$\begin{cases} \dot{z}^+(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(G_iz^+ + E_i^+(t)\varphi_i^+(t) - E_i^-(t)\varphi_i^-(t)) \\ \dot{z}^-(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(G_iz^- + E_i^+(t)\varphi_i^-(t) - E_i^-(t)\varphi_i^+(t)) \\ e^+(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^+(t) - F_i^-(t)z^-(t)) \\ e^-(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^-(t) - F_i^-(t)z^+(t)) \end{cases} \tag{7}$$

where $E_i^+(t) = max(E_i(t), 0), E_i^-(t) = E_i^+(t) - E_i(t)$ and the matrix $F_i(t)$ is the inverse of $E_i(t)$ with $F_i^+(t) = max(F_i(t), 0), F_i^-(t) = F_i^+(t) - F_i(t)$, is a T-S interval observer of system (6). *Disturbances functions $\varphi_i^-(t)$ and $\varphi_i^+(t)$ are known bounds of $\varphi_i(t) = \omega_{1i} - L_i\omega_2$ for $i = 1...M$.*

Consequently, the system state bounds are:

$$\begin{cases} x^+(t) = e^+(t) + \hat{x}(t) \\ x^-(t) = e^-(t) + \hat{x}(t) \end{cases}$$

Proof We use a time-varying change of coordinate $z(t) = E_i(t)e(t)$ for each local model $\dot{e} = (A_i - L_iC)e + \varphi_i(t)$ and from the convexity constraints of the membership functions μ_i we prove that system (7) is an T-S interval observer of system (6). Globally and from the T-S model:

$$\dot{z}(t) = \sum_{i=1}^{M} \mu_i(\xi(t))(G_iz(t) + E_i(t)\varphi_i(t))$$

let $A_i - L_iC = \bar{A}_i$, then locally, for $i = 1...M$, we have:

$$\begin{aligned} \dot{z} &= E_i(t)\dot{e} + \dot{E}_i(t)e(t) \\ &= E_i(t)(\bar{A}_ie + \varphi_i(t)) + (G_iE_i(t) - E_i(t)\bar{A}_i)e(t) \\ &= G_iE_i(t)e(t) + E_i(t)\varphi_i(t) \\ &= G_iz(t) + E_i(t)\varphi_i(t) \end{aligned}$$

The stability of (7) when both $\varphi_i^+(t)$ and $\varphi_i^-(t)$ are identically equal to zero is a consequence of the fact that $\bar{A}_i$ are Hurwitz for all $t \in \Re$.

Consider a solution $(z(t), e(t))$ of (7) with known initial conditions $e(t_0) = e_0$, $z(t_0) = (z^+(t_0), z^-(t_0))$ and e_0^+, e_0^- the state error vectors such that:

$$e_0^- \leqslant e_0 \leqslant e_0^+$$

Because the entries E_i^+ and E_i^- are nonnegative we get:

$$\begin{cases} E_i^+(t_0)e_0^- \leqslant E_i^+(t_0)e_0 \leqslant E_i^+(t_0)e_0^+ \\ E_i^-(t_0)e_0^- \leqslant E_i^-(t_0)e_0 \leqslant E_i^-(t_0)e_0^+ \end{cases}$$

for all $i = 1 \ldots M$. We get:

$$\begin{cases} E_i^+(t_0)e_0^- - E_i^-(t_0)e_0^+ \leqslant z(t_0) \\ z(t_0) \leqslant E_i^+(t_0)e_0^+ - E_i^-(t_0)e_0^- \end{cases}$$

for all $i = 1 \ldots M$. From the constraints of the membership functions in (2), we also get:

$$\begin{cases} \sum\limits_{i=1}^{M} \mu_i(\xi(t_0))(E_i^+(t_0)e_0^- - E_i^-(t_0)e_0^+) \leqslant z(t_0) \\ z(t_0) \leqslant \sum\limits_{i=1}^{M} \mu_i(\xi(t_0))(E_i^+(t_0)e_0^+ - E_i^-(t_0)e_0^-) \end{cases} \tag{8}$$

Then the initial conditions of the proposed observer are deduced:

$$\begin{cases} z^+(t_0) = \sum\limits_{i=1}^{M} \mu_i(\xi(t_0))(E_i^+(t_0)e_0^+ - E_i^-(t_0)e_0^-) \\ z^-(t_0) = \sum\limits_{i=1}^{M} \mu_i(\xi(t_0))(E_i^+(t_0)e_0^- - E_i^-(t_0)e_0^+) \end{cases}$$

Moreover, for all $i = 1 \ldots M$:

$$\begin{cases} E_i^+(t)\varphi_i^-(t) \leqslant E_i^+(t)\varphi_i(t) \leqslant E_i^+(t)\varphi_i^+(t) \\ E_i^-(t)\varphi_i^-(t) \leqslant E_i^-(t)\varphi_i(t) \leqslant E_i^-(t)\varphi_i^+(t) \end{cases}$$

and

$$\begin{cases} E_i^+(t)\varphi_i^-(t) - E_i^-(t)\varphi_i^+(t) \leqslant E_i(t)\varphi_i(t) \\ E_i(t)\varphi_i(t) \leqslant E_i^+(t)\varphi_i^+(t) - E_i^-(t)\varphi_i^-(t) \end{cases}$$

Since matrices G_i are cooperatives for all $i = 1 \ldots M$, membership functions μ_i satisfy constraints (2) and inequalities in (8) hold:

$$\begin{cases} \sum\limits_{i=1}^{M} \mu_i(\xi(t))(G_i z^-(t) + E_i^+(t)\varphi_i^-(t) - E_i^-(t)\varphi_i^+(t)) \\ \leqslant \sum\limits_{i=1}^{M} \mu_i(\xi(t))(G_i z(t) + E_i(t)\varphi_i(t)) \leqslant \\ \sum\limits_{i=1}^{M} \mu_i(\xi(t))(G_i z^+(t) + E_i^+(t)\varphi_i^+(t) - E_i^-(t)\varphi_i^-(t)) \end{cases}$$

and then:

$$\begin{cases} \dot{z}^-(t) \leqslant \dot{z}(t) \leqslant \dot{z}^+(t) \\ z^-(t) \leqslant z(t) \leqslant z^+(t) \end{cases}$$

Also and locally for all $i = 1 \dots M$:

$$z^-(t) \leqslant E_i(t)e(t) \leqslant z^+(t)$$

Since the matrices $F_i^+(t)$ and $F_i^-(t)$ for all $i = 1 \dots M$ are nonnegative, for all $t \geqslant 0$, we get:

$$\begin{cases} F_i^-(t)z^-(t) \leqslant F_i^-(t)E_i(t)e(t) \leqslant F_i^-(t)z^+(t) \\ F_i^+(t)z^-(t) \leqslant F_i^+(t)E_i(t)e(t) \leqslant F_i^+(t)z^+(t) \end{cases}$$

and

$$\begin{cases} F_i^+(t)z^-(t) - F_i^-(t)z^+(t) \leqslant F_i(t)E_i(t)e(t) \\ F_i(t)E_i(t)e(t) \leqslant F_i^+(t)z^+(t) - F_i^-(t)z^-(t) \end{cases}$$

From the fact that, $F_i(t)$ are inverse of $E_i(t)$, for all $i = 1 \dots M$ following inequalities hold:

$$\begin{cases} e(t) \leqslant F_i^+(t)z^+(t) - F_i^-(t)z^-(t) \\ F_i^+(t)z^-(t) - F_i^-(t)z^+(t) \leqslant e(t) \end{cases}$$

and from the properties of the membership functions in (2) we get:

$$\begin{cases} e(t) \leqslant \sum\limits_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^+(t) - F_i^-(t)z^-(t)) \\ \sum\limits_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^-(t) - F_i^-(t)z^+(t)) \leqslant e(t) \end{cases}$$

Finally, lower and upper bounds for the system states are directly deduced:

$$\begin{cases} x^+(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^+(t) - F_i^-(t)z^-(t)) + \hat{x}(t) \\ x^-(t) = \sum\limits_{i=1}^{M} \mu_i(\xi(t))(F_i^+(t)z^-(t) - F_i^-(t)z^+(t)) + \hat{x}(t) \end{cases}$$

□

4. Simulation

Let us consider the T-S system with two local models ($M = 2$):

$$\begin{cases} A_1 = \begin{bmatrix} -2 & -0.2 & 0.4 \\ 5 & -1 & 2 \\ 3 & -1 & -2 \end{bmatrix}; B_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}; \\ C_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}. \\ A_2 = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 3 \\ 0 & -0.2 & -1 \end{bmatrix}; B_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}; \\ C_2 = C_1. \end{cases}$$

The pairs (A_1, C_1) and (A_2, C_2) are detectables and the T-S proportional observer gains (L_1, L_2) are calculated by pole-placement in a stable complex plane region using LMIs. The two state estimation error matrices are Hurwitz and each one has two complex conjugate and one real eigen-values. Consequently, the corresponding Jordan matrices (9) are not cooperatives.

$$\begin{cases} J_1 = \begin{bmatrix} -1.8641 & 0 & 0 \\ 0 & -2.1082 & -2.6595 \\ 0 & 2.6595 & -2.1082 \end{bmatrix} \\ J_2 = \begin{bmatrix} -2.2162 & 0 & 0 \\ 0 & -2.433 & -1.1521 \\ 0 & 1.1521 & -2.433 \end{bmatrix} \end{cases} \tag{9}$$

The system state estimation error must be transformed into cooperative one using two linear time-varying change of coordinates. In Fig. 1, the input $u(t)$ is variable on its entire range $[-1, 1]$ in order to excite all local modes. It is also chosen the premisse variable $\xi(t)$ for the T-S system. Fig. 2 illustrates two membership functions that satisfy the convexity criterion (2) at each time. State and output disturbances $(\omega_{11}(t), \omega_{12}(t)$ and $\omega_2(t))$ are choosen uniformly distributed noise respectively in interval: $[-0.5, +0.5]$, $[-0.5, +0.5]$ and $[-1, +1]$. Like is shown in Figure 3, the states x_1, x_2 and x_3 remain inside the interval $[x_{i_{inf}}(t), x_{i_{sup}}(t)]$ respctively for $i = 1, 2, 3$.

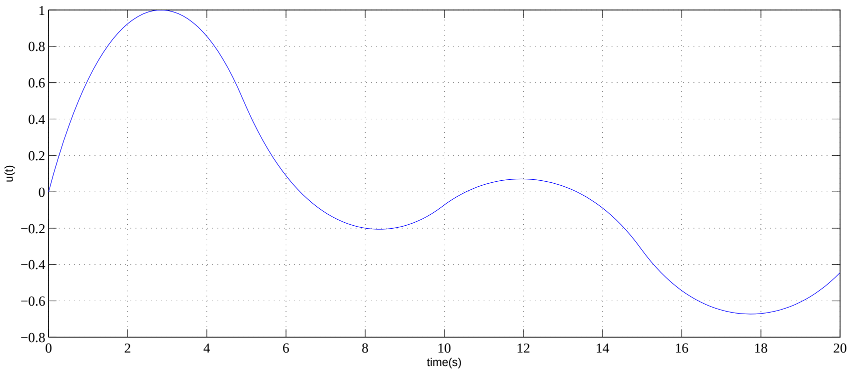

Figure 1: The input signal

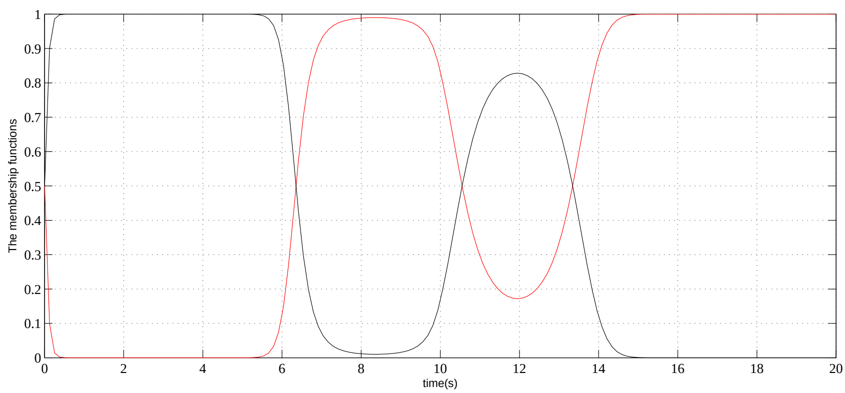

Figure 2: The membership functions: μ_1 in red line and μ_2 in black line

5. Conclusions

Based on interval analysis, which we believe to be an extremely promising approach for the investigation of the properties of nonlinear systems, a guaranteed technique for non-linear state estimation in a bounded error context have been presented. In First, the fuzzy proportional observer is built without uncertainties. Then Pole-placement ensures that state estimation error matrices are Hurwitz but rarely cooperatives. Finally, time-varying change of coordinates approach changes T-S error state estimation system with disturbances into cooperative one. By knowing initial state interval, T-S interval observer for error state estimation system is designed. From error state estimation bounds, system states bounds are deduced at each time. The fuzzy interval observer proposed in this paper can be applied to several practical systems with unknown disturbances but bounded within known bounds like waste water treatment plants.

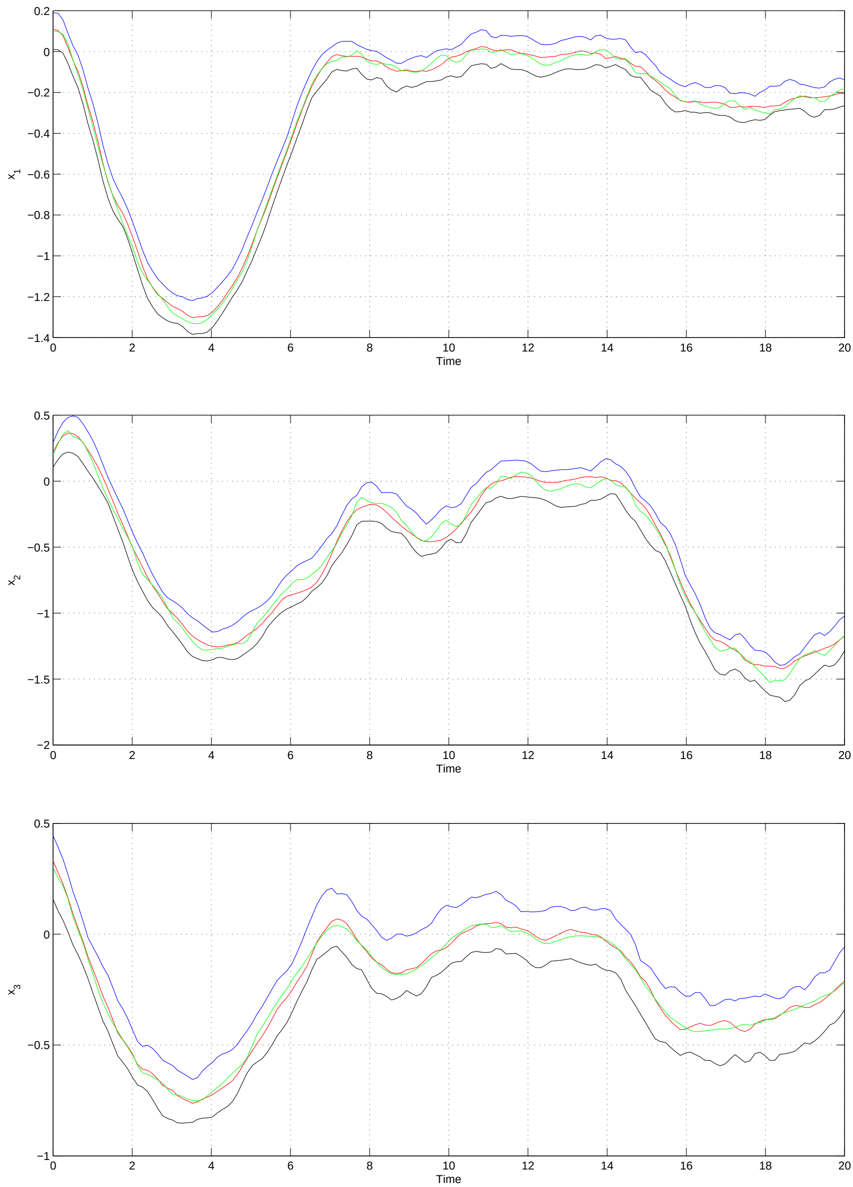

Figure 3: State x_1, x_2 and x_3 bounds in blue and black lines with the Luenberger observer (green lines) and comparison with the real state (red line).

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A new 3-D jerk chaotic system with two cubic nonlinearities and its adaptive backstepping control

SUNDARAPANDIAN VAIDYANATHAN

This paper presents a new seven-term 3-D jerk chaotic system with two cubic nonlinearities. The phase portraits of the novel jerk chaotic system are displayed and the qualitative properties of the jerk system are described. The novel jerk chaotic system has a unique equilibrium at the origin, which is a saddle-focus and unstable. The Lyapunov exponents of the novel jerk chaotic system are obtained as $L_1 = 0.2974$, $L_2 = 0$ and $L_3 = -3.8974$. Since the sum of the Lyapunov exponents of the jerk chaotic system is negative, we conclude that the chaotic system is dissipative. The Kaplan-Yorke dimension of the new jerk chaotic system is found as $D_{KY} = 2.0763$. Next, an adaptive backstepping controller is designed to globally stabilize the new jerk chaotic system with unknown parameters. Moreover, an adaptive backstepping controller is also designed to achieve global chaos synchronization of the identical jerk chaotic systems with unknown parameters. The backstepping control method is a recursive procedure that links the choice of a Lyapunov function with the design of a controller and guarantees global asymptotic stability of strict feedback systems. MATLAB simulations are shown to illustrate all the main results derived in this work.

Key words: chaos, chaotic systems, jerk systems, chaos control, adaptive control, backstepping control, synchronization.

1. Introduction

Modeling and applications of chaotic systems are active research areas in the literature [1, 2, 3]. The first famous chaotic system was discovered by Lorenz, when he was designing a weather model in 1963 [4]. Some well-known chaotic systems are Chen system [5], Lü system [6], Cai system [7], Tigan system [8], Sprott systems [9], etc.

Some well-known paradigms of 3-D chaotic systems are Arneodo system [10], Hénon-Heiles system [12], Lü-Chen system [13], Liu system [14], etc. Many new chaotic systems have been also discovered like Li system [15], Sundarapandian systems [16, 17], Vaidyanathan systems [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33], Pehlivan system [34], Tacha system [35], Jafari system [36], Sampath system [37], Pham systems [38, 39, 40, 41, 42, 43, 44], Volos system [45], Akif system [46], etc.

The author is with Research and Development Centre, Vel Tech University, Avadi, Chennai-600062, Tamil Nadu, India. E-mail: sundarvtu@gmail.com

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Chaos theory has applications in several fields of science and engineering such as oscillators [47, 48, 49, 50, 51, 52, 53, 54, 55], dynamos [56, 57, 58, 59], Tokamak systems [60, 61], chemical reactions [62, 63, 64, 65, 66, 67, 68, 69, 70, 71], neural networks [72, 73, 74, 75, 76, 77], neurology [78, 79, 80, 81, 82, 83], biology [84, 85, 86, 87, 88, 89, 90, 91, 92], electrical circuits [93, 94, 95], induction motors [96], cryptosystems [97, 98], memristors [99, 100, 101], random bit generator [102], etc.

In classical mechanics, a jerk system is expressed by an explicit third order differential equation describing the time evolution of a single scalar variable x according to the dynamics

$$\frac{d^3x}{dt^3} = f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right) \tag{1}$$

A particularly simple example of a jerk system is the famous Coullet system [103] given by

$$\frac{d^3x}{dt^3} + a\frac{d^2x}{dt^2} + \frac{dx}{dt} = g(x) \tag{2}$$

where $g(x)$ is a nonlinear function such as $g(x) = b(x^2 - 1)$. The Coullet system (2) exhibits chaos for $a = 0.6$ and $b = 0.58$.

A classical example of a cubic dissipative jerk chaotic flow was found by Sprott [104]. In this research work, we modify the dynamics of the jerk system in [104] by introducing two linear terms and taking different set values for the system parameters. Thus, we obtain a novel chaotic jerk system with two cubic nonlinearities.

In most of the synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of synchronization is to use the output of the master system to control the response of the slave system so that the slave system tracks the output of the master system asymptotically [105, 106, 107, 108].

In the chaos literature, an impressive variety of techniques have been proposed for chaos synchronization such as active control method [109, 110, 111, 112, 113, 114, 115], adaptive control method [116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127], backstepping control method [128, 129, 130, 131, 132, 133, 134, 135], sliding mode control method [136, 137, 138, 139, 140, 141, 142, 143, 144], etc.

All the main adaptive backstepping control results in this paper are proved using Lyapunov stability theory [145]. MATLAB simulations are depicted to illustrate the phase portraits of the novel jerk chaotic system, adaptive stabilization and synchronization results for the novel 3-D jerk chaotic system.

This research paper is organized as follows. Section 2 contains the dynamics and phase portraits of the novel chaotic jerk system. Section 3 details the qualitative properties of the novel chaotic jerk system. In Section 4, we apply adaptive backstepping control method to design an adaptive feedback control law that stabilizes the states of the novel jerk system.

In Section 5, we apply adaptive backstepping control method to design an adaptive feedback control law that achieves complete and exponential synchronization of the states of identical novel chaotic jerk systems. Finally, Section 6 contains a summary of the main results obtained in this work.

2. A new jerk chaotic system

A classical example of a cubic dissipative jerk chaotic flow was found by Sprott [104] and described by the third-order differential equation

$$\dddot{x} = -a\ddot{x} + x\dot{x}^2 - x^3 \quad (\text{with } a = 3.6) \tag{3}$$

In system form, Sprott's differential equation (3) corresponds to the jerk system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -ax_3 + x_1x_2^2 - x_1^3 \end{cases} \tag{4}$$

where $a = 3.6$ yields a chaotic attractor.

Using Wolf's algorithm [146], the Lyapunov exponents of the Sprott system (4) for $a = 3.6$ are numerically obtained as

$$L_1 = 0.1360, \quad L_2 = 0, \quad L_3 = -3.7367 \tag{5}$$

From (5), we see that the Maximal Lyapunov Exponent (MLE) of the Sprott system (4) is $L_1 = 0.1360$. Since $L_1 > 0$, the Sprott system (4) is *chaotic*.

The Kaplan-Yorke dimension of a chaotic system of order n is defined as

$$D_{KY} = j + \frac{L_1 + L_2 + \cdots + L_j}{|L_{j+1}|} \tag{6}$$

where $L_1 \geqslant L_2 \geqslant \cdots \geqslant L_n$ are the n Lyapunov exponents of the chaotic system and j is the largest integer for which $L_1 + L_2 + \cdots + L_j \geqslant 0$. Thus, the Kaplan-Yorke dimension of the Sprott jerk system (4) is calculated as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0364 \tag{7}$$

In this work, we propose a new jerk chaotic system, which is obtained by adding two linear systems $-bx$ and $c\dot{x}$, where $b, c > 0$, to the Sprott's jerk function in the ODE (3). Thus, our new jerk chaotic flow is described by the third order ODE

$$\dddot{x} = -a\ddot{x} + x\dot{x}^2 - x^3 - bx + c\dot{x} \tag{8}$$

In system form, the third order ODE (8) corresponds to the jerk system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -ax_3 - bx_1 + cx_2 + x_1x_2^2 - x_1^3 \end{cases} \tag{9}$$

where a, b and c are positive parameters.

In this paper, we shall show that the system (9) is *chaotic* when the parameters a and b take the values

$$a = 3.6, \quad b = 1.3, \quad c = 0.1 \tag{10}$$

Using Wolf's algorithm [146], the Lyapunov exponents of the novel system (9) for the parameter values (10) are numerically obtained as

$$L_1 = 0.2974, \;\; L_2 = 0, \;\; L_3 = -3.8974 \tag{11}$$

From (11), we see that the Maximal Lyapunov Exponent (MLE) of the novel system (9) is $L_1 = 0.2974$. Since $L_1 > 0$, the novel system (9) is *chaotic*. Moreover, we also note that the MLE of the novel jerk system (9) is greater than the MLE of the Sprott jerk system (4). Also, the Kaplan-Yorke dimension of the novel jerk system (9) is calculated as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0763, \tag{12}$$

which is greater than the Kaplan-Yorke dimension of the Sprott jerk system (4).

For numerical simulations, we take the initial conditions of the system (9) as

$$x_1(0) = 0.5, \;\; x_2(0) = 0.5, \;\; x_3(0) = 0.5 \tag{13}$$

The initial conditions in (13) have been chosen arbitrarily for the sake of simulations. For other initial conditions in $\mathbf{R}^3$ also, the system (9) is chaotic with a similar strange attractor.

Figure 1 depicts the chaotic attractor of the novel jerk system (9) in 3-D view. Figures 2-4 depict the 2-D projection of the strange chaotic attractor of the novel jerk chaotic system (9) on $(x_1, x_2), (x_2, x_3)$ and (x_3, x_1) planes, is shown, respectively.

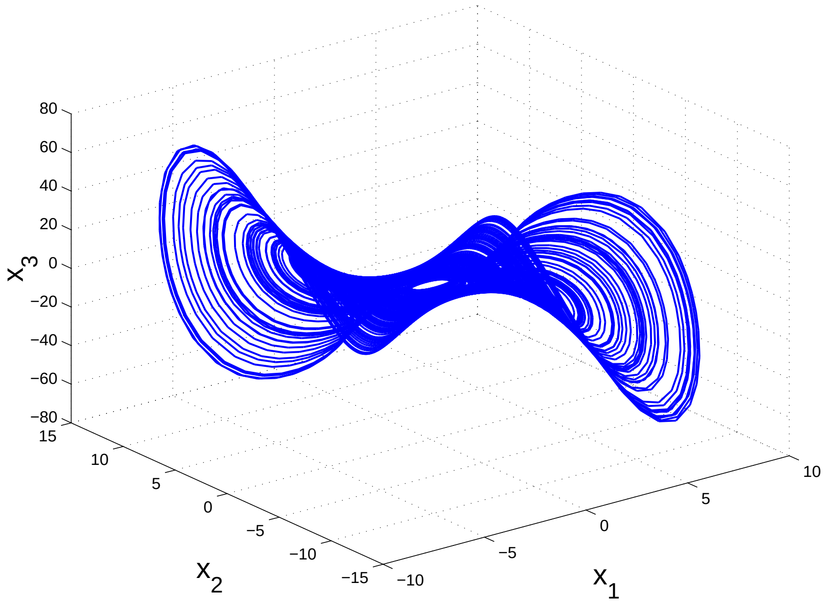

Figure 1: Strange attractor of the 3-D novel jerk chaotic System

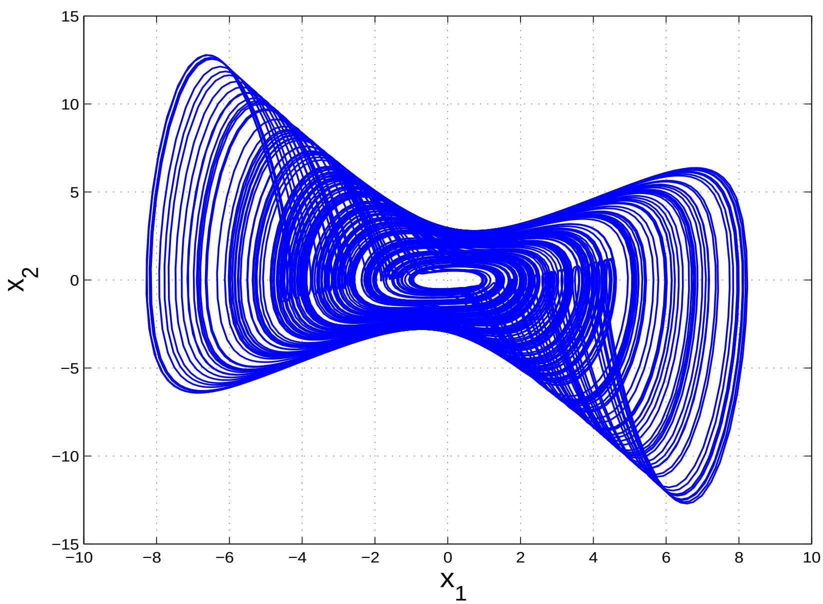

Figure 2: 2-D projection of the novel jerk chaotic system on the (x_1, x_2) plane

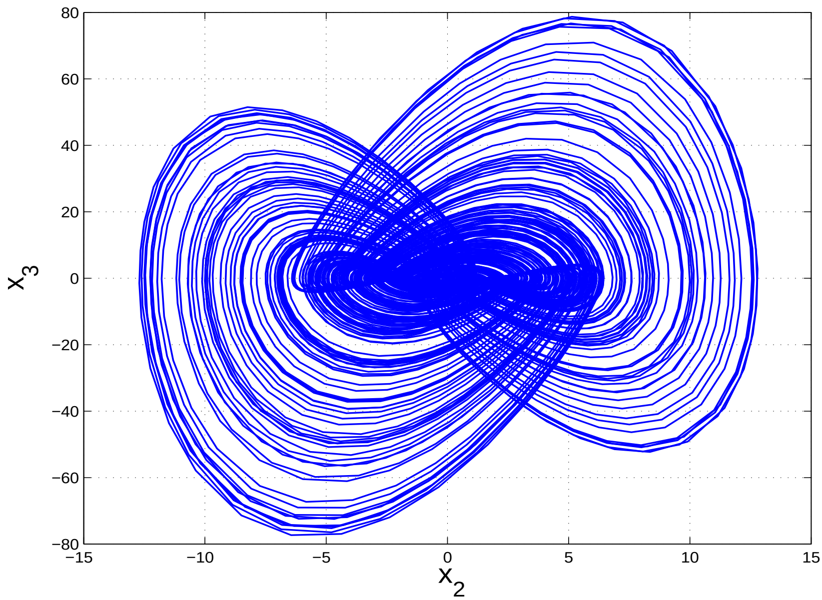

Figure 3: 2-D projection of the novel jerk chaotic system on the (x_2, x_3) plane

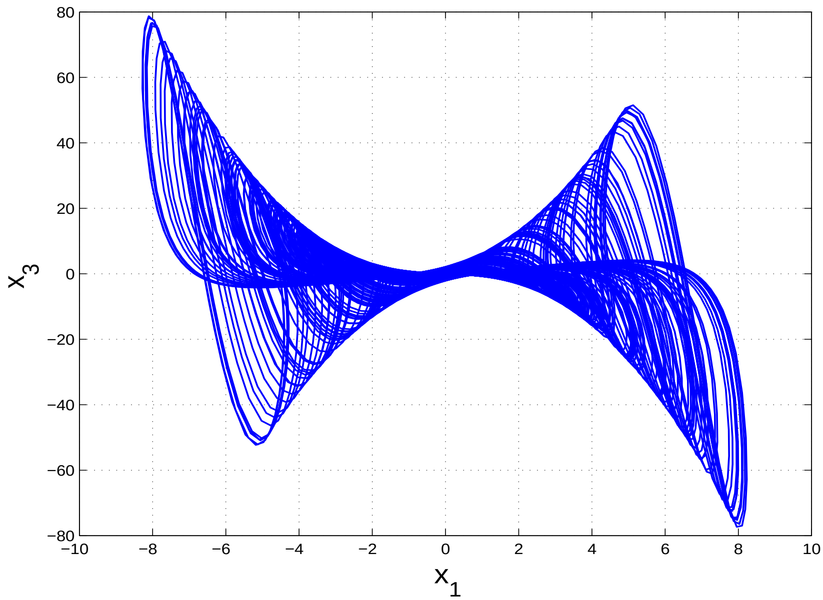

Figure 4: 2-D projection of the novel jerk chaotic system on the (x_1, x_3) plane

3. Analysis of the 3-D novel jerk chaotic system

3.1. Dissipativity

In vector notation, the new jerk system (9) can be expressed as

$$\dot{\mathbf{x}} = f(\mathbf{x}) = \begin{bmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix}, \tag{14}$$

where

$$\begin{cases} f_1(x_1,x_2,x_3) &= x_2 \\ f_2(x_1,x_2,x_3) &= x_3 \\ f_3(x_1,x_2,x_3) &= -ax_3 - bx_1 + cx_2 + x_1x_2^2 - x_1^3 \end{cases} \tag{15}$$

Let Ω be any region in $\mathbf{R}^3$ with a smooth boundary and also, $\Omega(t) = \Phi_t(\Omega)$, where Φ_t is the flow of f. Furthermore, let $V(t)$ denote the volume of $\Omega(t)$. By Liouville's theorem, we know that

$$\dot{V}(t) = \int\limits_{\Omega(t)} (\nabla \cdot f)\, dx_1\, dx_2\, dx_3 \tag{16}$$

The divergence of the novel jerk system (14) is found as:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -a < 0 \tag{17}$$

Inserting the value of $\nabla \cdot f$ from (17) into (16), we get

$$\dot{V}(t) = \int\limits_{\Omega(t)} (-a)\, dx_1\, dx_2\, dx_3 = -aV(t) \tag{18}$$

Integrating the first order linear differential equation (18), we get

$$V(t) = \exp(-at)V(0) \tag{19}$$

From Eq. (19), it is clear that $V(t) \to 0$ exponentially as $t \to \infty$. This shows that the novel 3-D jerk chaotic system (9) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the novel jerk chaotic system (9) settles onto a strange attractor of the system.

3.2. Equilibrium Points

The equilibrium points of the 3-D novel jerk chaotic system (9) are obtained by solving the equations

$$\begin{cases} f_1(x_1,x_2,x_3) &= x_2 &= 0 \\ f_2(x_1,x_2,x_3) &= x_3 &= 0 \\ f_3(x_1,x_2,x_3) &= -ax_3 - bx_1 + cx_2 + x_1x_2^2 - x_1^3 &= 0 \end{cases} \tag{20}$$

We take the parameter values as in the chaotic case (10), *i.e.*

$$a = 3.6, \quad b = 1.3, \quad c = 0.1 \tag{21}$$

Thus, the equilibrium points of the system (9) are characterized by the equations

$$x_2 = 0, \quad x_3 = 0, \quad x_1(x_1^2 + b) = 0 \tag{22}$$

Solving the system (22), we get the equilibrium points of the system (9) as

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

The Jacobian matrix of the novel jerk chaotic system (9) at E_0 is obtained as

$$J_0 = J(E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b & c & -a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.3 & 0.1 & -3.6 \end{bmatrix} \tag{24}$$

We find that J_0 has the eigenvalues

$$\lambda_1 = -3.7208, \quad \lambda_{2,3} = 0.0604 \pm 0.5880\, i \tag{25}$$

This shows that the equilibrium E_0 is a saddle-focus point, which is unstable.

3.3. Lyapunov exponents and Kaplan-Yorke dimension

We take the parameter values of the novel jerk system (9) as

$$a = 3.6, \quad b = 1.3, \quad c = 0.1 \tag{26}$$

Then the Lyapunov exponents are numerically obtained using Wolf's algorithm [146] as

$$L_1 = 0.2974, \quad L_2 = 0, \quad L_3 = -3.8974 \tag{27}$$

Thus, the maximal Lyapunov exponent (MLE) of the novel jerk system (9) is $L_1 = 0.2974 > 0$, which shows that the system (9) has chaotic behavior.

Since $L_1 + L_2 + L_3 = -3.6 = -a < 0$, it follows that the novel jerk chaotic system (9) is dissipative. Also, the Kaplan-Yorke dimension of the novel jerk chaotic system (9) is obtained as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0763, \tag{28}$$

which is fractional.

4. Adaptive control of the 3-D novel jerk chaotic system

In this section, we use backstepping control method to derive an adaptive feedback control law for globally stabilizing the 3-D novel jerk chaotic system with unknown parameters. Thus, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -ax_3 - bx_1 + cx_2 + x_1x_2^2 - x_1^3 + u \end{cases} \tag{29}$$

where a, b, c are unknown constant parameters, and u is a backstepping control law to be determined using estimates of the unknown system parameters.

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \end{cases} \tag{30}$$

Differentiating (30) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\ \dot{e}_c(t) &= -\dot{\hat{c}}(t) \end{cases} \tag{31}$$

Next, we shall state and prove the main result of this section.

Theorem 1 *The 3-D novel jerk chaotic system* (29)*, with unknown parameters a, b and c, is globally and exponentially stabilized by the adaptive feedback control law,*

$$u(t) = -[3 - \hat{b}(t)]x_1 - [5 + \hat{c}(t)]x_2 - [3 - \hat{a}(t)]x_3 - x_1x_2^2 + x_1^3 - kz_3 \tag{32}$$

where $k > 0$ is a gain constant,

$$z_3 = 2x_1 + 2x_2 + x_3, \tag{33}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= -z_3x_3 \\ \dot{\hat{b}}(t) &= -z_3x_1 \\ \dot{\hat{c}}(t) &= z_3x_2 \end{cases} \tag{34}$$

Proof We prove this result via Lyapunov stability theory [145].
First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{35}$$

where

$$z_1 = x_1 \tag{36}$$

Differentiating V_1 along the dynamics (29), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = x_1 x_2 = -z_1^2 + z_1(x_1 + x_2) \tag{37}$$

Now, we define

$$z_2 = x_1 + x_2 \tag{38}$$

Using (38), we can simplify the equation (37) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \tag{39}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{40}$$

Differentiating V_2 along the dynamics (29), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3) \tag{41}$$

Now, we define

$$z_3 = 2x_1 + 2x_2 + x_3 \tag{42}$$

Using (42), we can simplify the equation (41) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{43}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b, e_c) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 + \frac{1}{2}(e_a^2 + e_b^2 + e_c^2) \tag{44}$$

which is a positive definite function on $\mathbf{R}^6$. Differentiating V along the dynamics (29), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{45}$$

Eq. (45) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{46}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3 \tag{47}$$

A simple calculation gives

$$S = (3-b)x_1 + (5+c)x_2 + (3-a)x_3 + x_1x_2^2 - x_1^3 + u \tag{48}$$

Substituting the adaptive control law (32) into (48), we obtain

$$S = -[b - \hat{b}(t)]x_1 + [c - \hat{c}(t)]x_2 - [a - \hat{a}(t)]x_3 - kz_3 \tag{49}$$

Using the definitions (31), we can simplify (49) as

$$S = -e_b x_1 + e_c x_2 - e_a x_3 - kz_3 \tag{50}$$

Substituting the value of S from (50) into (46), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - (1+k)z_3^2 + e_a\left(-z_3x_3 - \dot{\hat{a}}\right) + e_b\left(-z_3x_1 - \dot{\hat{b}}\right) + e_c\left(z_3x_2 - \dot{\hat{c}}\right) \tag{51}$$

Substituting the update law (34) into (51), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1+k)z_3^2, \tag{52}$$

which is a negative semi-definite function on $\mathbf{R}^6$. From (52), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) & e_c(t) \end{bmatrix} \in \mathbf{L}_infty \tag{53}$$

Also, it follows from (52) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2 \tag{54}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{55}$$

Integrating the inequality (55) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \leqslant V(0) - V(t) \tag{56}$$

From (56), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$. From Eq. (29), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$. Thus, using Barbalat's lemma [145], we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathbf{R}^3$. Hence, it is immediate that $\mathbf{x}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{x}(0) \in \mathbf{R}^3$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (29) and (34), when the adaptive control law (32) is applied.

The parameter values of the novel jerk chaotic system (29) are taken as in the chaotic case (10), *i.e.*

$$a = 3.6, \quad b = 1.3, \quad c = 0.1 \tag{57}$$

The positive gain constant k is taken as $k = 10$. As initial conditions of the novel jerk chaotic system (29), we take

$$x_1(0) = 7.5, \quad x_2(0) = 12.1, \quad x_3(0) = 15.4 \tag{58}$$

Also, as initial conditions of the parameter estimates, we take

$$\hat{a}(0) = 3.1, \quad \hat{b}(0) = 6.8, \quad \hat{c}(0) = 9.2 \tag{59}$$

In Figure 5, the exponential convergence of the controlled states is depicted, when the adaptive control law (32) and parameter update law (34) are implemented.

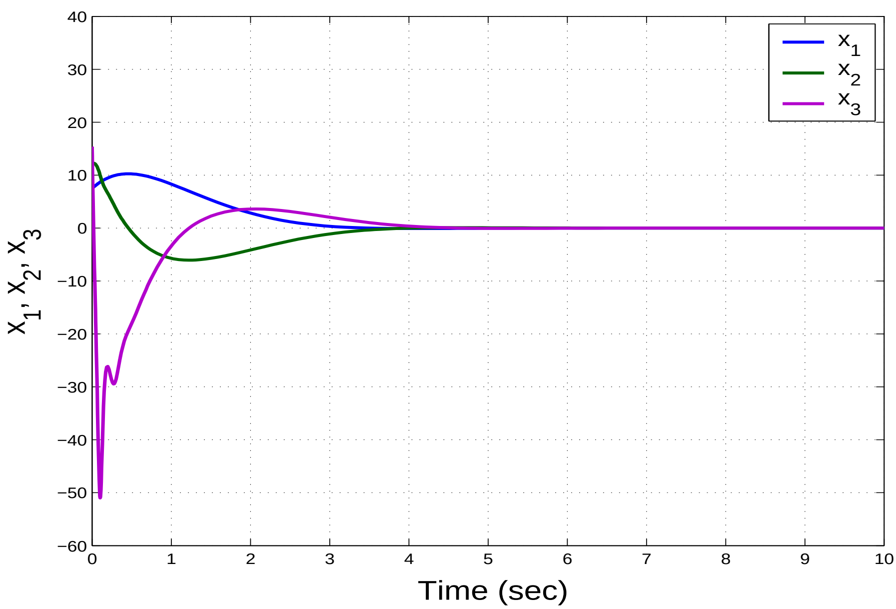

Figure 5: Time-history of the controlled states $x_1(t), x_2(t), x_3(t)$

5. Adaptive synchronization of the identical 3-D novel jerk chaotic systems

In this section, we use backstepping control method to derive an adaptive control law for globally and exponentially synchronizing the identical 3-D novel jerk chaotic systems with unknown parameters.

As the master system, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -ax_3 - bx_1 + cx_2 + x_1x_2^2 - x_1^3 \end{cases} \tag{60}$$

where x_1, x_2, x_3 are the states of the system, and a, b, c are unknown constant parameters.

As the slave system, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= -ay_3 - by_1 + cy_2 + y_1y_2^2 - y_1^3 + u \end{cases} \tag{61}$$

where y_1, y_2, y_3 are the states of the system, and u is a backstepping control to be determined using estimates of the unknown system parameters.

We define the synchronization errors between the states of the master system (60) and the slave system (61) as

$$\begin{cases} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \end{cases} \tag{62}$$

Then the error dynamics is easily obtained as

$$\begin{cases} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= -ae_3 - be_1 + ce_2 + y_1y_2^2 - x_1x_2^2 - y_1^3 + x_1^3 + u \end{cases} \tag{63}$$

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t) \\ e_c(t) &= c - \hat{c}(t) \end{cases} \tag{64}$$

Differentiating (64) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\ \dot{e}_c(t) &= -\dot{\hat{c}}(t) \end{cases} \tag{65}$$

Next, we shall state and prove the main result of this section.

Theorem 2 *The identical 3-D novel jerk chaotic systems* (60) *and* (61) *with unknown parameters a, b and c are globally and exponentially synchronized by the adaptive control law*

$$\begin{cases} u(t) & = & -[3-\hat{b}(t)]e_1 - [5+\hat{c}(t)]e_2 - [3-\hat{a}(t)]e_3 \\ & & -y_1y_2^2 + x_1x_2^2 + y_1^3 - x_1^3 - kz_3 \end{cases} \tag{66}$$

where $k > 0$ is a gain constant,

$$z_3 = 2e_1 + 2e_2 + e_3, \tag{67}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) & = & -z_3e_3 \\ \dot{\hat{b}}(t) & = & -z_3e_1 \\ \dot{\hat{c}}(t) & = & z_3e_2 \end{cases} \tag{68}$$

Proof We prove this result via backstepping control method and Lyapunov stability theory.

First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2}z_1^2 \tag{69}$$

where

$$z_1 = e_1 \tag{70}$$

Differentiating V_1 along the error dynamics (63), we get

$$\dot{V}_1 = z_1\dot{z}_1 = e_1e_2 = -z_1^2 + z_1(e_1 + e_2) \tag{71}$$

Now, we define

$$z_2 = e_1 + e_2 \tag{72}$$

Using (72), we can simplify the equation (71) as

$$\dot{V}_1 = -z_1^2 + z_1z_2 \tag{73}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{74}$$

Differentiating V_2 along the error dynamics (63), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3) \tag{75}$$

Now, we define

$$z_3 = 2e_1 + 2e_2 + e_3 \tag{76}$$

Using (76), we can simplify the equation (75) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \tag{77}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b, e_c, e_p) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 + \frac{1}{2}\left(e_a^2 + e_b^2 + e_c^2\right) \tag{78}$$

which is a positive definite function on $\mathbf{R}^6$. Differentiating V along the error dynamics (63), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{79}$$

Eq. (79) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} - e_c\dot{\hat{c}} \tag{80}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 \tag{81}$$

A simple calculation gives

$$S = (3-b)e_1 + (5+c)e_2 + (3-a)e_3 + y_1 y_2^2 - x_1 x_2^2 - y_1^3 + x_1^3 + u \tag{82}$$

Substituting the adaptive control law (66) into (48), we obtain

$$S = -[b - \hat{b}(t)]e_1 + [c - \hat{c}(t)]e_2 - [a - \hat{a}(t)]e_3 - kz_3 \tag{83}$$

Using the definitions (65), we can simplify (83) as

$$S = -e_b e_1 + e_c e_2 - e_a e_3 - kz_3 \tag{84}$$

Substituting the value of S from (84) into (80), we obtain

$$\left\{ \begin{array}{lcl} \dot{V} & = & -z_1 - z_2 - (1+k)z_3^2 + e_a[-z_3 e_3 - \dot{\hat{a}}] + e_b[-z_3 e_1 - \dot{\hat{b}}] \\ & & +e_c[z_3 e_2 - \dot{\hat{c}}] \end{array} \right. \tag{85}$$

Substituting the update law (68) into (85), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1+k)z_3^2, \tag{86}$$

which is a negative semi-definite function on $\mathbf{R}^6$. From (86), it follows that the vector $\mathbf{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) & e_c(t) \end{bmatrix} \in \mathbf{L}_infty \tag{87}$$

Also, it follows from (86) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2 \tag{88}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V} \tag{89}$$

Integrating the inequality (89) from 0 to t, we get

$$\int_0^t |\mathbf{z}(\tau)|^2 \, d\tau \leqslant V(0) - V(t) \tag{90}$$

From (90), it follows that $\mathbf{z}(t) \in \mathbf{L}_2$. From Eq. (63), it can be deduced that $\dot{\mathbf{z}}(t) \in \mathbf{L}_\infty$. Thus, using Barbalat's lemma, we conclude that $\mathbf{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{z}(0) \in \mathbf{R}^3$. Hence, it is immediate that $\mathbf{e}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\mathbf{e}(0) \in \mathbf{R}^3$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (60) and (61).

The parameter values of the novel jerk chaotic systems are taken as in the chaotic case, (10), i.e.

$$a = 3.6, \quad b = 1.3, \quad c = 0.1 \tag{91}$$

The positive gain constant is taken as $k = 10$. As initial conditions of the master chaotic system (60), we take

$$x_1(0) = -5.8, \quad x_2(0) = 3.7, \quad x_3(0) = -4.9 \tag{92}$$

As initial conditions of the slave chaotic system (61), we take

$$y_1(0) = 4.5, \quad y_2(0) = 8.4, \quad y_3(0) = -8.5 \tag{93}$$

Also, as initial conditions of the parameter estimates, we take

$$\hat{a}(0) = 11.2, \quad \hat{b}(0) = 6.1, \quad \hat{c}(0) = 12.6 \tag{94}$$

In Figs. 6-8, the complete synchronization of the identical 3-D jerk chaotic systems (60) and (61) is shown, when the adaptive control law (66) and the parameter update law (68) are implemented.

Also, in Fig. 9, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$, is shown.

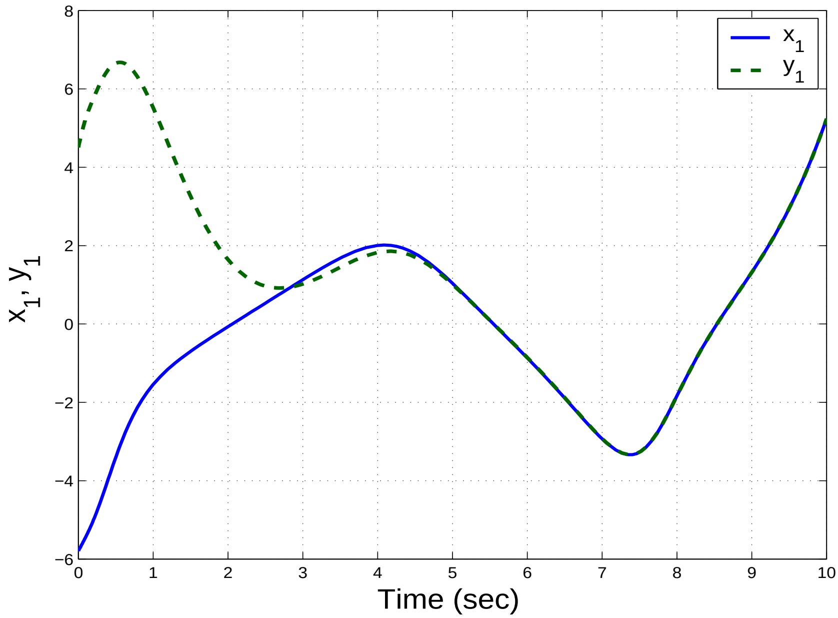

Figure 6: Synchronization of the states $x_1(t)$ and $y_1(t)$

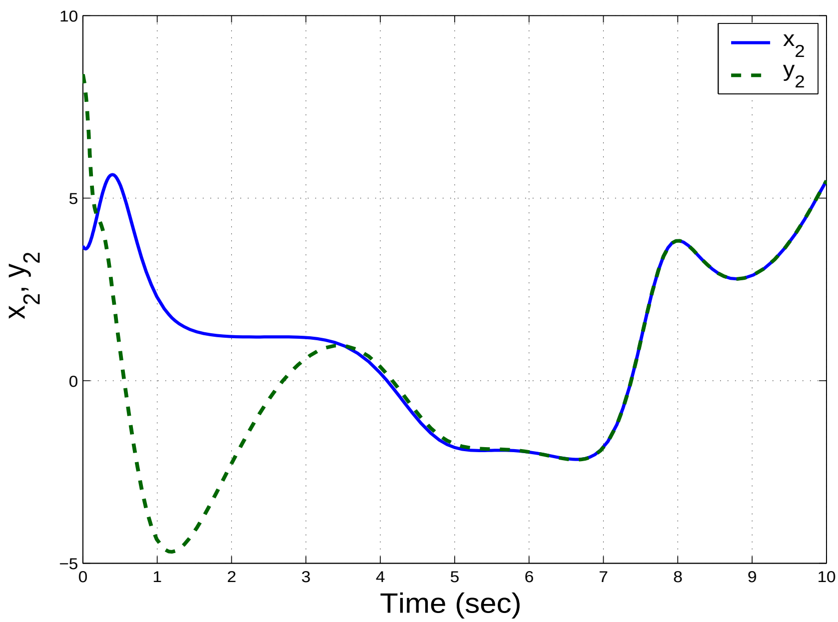

Figure 7: Synchronization of the states $x_2(t)$ and $y_2(t)$

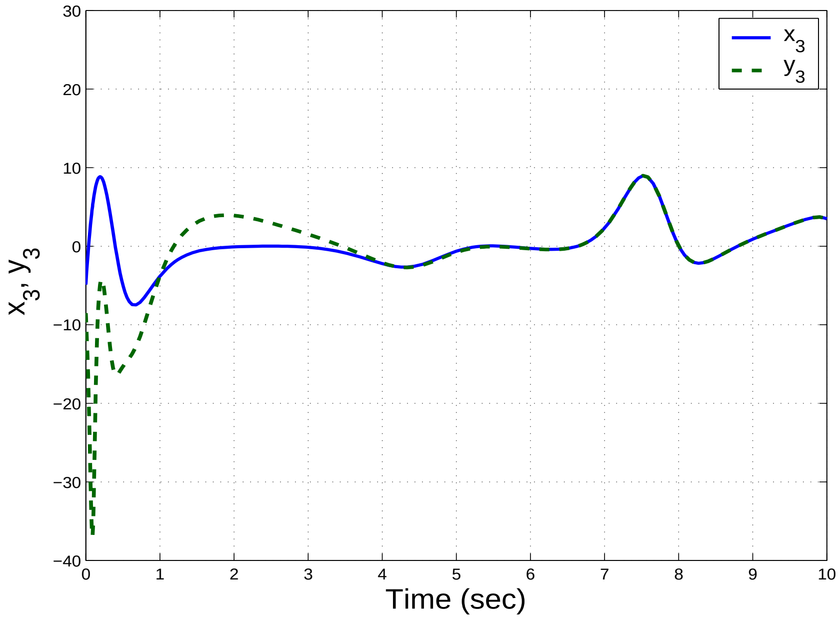

Figure 8: Synchronization of the states $x_3(t)$ and $y_3(t)$

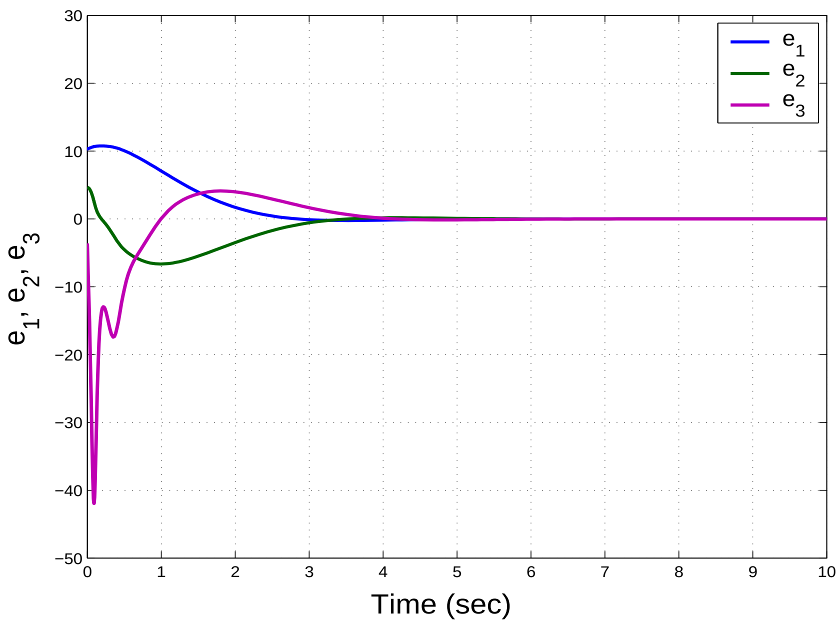

Figure 9: Time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$

6. Conclusions

In this paper, we announced a seven-term novel 3-D jerk chaotic system with two cubic nonlinearities. The phase portraits of the novel jerk chaotic system were displayed and the qualitative properties were discussed. Next, an adaptive backstepping controller was designed to globally stabilize the novel jerk chaotic system with unknown parameters. Moreover, an adaptive backstepping controller was also designed to achieve global chaos synchronization of the identical jerk chaotic systems with unknown parameters. MATLAB simulations were depicted to illustrate the phase portraits of the novel jerk chaotic system and also the adaptive backstepping control results.

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Relationship between the observability of standard and fractional linear systems

TADEUSZ KACZOREK

The relationship between the observability of standard and fractional discrete-time and continuous-time linear systems are addressed. It is shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

Key words: fractional, standard, linear, discrete-time, continuous-time, system, observability.

1. Introduction

The notion of controllability and observability of linear systems have been introduced by Kalman [14, 15]. Those notions are the basic concepts of the modern control theory [1, 6, 13, 16, 21, 24, 25]. They have been extended to positive and fractional linear and nonlinear systems [2, 4, 5, 7-11, 22, 23]. The mathematical fundamentals of fractional calculus are given in the monographs [18-20]. The positive fractional linear systems have been introduced in [8, 11].

In the paper [17] it has been shown that the fractional discrete-time and continuous-time linear systems are controllable if and only if the standard discrete-time and continuous-time systems are controllable.

In this paper it will be shown that the fractional discrete-time and continuous-time linear systems are observable if and only if the standard discrete-time and continuous-time linear systems are observable.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning standard and fractional discrete-time and continuous-time linear systems are recalled. The relationship between the observability of the standard and fractional discrete-time linear systems is considered in section 3 and of continuous-time linear systems in section 4. Concluding remarks are given in section 5.

The authors is with Bialystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Bialystok, Poland. E-mail: kaczorek@ee.pw.edu.pl

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The following notation will be used: $\Re^{n\times m}$ is the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ is the set of nonnegative integers, I_n is the $n\times n$ identity matrix.

2. Preliminaries

Consider the standard discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i\in Z_+ = \{0,1,...\}, \tag{1a}$$

$$y_i = Cx_i, \tag{1b}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

The solution to the equation (1a) is given by

$$x_i = A^i x_0 + \sum_{j=0}^{i-1} A^{i-j-1}Bu_j. \tag{2}$$

Substituting (2) into (1b) we obtain

$$y_i = CA^i x_0 + \sum_{j=0}^{i-1} CA^{i-j-1}Bu_j. \tag{3}$$

Now let us consider the fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad 0<\alpha<2, \tag{4a}$$

$$y_i = Cx_i, \tag{4b}$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i}(-1)^j \binom{\alpha}{j} x_{i-j}, \tag{4c}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j=0 \\ \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,... \end{cases} \tag{4d}$$

is the fractional α-order difference of x_i and $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

Substitution of (4c) into (4a) yields

$$x_{i+1} = (A+I_n\alpha)x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, \quad i\in Z_+, \tag{5a}$$

where

$$c_j = c_j(\alpha) = (-1)^{j+1}\begin{pmatrix}\alpha\\ j\end{pmatrix}, \quad j = 2,3,\ldots \tag{5b}$$

The solution to the equation (5a) has the form [11]

$$x_{i+1} = (A + I_n\alpha)x_i + \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, \quad i \in Z_+, \tag{6a}$$

where

$$\Phi_{j+1} = \Phi_j(A + I_n\alpha) + \sum_{k=2}^{j+1} c_k\Phi_{j-k+1}, \quad \Phi_0 = I_n \tag{6b}$$

and c_k is defined by (5b).

Substituting (6a) into (4b) we obtain

$$y_i = C\Phi_i x_0 + \sum_{j=0}^{i-1} C\Phi_{i-j-1}Bu_j. \tag{7}$$

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{8a}$$

$$y(t) = Cx(t), \tag{8b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

The solution to the equation (8a) has the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{9}$$

and

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau. \tag{10}$$

Now let us consider the fractional continuous-time linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha < 2 \tag{11a}$$

$$y(t) = Cx(t), \tag{11b}$$

where

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \quad x^{(n)}(\tau) = \frac{d^n x(\tau)}{d\tau^n} \tag{12}$$

is the Caputo fractional derivative of order $n-1<\alpha<n$ $(n\in N)$ of $x(t)$, $\Gamma(x)$ is the Euler gamma function, $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

The solution of the equation (11a) is given by [11]

$$x(t)=\Phi_0(t)x_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau,\quad x_0=x(0), \tag{13a}$$

where

$$\Phi_0(t)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{13b}$$

$$\Phi(t)=\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{13c}$$

and

$$y(t)=C\Phi_0(t)x_0+\int_0^t C\Phi(t-\tau)Bu(\tau)d\tau. \tag{14}$$

Theorem 4 (Cayley-Hamilton) *Let* $A\in\Re^{n\times n}$ *and*

$$\det[I_n\lambda-A]=\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_1\lambda+a_0. \tag{15}$$

Then

$$A^n+a_{n-1}A^{n-1}+\ldots+a_1A+a_0I_n=0. \tag{16}$$

Proof Proof is given in [3, 12].

Theorem 5 (Kronecker-Capelli) *The linear matrix equation*

$$Ax=b,\quad A\in\Re^{n\times n},\quad b\in\Re^n \tag{17}$$

has a solution $x\in\Re^n$ *if and only if*

$$\operatorname{rank}[A,b]=\operatorname{rank}A. \tag{18}$$

Proof Proof is given in [12].

3. Observability of standard and fractional discrete-time linear systems

It is well-known [1, 2, 7] that the observability of the standard and fractional linear systems depends only of the pair (A,C) and it is independent of the matrix B.

Definition 13 *The standard linear discrete-time linear system* (1) *is called observable in the interval* $[0,q]$ *if knowing the output* y_i *for* $i = 0,1,\dots,q-1$, $q \leqslant n$, *it is possible to find the unique* x_0 *of the system.*

Theorem 6 *The standard linear discrete-time linear system* (1) *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{19}$$

Proof Proof is given in [1, 6, 13].

Definition 14 *The fractional discrete-time linear system* (4) *is called observable in the interval* $[0,q]$ *if knowing the output* y_i *for* $i = 0,1,\dots,q-1$, $q < n$, *it is possible to find the unique* x_0 *of the system.*

We shall show that the fractional discrete-time linear system (4) is observable in the interval $[0,q]$ if and only if the standard linear discrete-time system (1) is observable in the same interval.

From (7) for $B = 0$ and (6b) for $i = 0,1,\dots,q-1$ we have

$$y_{0q} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{q-1} \end{bmatrix} = \begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ \vdots \\ C\Phi_{q-1} \end{bmatrix} x_0 = O_{0q}x_0, \tag{20a}$$

where

$$O_{0q} = \begin{bmatrix} C \\ C(A + I_n\alpha) \\ C[(A + I_n\alpha)^2 + c_2 I_n] \\ \vdots \\ C[(A + I_n\alpha)^{q-1} + \dots + (\alpha^{q-1} + \dots + c_{q-1})I_n] \end{bmatrix}. \tag{20b}$$

By Kronecker-Capelli theorem the equation (20a) has a unique solution x_0 for any given y_{0q} if and only if

$$\operatorname{rank} O_{0q} = n. \tag{20c}$$

Therefore, the following theorem has been proved.

Theorem 7 *The fractional discrete-time linear system* (4) *or equivalently* (5a), (4b), *is observable in the interval* $[0,q]$ *if and only if the condition* (20c) *is satisfied.*

It will be shown that the condition (20c) is equivalent to the condition (19). Note that

$$O_{0q} = \begin{bmatrix} C \\ C(A+I_n\alpha) \\ C[(A+I_n\alpha)^2+c_2I_n] \\ \vdots \\ C[(A+I_n\alpha)^{q-1}+\ldots+(\alpha^{q-1}+\ldots+c_{q-1})I_n] \end{bmatrix}$$
$$= \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ \alpha I_n & I_n & 0 & \cdots & 0 \\ (c_2+\alpha^2)I_n & 2\alpha I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (c_{q-1}+\ldots+\alpha^{q-1})I_n & \cdots & \cdots & \cdots & I_n \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \tag{21}$$

since

$$(A+I_n\alpha)^k = A^k + k\alpha A^{k-1} + \ldots + \alpha^k I_n \text{ for } k = 2,3,\ldots,q-1. \tag{22}$$

From (21) it follows that

$$\operatorname{rank} O_{0q} = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} \tag{23}$$

since the matrix

$$\begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ \alpha I_n & I_n & 0 & \cdots & 0 \\ (c_2+\alpha^2)I_n & 2\alpha I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (c_{q-1}+\ldots+\alpha^{q-1})I_n & \cdots & \cdots & \cdots & I_n \end{bmatrix} \tag{24}$$

is nonsingular for all values of α and c_k, $k = 1,2,\ldots,q-1$. Therefore, the following theorem has been proved.

Theorem 8 *The fractional discrete-time linear system* (4) *is observable in the interval* $[0,q]$, $q \leqslant n$, *if and only if the standard discrete-time linear system* (1) *is observable in the same interval* $[0,q]$.

Example 1 Consider the standard system (1) and the fractional system (4) for $\alpha = 0.5$ with the same matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \quad C = [\ 1 \quad 1\ ]. \tag{25}$$

Using (19) and (25) for $q = 2$ we obtain

$$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = 2 \tag{26}$$

and by Theorem 6 the standard system is observable in the interval $[0,2]$.

For the fractional system with (25) using (20b) we obtain

$$\operatorname{rank}\begin{bmatrix} C \\ C(A+\alpha I_2) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ -0.5 & -1.5 \end{bmatrix} = 2. \tag{27}$$

By Theorem 7 the fractional system with (25) is also observable in the interval $[0,2]$.

4. Observability of standard and fractional continuous-time linear systems

Definition 15 *The standard continuous-time linear system* (8) *is called observable in the interval* $[0,t_f]$ *if knowing the output* $y(t)$ *for* $t \in [0,t_f]$ *it is possible to find the unique* x_0 *of the system.*

Theorem 9 *The standard continuous-time linear system* (8) *is observable if and only if*

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{28}$$

Proof Proof is given in [1, 6, 13].

Definition 16 *The fractional continuous-time linear system* (11) *is called observable in the interval* $[0,t_f]$ *if knowing the output* $y(t)$ *for* $t \in [0,t_f]$ *it is possible to find the unique* x_0 *of the system.*

We shall show that the fractional continuous-time linear system (11) is observable in the interval $[0,t_f]$ if and only if the standard continuous-time linear system (8) is observable in the same interval.

Using the Cayley-Hamilton theorem (the equality (10)) it is possible to eliminate the powers $k = n, n+1, \ldots$ of the matrix A^k in (13b) and we obtain

$$\Phi_0(t) = \sum_{k=0}^{n-1} c_k(t)A^k. \tag{29}$$

The coefficients c_k in (29) can be computed as follows.

To simplify the calculations it is assumed the eigenvalues λ_k of the matrix A are distinct, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$. In this case using (29) we obtain

$$\begin{bmatrix} \Phi_0(\lambda_1) \\ \Phi_0(\lambda_2) \\ \vdots \\ \Phi_0(\lambda_n) \end{bmatrix} = H \begin{bmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix}, \tag{30}$$

where

$$H = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix}. \tag{31}$$

If the eigenvalues are distinct, then the matrix (31) is nonsingular and from (30) we have

$$\begin{bmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix} = H^{-1} \begin{bmatrix} \Phi_0(\lambda_1) \\ \Phi_0(\lambda_2) \\ \vdots \\ \Phi_0(\lambda_n) \end{bmatrix}. \tag{32}$$

The coefficients $c_k(t)$, $k = 0, 1, \ldots, n-1$ can be also found using the well-known Lagrange-Sylvester formula [3, 12].

Substitution of (29) into (14) for $B = 0$ yields

$$y(t) = C\Phi_0(t)x_0 = \sum_{k=0}^{n-1} c_k(t)CA^k = \begin{bmatrix} c_0(t) & c_1(t) & \cdots & c_{n-1}(t) \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0. \tag{33}$$

From (33) it follows that it is possible to find $y(t)$ for given $t \in [0, t_f]$, if and only if

$$\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{34}$$

since $c_k(t) \neq 0$ for $t \in [0, t_f]$. Therefore, the following theorem has been proved.

Theorem 10 *The fractional continuous-time linear system* (11) *is observable in the interval* $[0, t_f]$ *if and only if the standard continuous-time linear system* (8) *is observable in the same interval.*

Example 2 Consider the standard system (8) and the fractional system (11) with the same matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{35}$$

Using (28) and (35) we obtain

$$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 \tag{36}$$

and by Theorem 10 the standard system is observable. In this case for the fractional system (11) with (35) we obtain

$$\Phi_0(t) = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)} = I_2 + \frac{At^\alpha}{\alpha} = \begin{bmatrix} 1 & \frac{t^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix} = c_0(t)I_2 + c_1(t)A, \tag{37}$$

where

$$c_0(t) = 1, \ \ c_1(t) = \frac{t^\alpha}{\alpha}. \tag{38}$$

By Theorem 10 the fractional system is also observable.

5. Concluding remarks

The relationship between the observability of the standard and fractional discrete-time and continuous-time linear systems has been addressed. It has been shown that: 1) the fractional discrete-time linear systems are observable if and only if the standard discrete-time linear systems are observable (Theorem 8); 2) the fractional continuous-time linear systems are observable if and only if the standard continuous-time linear systems are observable (Theorem 10). The considerations have been illustrated by numerical examples. The considerations can be extended to the standard and fractional time-varying linear systems.

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Stabilization of a certain class of fuzzy control systems with uncertainties

NIZAR HADJ TAIEB, MOHAMED ALI HAMMAMI and FRANÇOIS DELMOTTE

In this paper, we investigate the global uniform practical exponential stability for a class of uncertain Takagi-Sugeno fuzzy systems. The uncertainties are supposed uniformly to be bounded by some known integrable functions to obtain an exponential convergence toward a neighborhood of the origin. Therefore, we use common quadratic Lyapunov function (CQLF) and parallel distributed compensation (PDC) controller techniques to show the global uniform practical exponential stability of the closed-loop system. Numeric simulations are given to validate the proposed approach.

Key words: Takagi-Sugeno fuzzy systems, PDC controller, global uniform practical exponential stability, Lyapunov stability, parametric uncertainty.

1. Introduction

It is well known that most plants in industry show significant nonlinearities, which usually make the analysis and controller design difficult. In order to overcome such difficulties, various schemes have been developed in the past two decades, among which a successful approach is the fuzzy control ([20], [22], [23], [30]). In recent years, Takagi-Sugeno (T-S) fuzzy models [22] have become a useful tool to deal with a class of nonlinear systems. The models can be described by a set of "if-then" rules which gives local linear approximations of an underlying system.

The stability analysis and control design for T-S fuzzy systems keep attracting researchers for decades ([1], [6], [7], [26], [27], [31]). The Lyapunov stability theory is the main approach for these kinds of problems. Among them, the simplest approaches consists in looking for a common quadratic Lyapunov function (CQLF) by using the concept of the parallel distributed compensation (PDC) technique ([19], [26], [27], [29]) to design a stabilizing controller. However, another important issue in stability analysis of nonlinear systems may be how to study the behavior of the solutions in the case when

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N.H. Taieb and M.A. Hammami are with University of Sfax, Tunisia, Faculty of Sciences of Sfax, Department of Mathematics. E-mails: Nizar.HadjTaieb@yahoo.fr, MohamedAli.Hammami@fss.rnu.tn. F. Delmotte is with University of Artois, France, e-mail: francois.delmotte@univ-artois.fr. Corresponding author is M.A. Hammani.

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they converge to a small neighborhood of the origin. To deal with these situation, the concept of practical stability ([12], [13], [16]), which is derived from the so called, finite time stability, is more useful. Indeed, the practical stability for nonlinear systems has been widely investigated in mathematical theory. In these studies, the origin was not supposed to be an equilibrium point of the system. So, we can no longer expect to design a controller that guarantees the stability of the origin as an equilibrium point. In [3], [17] and [18], some controllers are constructed to guarantee exponential stability of a ball containing the origin of the state space where the radius of this ball can be made arbitrary small. The authors in [33], introduced the notion of input to state practical stability to design of robust adaptive controllers for nonlinear systems with dynamic uncertainties. In [32], the concept of input to state practical stability is extended to stochastic case and an output feedback controller is proposed for a class of stochastic nonlinear systems with uncertain nonlinear functions. By using the fuzzy approach the authors in [14], have investigated the practical stability of a class of uncertain T-S fuzzy systems where the uncertainties satisfy the so called matching conditions.

In this paper, we deal with the uniform ultimate boundedness for a class of Takagi-Sugeno fuzzy systems in presence of external disturbances. The objective is to guarantee, no matter how we select the uncertain external disturbances, that the state will eventually end up and remain within some pre-specified region. When this region is a small neighborhood about the origin, the concept of uniform ultimate boundedness is equivalent to practical stability. Therefore, we are interested in studying the global uniform practical exponential stability for a class of uncertain Takagi-Sugeno fuzzy systems in term of convergence toward a neighborhood of the origin. The main novelty of this paper relies on the fact that the proposed approach for stability analysis allows for the computation of the bound which characterize the exponential rate of convergence of the solutions. The common quadratic Lyapunov function and parallel distributed compensation controller are used to show the ultimate boundedness of the solutions of the uncertain T-S fuzzy systems, even when the origin is not an equilibrium point of the system, provided that the uncertainties are supposed uniformly bounded by known integrable functions. Compared to classical LMIs conditions, the new LMIs are a little bit more severe in order to handle the uncertainties. Then, it is possible to prove systems performance by adjusting the practical stability conditions.

The remainder of this paper is organized as follows: section 2 reviews the conventional T-S fuzzy model and issues about stability. Section 3 presents the global uniform practical exponential stability for T-S fuzzy uncertain systems in term of convergence toward a neighborhood of the origin, furthermore new LMIs are presented in order to handel the uncertainties. Section 4 presents the numerical examples.

2. Takagi-Sugeno fuzzy systems

Consider a class of the continuous-time T-S fuzzy control system which can be described by the following fuzzy rules,

$$\text{Rule } i: \text{ If } z_1(t) \text{ is } M_{i1} \text{ and } z_2(t) \text{ is } M_{i2} \dots \text{and } z_p(t) \text{ is } M_{ip}, \text{ then}$$

$$\dot{x}(t) = A_i x(t) + B_i u(t), \; i = 1,2,\dots,r,$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are the system matrix input matrix, $i = 1,\dots,r$ is the number of fuzzy rules, M_{ij} are the inputs fuzzy sets, $z(t) = [z_1(t),\dots,z_p(t)]^T$ are measurable variables, i.e., the premise variables. Using weighted average defuzzifiers, the aggregated fuzzy model is given by

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} w_i(z)\big(A_i x(t) + B_i u(t)\big)}{\sum_{i=1}^{r} w_i(z)},$$

where

$$w_i(z) = \prod_{i=1}^{r} M_{ij}(z_j).$$

Let $\mu_i(z)$ be the membership functions that belong to class $\mathcal{C}^1$, i,e., they are continuous differentiable and defined as

$$\mu_i(z) = \frac{w_i(z)}{\sum_{i=1}^{r} w_i(z)}.$$

Then the fuzzy system has the state-space form

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big(A_i x(t) + B_i u(t)\big). \tag{1}$$

μ_i are such that $\mu_i(z) \geqslant 0$ for $i = 1,2,\dots,r$ and $\sum_{i=1}^{r} \mu_i(z) = 1$.

Many published results, concerning the control of the fuzzy system, are based on the PDC principle. The design of the fuzzy controller shares the same antecedent as the fuzzy system and employs a linear state feedback control in the consequent part. For each local dynamics the controller is defined as

$$\text{Rule } i: \text{ If } z_1(t) \text{ is } M_{i1} \text{ and } z_2(t) \text{ is } M_{i2} \dots \text{and } z_p(t) \text{ is } M_{ip}, \text{ then}$$

$$u(t) = -K_i x(t), \; i = 1,2,\dots,r, \tag{2}$$

where K_i is the local state feedback gain. Consequently, the defuzzified result is

$$u(t) = -\sum_{i=1}^{r} \mu_i(z) K_i x(t). \tag{3}$$

The system (1) in closed-loop with the fuzzy controller (3) yields the following fuzzy system,

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z)\mu_j(z)\big(A_i - B_i K_j\big)x(t). \tag{4}$$

A sufficient condition for the stability is deduced using Lyapunov's direct method. Suppose that a common positive definite matrix P exists, so that the following conditions are satisfied [9].

$$(A_i - B_i K_i)^T P + P(A_i - B_i K_i) < 0, \ i = 1, 2, \ldots, r,$$

and

$$\frac{1}{2}(A_i - B_i K_j + A_j - B_j K_i)^T P + \frac{1}{2}P(A_i - B_i K_j + A_j - B_j K_i) < 0, \ 1 \leqslant i < j \leqslant r.$$

When these conditions are satisfied, the fuzzy system (4) is asymptotically stable. The design work can be transformed into a convex problem [8], which is efficiently solved by linear matrix inequalities optimization. If the solution is feasible, meaning that the stabilization constraints are met, then local state feedback gains are obtained. Relaxed results on stabilization and state feedback H_∞ Control conditions for T-S Fuzzy systems were given in [24].

3. Control of uncertain fuzzy systems

Motivated by the results of the above section concerning the control of fuzzy-model, we will extend the T-S fuzzy system with the presence of external disturbances [21]. Consider the following T-S fuzzy uncertain model,

Rule i : If $z_1(t)$ is M_{i1} and $z_2(t)$ is M_{i2} ... and $z_p(t)$ is M_{ip}, then

$$\dot{x}(t) = A_i x(t) + B_i u(t) + f_i(t, x(t)), \ i = 1, 2, \ldots, r. \tag{5}$$

The fuzzy system is then inferred to be

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\Big(A_i x(t) + B_i u(t) + f_i(t, x(t))\Big). \tag{6}$$

The function f_i represent the uncertain external disturbance of each fuzzy subsystem and are time-varying satisfying the following inequality,

$$\|f_i(t, x(t))\| \leqslant \alpha_i(t)\|x(t)\| + \beta_i(t), \ \ i = 1, 2, \ldots, r, \tag{7}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}^n$, where α_i and β_i are known nonnegative continuous functions.

Remark 2 Inequality (7) means that the time-varying function f_i may be bounded and/or unbounded on time. For the knowledge of the authors, this is new.

Suppose the following assumption,

$(\mathcal{H}_1)$ The pairs (A_i, B_i), $i = 1, \ldots, r$, are controllable, that is each nominal local model is controllable.

The fuzzy control rule is defined as above and we will consider the fuzzy uncertain system (6) Therefore, the closed-loop system with respect the fuzzy control (2 - 3) is given by

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z)\mu_j(z)\big(A_i - B_iK_j\big)x(t) + \sum_{i=1}^{r}\mu_i(z)f_i(t, x(t)). \tag{8}$$

Thus,

$$\dot{x}(t) = \sum_{i=1}^{r}\mu_i^2 G_{ii}x(t) + 2\sum_{i<j}\mu_i\mu_j G_{ij}x(t) + \sum_{i=1}^{r}\mu_i f_i(t, x(t)),$$

where

$$G_{ii} = A_i - B_iK_i$$

and

$$G_{ij} = \frac{1}{2}(A_i - B_iK_j + A_j - B_jK_i).$$

The controller synthesis initially considers the stability of the local fuzzy dynamics. That is, the stable feedback gains are determined for every subsystem. Suppose that there exist positive symmetric and definite matrices P, Q_i, and Q_{ij} $(i < j)$, and some matrices K_i, $i = 1, \ldots, r$, such that the following inequalities [15] hold,

$$G_{ii}^T P + PG_{ii} < -Q_i,\ i = 1, 2, \ldots, r, \tag{9}$$

and

$$G_{ij}^T P + PG_{ij} < -Q_{ij},\ 1 \leqslant i < j \leqslant r. \tag{10}$$

Based on this assumption, each nominal local model is controllable and a suitable feedback gain can be obtained.

As a first step, we need to recall what is meant by uniformly ultimately bounded and uniform global practical exponential stability of dynamic systems ([2], [4], [5]). Consider a system described by

$$\dot{x} = F(t, x) \tag{11}$$

with $t \in \mathbb{R}_+$ is the time and $x \in \mathbb{R}^n$ is the state.

Definition 1 *The system* (11) *is said uniformly ultimately bounded if there exists* $R > 0$, *such that for all* $R_1 > 0$, *there exists a* $T = T(R_1) > 0$ *such that*

$$\|x(t_0)\| \leqslant R_1 \Rightarrow \|x(t)\| \leqslant R \ \text{ for all } \ t \geqslant t_0 + T \ \text{ and } \ t_0 \geqslant 0.$$

Definition 2 *The system* (11) *is said uniformly globally practically exponentially stable, if there exists a ball*

$$\mathcal{B}_\eta = \{x \in \mathbb{R}^n \ / \ \|x\| \leqslant \eta\},$$

such that $\mathcal{B}_\eta$ *is uniformly globally practically exponentially stable, it means that, there exists* $\eta > 0$ *such that, for all* $\varepsilon > \eta$, *there exists* $\varepsilon = \varepsilon(\varepsilon) > 0$ *such that, for all* $t_0 \geqslant 0$, $\|x(t_0)\| \leqslant \varepsilon$, *we have*

$$\|x(t)\| \leqslant \gamma \|x(t_0)\| e^{-\upsilon(t-t_0)} + \eta, \quad \textit{for all } \ t \geqslant t_0, \tag{12}$$

with $\gamma > 0$, $\upsilon > 0$.

The inequality (12) implies that $x(t)$ will be bounded by a small bound $\eta > 0$, that is, $\|x(t)\|$ will be small for sufficiently large t. It means that (11) will be uniformly ultimately bounded for sufficiently large t. If in (12) η can be replaced by a smooth map $\eta(t)$ as a function of t which tends to zero as t tends to $+\infty$, then the ultimate bound approaches to zero.

Remark 3 The goal of this paper is to find some conditions on the functions $\alpha_i(t)$ and $\beta_i(t)$ such that the fuzzy system (8) is globally uniformly practically exponentially stable. If $\beta_i(t) = 0$, for all $i = 1, \ldots, r$, the fuzzy uncertain system (8) has an equilibrium point at the origin. In this case, we can analyze the stability of the closed-loop system behavior for the origin as an equilibrium point. If $\beta_i(t) \neq 0$, for some $i = 1, \ldots, r$, then the origin can will not be an equilibrium point of the fuzzy uncertain system (8). In this case, we study the convergence of the solutions toward a neighborhood of the origin.

Let

$$\alpha(t) := \left(\sum_{i=1}^{r} \alpha_i(t)^2 \right)^{\frac{1}{2}},$$

such that α is bounded and to satisfy: there exists M_α a positive scalar constant satisfy,

$$\int_0^{+\infty} \alpha(t)\, dt \leqslant M_\alpha < +\infty.$$

In the first part, let consider the following assumption,

$(\mathcal{H}_2)$ There exists M_β a positive scalar constant satisfy,

$$\int_0^{+\infty} \beta^2(t) dt \leqslant M_\beta < +\infty,$$

where

$$\beta(t) := \left(\sum_{i=1}^{r} \beta_i(t)^2 \right)^{\frac{1}{2}}.$$

To find an estimation as in (12), we will impose a restriction on the upper bound of the uncertain term formulated in the following condition,

$(\mathcal{H}_3)$

$$\alpha(t) < \frac{1}{2}\frac{\lambda_0}{\lambda_{\max}(P)} \tag{13}$$

where $\lambda_0 = \inf\{(\lambda_{\min}(Q_i); i = 1, \ldots, r); (\lambda_{\min}(Q_{ij}); 1 \leqslant i < j \leqslant r)\}$, $\lambda_{\min(\max)}$ denotes the smallest (largest) eigenvalue of the matrix.

Remark 4 The inequality (13) is equivalent to the following LMIs,

$$P < \frac{1}{2\alpha(t)} Q_i, \quad i = 1, \ldots, r, \tag{14}$$

and

$$P < \frac{1}{2\alpha(t)} Q_{ij}, \quad 1 \leqslant i < j \leqslant r. \tag{15}$$

Compared to classical LMI conditions, the new LMIs are a little bit more sever in order to handle the time varying uncertain term.

Remark 5 The matrices P, Q_i, Q_{ij} $(i < j)$ and K_i can be obtained using the following LMIs,

$$X > 0,$$

$$X < \frac{1}{2\alpha(t)} X Q_i X, \quad i = 1, \ldots, r,$$

$$X < \frac{1}{2\alpha(t)} X Q_{ij} X, \quad 1 \leqslant i < j \leqslant r,$$

$$X A_i^T + A_i X - M_i^T B_i^T - B_i M_i < -X Q_i X, \quad i = 1, \ldots, r,$$

$$X A_i^T + A_i X + X A_j^T + A_j X - M_j^T B_i^T - M_i^T B_j^T - B_i M_j - B_j M_i < -2 X Q_{ij} X, \quad 1 \leqslant i < j \leqslant r,$$

and

$$\begin{bmatrix} Q_1 & Q_{12} & \cdots & Q_{1r} \\ Q_{12} & Q_{22} & & \vdots \\ \vdots & & \ddots & Q_{r(r-1)} \\ Q_{1r} & \cdots & Q_{r(r-1)} & Q_r \end{bmatrix} > 0,$$

where $X = P^{-1}$, $K_i = M_i P$.

Now, one can state the following theorem.

Theorem 3 *Suppose that the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *hold and there exist a common positive definite matrix* P *and some feedback gain matrices* K_i, $i = 1,\ldots,r$, *such that the stability conditions* (9-10) *are satisfied, then the fuzzy closed-loop system* (8) *with the control laws* (2-3) *is guaranteed to be globally uniformly practically exponentially stable.*

Proof. Consider the Lyapunov function candidate $V(t,x) = x^T Px$. It's derivative with respect to time is given by,

$$\dot{V}(t,x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + PG_{ii})x + 2\sum_{i<j}^{r} \mu_i\mu_j x^T (G_{ij}^T P + PG_{ij})x + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

The first two terms on the right-hand side constitute the derivative of the Lyapunov function $V(x)$ with respect the nominal system, while the third term is the effect of the perturbation. On the one hand, we have

$$x^T (G_{ii}^T P + PG_{ii})x \leqslant -\lambda_{min}(Q_i)\|x\|^2, \ i = 1,2,\ldots,r,$$

and

$$x^T (G_{ij}^T P + PG_{ij})x \leqslant -\lambda_{min}(Q_{ij})\|x\|^2, \ 1 \leqslant i < j \leqslant r.$$

It follows that,

$$\dot{V}(t,x) \leqslant -\sum_{i=1}^{r} \mu_i^2 \lambda_{min}(Q_i)\|x\|^2 - 2\sum_{i<j}^{r} \mu_i\mu_j \lambda_{min}(Q_{ij})\|x\|^2 + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

Thus,

$$\dot{V}(t,x) \leqslant -\Big(\sum_{i=1}^{r} \mu_i^2 \lambda_{min}(Q_i) + 2\sum_{i<j}^{r} \mu_i\mu_j \lambda_{min}(Q_{ij})\Big)\|x\|^2 + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

Then, one gets

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 \sum_{i=1}^{r}\sum_{i=1}^{r} \mu_i\mu_j + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

Since,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j = 1,$$

then, we have

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

On the other hand, we have

$$\|\sum_{i=1}^{r} \mu_i f_i(t,x(t))\| \leqslant \sum_{i=1}^{r} \mu_i(\alpha_i(t)\|x\| + \beta_i(t)).$$

Taking into account the above expressions, it follows that

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\|x\|\|P\|\sum_{i=1}^{r}\mu_i(\alpha_i(t)\|x\| + \beta_i(t)).$$

Thus, by using the Cauchy-Schwartz inequality, one has

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\|x\|\|P\|\left((\sum_{i=1}^{r}\mu_i^2)^{\frac{1}{2}}(\sum_{i=1}^{r}\alpha_i(t)^2)^{\frac{1}{2}}\|x\| + (\sum_{i=1}^{r}\mu_i^2)^{\frac{1}{2}}(\sum_{i=1}^{r}\beta_i(t)^2)^{\frac{1}{2}}\right).$$

It follows that,

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\|P\|(\sum_{i=1}^{r}\alpha_i(t)^2)^{\frac{1}{2}}\|x\|^2 + 2\|P\|(\sum_{i=1}^{r}\beta_i(t)^2)^{\frac{1}{2}}\|x\|.$$

Hence,

$$\dot{V}(t,x) \leqslant -\left(\lambda_0 - 2\|P\|(\sum_{i=1}^{r}\alpha_i(t)^2)^{\frac{1}{2}}\right)\|x\|^2 + 2\|P\|(\sum_{i=1}^{r}\beta_i(t)^2)^{\frac{1}{2}}\|x\|.$$

Since,

$$\lambda_{min}(P)\|x\|^2 \leqslant V(t,x) = x^T Px \leqslant \lambda_{max}(P)\|x\|^2,$$

then, by taking $\|P\| = \lambda_{max}(P)$, yields

$$\dot{V}(t,x) \leqslant -\frac{1}{\lambda_{max}(P)}\left(\lambda_0 - 2\lambda_{max}(P)\alpha(t)\right)V(t,x) + 2\frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\beta(t)V(t,x)^{\frac{1}{2}}.$$

Let,

$$a(t) = \frac{1}{\lambda_{max}(P)}\left(\lambda_0 - 2\lambda_{max}(P)\alpha(t)\right),$$

$$b(t) = 2\frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\beta(t).$$

With the previous notations, it follows that

$$\dot{V}(t,x) \leqslant -a(t)V(t,x) + b(t)V(t,x)^{\frac{1}{2}}.$$

In the last expression, we make the following change of variable, $w(t) = V(t,x)^{\frac{1}{2}}$. The derivative with respect to time is given by

$$\dot{w}(t) = \frac{\dot{V}(t,x)}{2V(t,x)^{\frac{1}{2}}}.$$

This implies that,

$$\dot{w}(t) \leqslant -\frac{1}{2}a(t)w(t) + \frac{1}{2}b(t).$$

Letting

$$z(t) = w(t)e^{\frac{1}{2}\int_{t_0}^{t} a(s)ds}$$

it follows that,

$$\dot{z}(t) = \left(\dot{w}(t) + \frac{1}{2}a(t)w(t)\right)e^{\frac{1}{2}\int_{t_0}^{t} a(s)ds} \leqslant \frac{1}{2}b(t)e^{\frac{1}{2}\int_{t_0}^{t} a(s)ds}.$$

Integrating between t_0 and t, one obtains for all $t \geqslant t_0$,

$$z(t) \leqslant z(t_0) + \frac{1}{2}\int_{t_0}^{t} b(s)e^{\frac{1}{2}\int_{t_0}^{s} a(\xi)d\xi}ds.$$

By the fact that $z(t) = w(t)e^{\frac{1}{2}\int_{t_0}^{t} a(s)ds}$, we obtain

$$w(t) \leqslant w(t_0)e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} + \frac{1}{2}\left(\int_{t_0}^{t} b(s)e^{\frac{1}{2}\int_{t_0}^{s} a(\xi)d\xi}ds\right)e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds}. \tag{16}$$

Using the forms of $a(t)$ and $b(t)$, we first compute $-\frac{1}{2}\int_{t_0}^{t} a(s)ds$.

$$-\frac{1}{2}\int_{t_0}^{t} a(s)ds = -\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0) + \int_{t_0}^{t} \alpha(s)ds \leqslant -\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0) + M_\alpha.$$

It follows that, the first term on the right-hand side of (3.12) satisfies,

$$e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} \leqslant e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)}. \tag{17}$$

Next, consider the second term on the right-hand side of (3.12). We have,

$$\frac{1}{2}\left(\int_{t_0}^{t} b(s)e^{\frac{1}{2}\int_{t_0}^{s} a(\xi)d\xi}ds\right)e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds}$$
$$= \left(\int_{t_0}^{t} \frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\beta(s)e^{\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(s-t_0)-\int_{t_0}^{s}\alpha(\xi)d\xi}ds\right)e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)}.$$

Note that, since $\alpha(t) \geqslant 0$ for all $t \geqslant 0$, then it is clear that

$$e^{-\int\limits_{t_0}^{t} \alpha(\xi)d\xi} \leqslant 1.$$

Thus,

$$\frac{1}{2}\left(\int\limits_{t_0}^{t} b(s)e^{\frac{1}{2}\int_{t_0}^{s} a(\xi)d\xi}ds\right)e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} \leqslant \left(\int\limits_{t_0}^{t} \frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\beta(s)e^{\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(s-t_0)}ds\right)e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)}$$

$$\leqslant \frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)}\left(\int\limits_{t_0}^{t}(\beta(s))^2ds\right)^{\frac{1}{2}}\left(\int\limits_{t_0}^{t} e^{\frac{\lambda_0}{\lambda_{max}(P)}(s-t_0)}ds\right)^{\frac{1}{2}}.$$

Hence,

$$\frac{1}{2}\left(\int\limits_{t_0}^{t} b(s)e^{\frac{1}{2}\int_{t_0}^{s} a(\xi)d\xi}ds\right)e^{-\frac{1}{2}\int_{t_0}^{t} a(s)ds} \leqslant M_\beta^{\frac{1}{2}}e^{M_\alpha}\frac{\lambda_{max}^{\frac{3}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)\lambda_0^{\frac{1}{2}}}. \tag{18}$$

The inequality (16) in conjunction with (17) and (18), yields

$$w(t) \leqslant w(t_0)e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)} + M_\beta^{\frac{1}{2}}e^{M_\alpha}\frac{\lambda_{max}^{\frac{3}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)\lambda_0^{\frac{1}{2}}}.$$

It follows that,

$$V(t,x)^{\frac{1}{2}} \leqslant V(t_0,x(t_0))^{\frac{1}{2}}e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)} + M_\beta^{\frac{1}{2}}e^{M_\alpha}\frac{\lambda_{max}^{\frac{3}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)\lambda_0^{\frac{1}{2}}}.$$

Therefore,

$$\|x(t)\| \leqslant \frac{\lambda_{max}^{\frac{1}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}e^{M_\alpha}\|x(t_0)\|e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)} + M_\beta^{\frac{1}{2}}e^{M_\alpha}\frac{\lambda_{max}^{\frac{3}{2}}(P)}{\lambda_{min}(P)\lambda_0^{\frac{1}{2}}}.$$

Hence, we obtain an estimation as in (12) with

$$\gamma = \frac{\lambda_{max}^{\frac{1}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}e^{M_\alpha},$$

$$\upsilon = \frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)},$$

and

$$\eta_f = M_\beta^{\frac{1}{2}} e^{M_\alpha} \frac{\lambda_{max}^{\frac{3}{2}}(P)}{\lambda_{min}(P)\lambda_0^{\frac{1}{2}}}.$$

Therefore, $\mathcal{B}_{\eta_f}$ is globally uniformly practically exponentially stable. □

Remark 6 This bound can be minimized by solving the following optimization problem: Find P, Q_i, Q_{ij} $i<j$ and K_i, $i,j=1,\ldots,r$, and maximize ε_1, ε_2, ε_3, ε_4, ε_5 and ε_6 subject to:

$$P = P^T > 0, \quad P > \varepsilon_1 I, \quad -P > -\varepsilon_2 I,$$

$$P < \frac{1}{2\alpha(t)} Q_i - \varepsilon_3 I, \quad i = 1,\ldots,r,$$

$$P < \frac{1}{2\alpha(t)} Q_{ij} - \varepsilon_4 I, \quad 1 \leqslant i < j \leqslant r,$$

$$G_{ii}^T P + P G_{ii} < -Q_i - \varepsilon_5 I, \; i = 1,\ldots,r,$$

$$G_{ij}^T P + P G_{ij} < -Q_{ij} - \varepsilon_6 I, \; 1 \leqslant i < j \leqslant r,$$

and

$$\begin{bmatrix} Q_1 & Q_{12} & \cdots & Q_{1r} \\ Q_{12} & Q_{22} & & \vdots \\ \vdots & & \ddots & Q_{r(r-1)} \\ Q_{1r} & \cdots & Q_{r(r-1)} & Q_r \end{bmatrix} > 0.$$

Where I is the matrix identity.

Remark 7 Compared to the existing results, such as the input-output methods and slack matrix method as in ([10], [11], [25]), in this work the quadratic Lyapunov function and the PDC controller techniques can be used to show the ultimate boundedness of the solutions of the uncertain T-S fuzzy systems, even when the origin is not an equilibrium point of the system. Therefore, we can study the convergence of the solutions toward a neighborhood of the origin and this is what we mean by practical stability.

In the second part, we suppose the following assumption.

$(\mathcal{H}'_2)$

$$\delta(t) \leqslant M_\delta, \;\; \text{for all} \;\; i = 1,2,\ldots,r \;\; \text{and} \;\; t \geqslant 0, \tag{19}$$

where

$$\delta(t) := \sum_{i=1}^{r} \mu_i \beta_i(t)$$

and M_δ is a positive constant.

Theorem 4 *Suppose that the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H'}_2)$ *and* $(\mathcal{H}_3)$ *hold and there exist a common positive definite matrix* P *and some feedback gain matrices* K_i, $i = 1,\dots,r$, *such that the stability conditions* (9 - 10) *are satisfied, then the fuzzy closed-loop system* (8) *with the control laws* (2 - 3) *is guaranteed to be globally uniformly practically exponentially stable.*

Proof. Let consider the Lyapunov function candidate $V(t,x) = x^T Px$. It's derivative with respect to time is given by,

$$\dot{V}(t,x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + PG_{ii})x + 2\sum_{i<j}^{r} \mu_i\mu_j x^T (G_{ij}^T P + PG_{ij})x + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)),$$

then, we have

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

Thus, by using the following inequality,

$$\|f_i(t,x(t))\| \leqslant \alpha_i(t)\|x\| + \beta_i(t),$$

one has

$$\dot{V}(t,x) \leqslant -\Big(\lambda_0 - 2\|P\|(\sum_{i=1}^{r} \alpha_i(t)^2)^{\frac{1}{2}}\Big)\|x\|^2 + 2M_\delta\|P\|\|x\|.$$

Then, by taking $\|P\| = \lambda_{max}(P)$, yields

$$\dot{V}(t,x) \leqslant -\frac{1}{\lambda_{max}(P)}\Big(\lambda_0 - 2\lambda_{max}(P)\alpha(t)\Big)V(t,x) + 2M_\delta \frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)} V(t,x)^{\frac{1}{2}}.$$

By using the same idea as in the proof of theorem 1, we obtain the following estimation

$$\|x(t)\| \leqslant \frac{\lambda_{max}^{\frac{1}{2}}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\|x(t_0)\|e^{M_\alpha}e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)} + 2M_\delta e^{M_\alpha}\frac{\lambda_{max}^2(P)}{\lambda_{min}(P)\lambda_0}.$$

It follows that,

$$\mathcal{B}_\eta = \{x \in \mathbb{R}^n \ / \ \|x\| \leqslant \eta = 2M_\delta e^{M_\alpha}\frac{\lambda_{max}^2(P)}{\lambda_{min}(P)\lambda_0}\},$$

is globally uniformly practically exponentially stable. □

Motivated by the above results, the design principle can be extended to the T-S fuzzy system with parametric uncertainties. Indeed, one can consider the following T-S fuzzy uncertain model,

Rule i: If $z_1(t)$ is M_{i1} and $z_2(t)$ is M_{i2} ... and $z_p(t)$ is M_{ip}, then

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + B_i u(t) + f_i(t,x(t)),\ i = 1,2,\dots,r. \tag{20}$$

Notably, the model is almost the same as (5) except for the term ΔA_i which stand for the parametric uncertainties for each subsystem and time-varying with appropriate dimensions. The fuzzy system is then inferred to be

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big((A_i + \Delta A_i)x(t) + B_i u(t) + f_i(t, x(t))\big). \tag{21}$$

Then, let us consider the following assumptions.

$(\mathcal{H}_4)$ The parametric uncertainties ΔA_i is norm bounded and structured, in the form

$$\Delta A_i = \rho_i(t) D_i E_i(t) F_i,$$

where D_i, and F_i are known real constant matrices with appropriate dimensions, $E_i(t)$, is unknown matrix function which satisfy,

$$E_i^T(t)E_i(t) \leqslant I, \quad \text{and} \quad E_i(t)E_i^T(t) \leqslant I \quad \text{for all} \quad t \geqslant 0,$$

and $\rho_i(t)$ is a known continuous nonnegative scalar function and I is the identity matrix of appropriate dimension. Let

$$\rho(t) := \left(\sum_{i=1}^{r} \rho_i^2(t)\right)^{\frac{1}{2}}$$

such that, there exists M_ρ a positive scalar constant satisfy,

$$\int_0^{+\infty} \rho(t)dt \leqslant M_\rho < +\infty.$$

$(\mathcal{H}_5)$ We suppose that $\rho(t)$ satisfies the following restriction,

$$\Big(\lambda_0 - 2\sigma_1(\lambda_{\max}^2(P) + \sigma_2)\rho(t) - 2\lambda_{\max}(P)\alpha(t)\Big) > 0, \;\; \text{for all} \;\; t \geqslant 0, \tag{22}$$

where $\lambda_0 = \inf\{(\lambda_{\min}(Qi); i = 1, \ldots, r), (\lambda_{\min}(Qij); 1 < i \leqslant j < r)\}$, $\sigma_1 = \max(\|D_i\|^2, i = 1, \ldots, r)$ and $\sigma_2 = \max(\|F_i\|^2, i = 1, \ldots, r)$.

Remark 8 The inequality (22) is equivalent to the following LMIs,

$$P < \frac{1}{2\sigma\rho(t)} Q_i, \quad i = 1, \ldots, r, \tag{23}$$

$$P < \frac{1}{2\alpha(t)} Q_i, \quad i = 1, \ldots, r, \tag{24}$$

$$P < \frac{1}{2\sigma\rho(t)} Q_{ij}, \quad 1 \leqslant i < j \leqslant r, \tag{25}$$

and

$$P < \frac{1}{2\alpha(t)} Q_{ij}, \quad 1 \leqslant i < j \leqslant r, \tag{26}$$

where $\sigma = \inf(\sigma_1, \sigma_2)$.

Remark 9 Similar to the remark (8), in this case the matrices P, Q_i, Q_{ij} $(i < j)$ and K_i can be obtained using the following LMIs,

$$X > 0,$$

$$X < \frac{1}{2\alpha(t)} X Q_i X, \quad i = 1, \dots, r,$$

$$X < \frac{1}{2\sigma\rho(t)} X Q_i X, \quad i = 1, \dots, r,$$

$$X < \frac{1}{2\alpha(t)} X Q_{ij} X, \quad 1 \leqslant i < j \leqslant r,$$

$$X < \frac{1}{2\sigma\rho(t)} X Q_{ij} X, \quad 1 \leqslant i < j \leqslant r,$$

$$X A_i^T + A_i X - M_i^T B_i^T - B_i M_i < -X Q_i X, \quad i = 1, \dots, r,$$

$$X A_i^T + A_i X + X A_j^T + A_j X - M_j^T B_i^T - M_i^T B_j^T - B_i M_j - B_j M_i < -2 X Q_{ij} X, \quad 1 \leqslant i < j \leqslant r,$$

and

$$\begin{bmatrix} Q_1 & Q_{12} & \cdots & Q_{1r} \\ Q_{12} & Q_{22} & & \vdots \\ \vdots & & \ddots & Q_{r(r-1)} \\ Q_{1r} & \cdots & Q_{r(r-1)} & Q_r \end{bmatrix} > 0,$$

where $X = P^{-1}$, $K_i = M_i P$.

Then, let consider the following theorem.

Theorem 5 *Suppose that the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_4)$ *and* $(\mathcal{H}_5)$ *hold and there exist a common positive definite matrix P and some feedback gain matrices K_i, $i = 1, \dots, r$, such that the stability conditions* (9-10) *are satisfied, then the fuzzy closed-loop system* (21) *with the control laws* (2-3) *is guaranteed to be globally uniformly practically exponentially stable.*

Proof. Consider the Lyapunov function candidate $V(t,x) = x^T Px$. It's derivative with respect to time is given by,

$$\dot{V}(t,x) = \sum_{i=1}^{r} \mu_i^2 x^T (G_{ii}^T P + PG_{ii})x + 2\sum_{i<j}^{r} \mu_i\mu_j x^T (G_{ij}^T P + PG_{ij})x + 2x^T \sum_{i=1}^{r} \mu_i P\Delta A_i x$$
$$+ 2x^T P \sum_{i=1}^{r} \mu_i f_i(t,x(t)).$$

By the fact that,

$$\begin{aligned} 2x^T P\Delta A_i x &= x^T (\Delta A_i^T P + P\Delta A_i)x \\ 2x^T P\Delta A_i x &\leqslant \rho_i(t)x^T PD_iD_i^T Px + \rho_i(t)F_i^T F_i x \\ &\leqslant \rho_i(t)x^T (PD_iD_i^T P + F_i^T F_i)x. \end{aligned}$$

we have

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\sum_{i=1}^{r} \mu_i\rho_i(t)\|P\|^2\|D_i\|^2\|x\|^2 + 2\sum_{i=1}^{r} \mu_i\rho_i(t)\|F_i\|^2\|x\|^2$$
$$+ 2\lambda_{max}(P)\alpha(t)\|x\|^2 + 2\lambda_{max}(P)\beta(t)\|x\|.$$

Then,

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\sigma_1\lambda_{max}^2(P)\sum_{i=1}^{r} \mu_i\rho_i(t)\|x\|^2 + 2\sigma_2\sum_{i=1}^{r} \mu_i\rho_i(t)\|x\|^2$$
$$+ 2\lambda_{max}(P)\alpha(t)\|x\|^2 + 2\lambda_{max}(P)\beta(t)\|x\|.$$

By using the Cauchy-Schwartz inequality, one has

$$\dot{V}(t,x) \leqslant -\lambda_0\|x\|^2 + 2\sigma_1\lambda_{max}^2(P)\rho(t)\|x\|^2 + 2\sigma_2\rho(t)\|x\|^2 + 2\lambda_{max}(P)\alpha(t)\|x\|^2$$
$$+ 2\lambda_{max}(P)\beta(t)\|x\|.$$

It follows that,

$$\dot{V}(t,x) \leqslant -\Big(\lambda_0 - 2\big(\sigma_1\lambda_{max}^2(P) + \sigma_2\big)\rho(t) - 2\lambda_{max}(P)\alpha(t)\Big)\|x\|^2$$
$$+ 2\lambda_{max}(P)\beta(t)\|x\|.$$

Then,

$$\dot{V}(t,x) \leqslant -\frac{1}{\lambda_{max}(P)}\Big(\lambda_0 - 2\big(\sigma_1\lambda_{max}^2(P) + \sigma_2\big)\rho(t) - 2\lambda_{max}(P)\alpha(t)\Big)V(t,x)$$
$$+2\frac{\lambda_{max}(P)}{\lambda_{min}^{\frac{1}{2}}(P)}\beta(t)V(t,x)^{\frac{1}{2}}.$$

Let,

$$a(t) = \frac{1}{\lambda_{max}(P)}\Big(\lambda_0 - 2\big(\sigma_1\lambda^2_{max}(P)+\sigma_2\big)\rho(t) - 2\lambda_{max}(P)\alpha(t)\Big),$$

$$b(t) = 2\frac{\lambda_{max}(P)}{\lambda^{\frac{1}{2}}_{min}(P)}\beta(t).$$

It follows that

$$\dot{V}(t,x) \leqslant -a(t)V(t,x) + b(t)V(t,x)^{\frac{1}{2}}.$$

Using the same idea as in the proofs of theorems 1 and 2 we obtain the following estimation of the state,

$$\|x(t)\| \leqslant \frac{\lambda^{\frac{1}{2}}_{max}(P)}{\lambda^{\frac{1}{2}}_{min}(P)} e^{\rho M_\rho} e^{M_\alpha} \|x(t_0)\| e^{-\frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)}(t-t_0)} + M_\beta^{\frac{1}{2}} e^{\rho M_\rho} e^{M_\alpha} \frac{\lambda^{\frac{3}{2}}_{max}(P)}{\lambda_{min}(P)\lambda_0^{\frac{1}{2}}},$$

where

$$\rho = \frac{\sigma_1\lambda^2_{max}(P)+\sigma_2}{\lambda_{max}(P)}.$$

Here, we obtain an estimation as in (12) with

$$\gamma = \frac{\lambda^{\frac{1}{2}}_{max}(P)}{\lambda^{\frac{1}{2}}_{min}(P)} e^{\rho M_\rho} e^{M_\alpha},$$

$$\upsilon = \frac{1}{2}\frac{\lambda_0}{\lambda_{max}(P)},$$

and

$$\eta_{\rho\alpha} = M_\beta^{\frac{1}{2}} e^{\rho M_\rho} e^{M_\alpha} \frac{\lambda^{\frac{3}{2}}_{max}(P)}{\lambda_{min}(P)\lambda_0^{\frac{1}{2}}}.$$

So, $\mathcal{B}_{\eta_{\rho\alpha}}$ is uniformly globally practically exponentially stable. □

Corollary 1 *If we suppose that the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}'_2)$, $(\mathcal{H}_4)$ *and* $(\mathcal{H}_5)$ *hold and there exist a common positive definite matrix P and some feedback gain matrices K_i, $i = 1,\ldots,r$, such that the stability conditions* (9-10) *are satisfied, then the fuzzy closed-loop system* (21) *with the control laws* (2-3) *is guaranteed to be uniformly globally practically exponentially stable such that the ball ,*

$$\mathcal{B}_{\eta_{\rho\alpha}} = \{x \in \mathbb{R}^n \ / \ \|x\| \leqslant \eta = 2M_\delta e^{\rho M_\rho} e^{M_\alpha} \frac{\lambda^2_{max}(P)}{\lambda_{min}(P)\lambda_0}\},$$

is globally uniformly practically exponentially stable.

According to the above analysis, the design procedure for uncertain Takagi-Sugeno fuzzy systems is summarized as follows.

Step 1: Confirm that assumption $(\mathcal{H}_1)$ is satisfied for the designed system.
Step 2: Verify that the functions $\alpha(t)$, $\rho(t)$ and $\beta(t)$ satisfy the assumptions of integrability.
Step 3: Solve the LMI problem indicated in remark 3.8 and obtain P, Q_i, Q_{ij} $(i < j)$, and K_i, $i = 1, \ldots, r$.
Step 4: Simulate the system in order to plot its trajectories.

4. Simulation examples

To illustrate the proposed fuzzy control approach we propose the following examples.

Example 1 Consider a flexible-joint robot arm. The system is described by the following equations ([28]):

$$I_1\ddot{\theta}_1(t) + mglsin(\theta_1) + k(\theta_1 - \theta_2) = 0, \tag{27}$$

$$I_2\ddot{\theta}_2(t) + k(\theta_2 - \theta_1) = u. \tag{28}$$

where u is the torque input, I_1 is the link inertia, I_2 is the motor inertia, m is the mass, g is the gravity constant, l is the link length, k is the stiffness, θ_1 and θ_2 are the angular positions of the first and second joints respectively. Let $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3(t) = \theta_2$, $x_4 = \dot{\theta}_2$. The dynamic equations (27) and (28) can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = I_1^{-1}(-mglsin(x_1(t)) + kx_3(t) - kx_1(t)) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = I_2^{-1}\big(k(x_1 - x_3) + u(t)\big), \end{cases}$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix}^T$, is the state vector. One can represent exactly the system by the following two-rule fuzzy model:

Rule 1 : If x_1 is M_{11} then

$$\dot{x}(t) = A_1x(t) + B_1u(t)$$

Rule 2 : If x_1 is M_{21} then

$$\dot{x}(t) = A_2x(t) + B_2u(t),$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (mgl-k)I_1^{-1} & 0 & kI_1^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ kI_2^{-1} & 0 & -kI_2^{-1} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^{-1} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (-mgl-k)I_1^{-1} & 0 & kI_1^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ kI_2^{-1} & 0 & -kI_2^{-1} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_2^{-1} \end{bmatrix}.$$

The membership functions for rule 1 and 2 are respectively:

$$\mu_1(x_1) = \begin{cases} \dfrac{1}{2} - \dfrac{\sin(x_1)}{2x_1} & \text{if } x_1 \neq 0 \\ 0 & \text{if } x_1 = 0 \end{cases} \quad \text{and } \mu_2(x_1) = 1 - \mu_1(x_1).$$

In this simulation, we choose $I_1 = I_2 = 1\text{kgm}^2$, $m = 0.01\text{kg}$, $k = 0.05Nm/rad$, $l = 1m$, $g = 9.8ms^{-2}$. Using an LMI optimisation algorithm, we obtain:

$$P = \begin{bmatrix} 0.0439 & 0.1845 & 0.0155 & 0.0081 \\ 0.1845 & 0.8899 & 0.0754 & 0.0424 \\ 0.0155 & 0.0754 & 0.0067 & 0.0037 \\ 0.0081 & 0.0424 & 0.0037 & 0.0026 \end{bmatrix},$$

the following feedback gains:

$$K_1 = \begin{bmatrix} 26.2016 & 131.2200 & 11.7453 & 6.7109 \end{bmatrix}$$

and

$$K_2 = \begin{bmatrix} 17.7564 & 90.4835 & 8.2953 & 4.7677 \end{bmatrix}.$$

and the following positive definite matrices:

$$Q_1 = \begin{bmatrix} 46.8501 & -23.1530 & 11.5634 & -0.2938 \\ -23.1530 & 13.6945 & -12.3291 & 0.1252 \\ 11.5634 & -12.3291 & 228.6668 & 0.8527 \\ -0.2938 & 0.1252 & 0.8527 & 292.7310 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 46.8501 & -2.8481 & 11.5634 & -0.2282 \\ -2.8481 & 2.4814 & -8.6590 & -0.0128 \\ 11.5634 & -8.6590 & 228.6668 & 0.8683 \\ -0.2282 & -0.0128 & 0.8683 & 292.7303 \end{bmatrix},$$

and

$$Q_{12} = \begin{bmatrix} 416.0678 & 21.4119 & -4.9959 & -2.3294 \\ 21.4119 & 482.9281 & 9.8489 & 1.4868 \\ -4.9959 & 9.8489 & 297.6419 & 1.3142 \\ -2.3294 & 1.4868 & 1.3142 & 292.7490 \end{bmatrix}.$$

It can be easily shown that the following stability conditions are satisfied:

$$G_{ii}^T P + PG_{ii} < -Q_i,\ i = 1,2,$$

and

$$G_{12}^T P + PG_{12} < -Q_{12}.$$

Then, we have

$$\lambda_{min}(P) = 0.0003,\ \ \lambda_{max}(P) = \|P\| = 0.9367$$

and

$$\lambda_0 = \inf\{(\lambda_{min}(Q_i); i = 1,2), (\lambda_{min}(Q_{12}))\} = 1.6513.$$

The resulting PDC control law is as follows:

Rule 1: If x_1 is M_{11} then

$$u(t) = -K_1 x(t)$$

Rule 2: If x_1 is M_{21} then

$$u(t) = -K_2 x(t).$$

That is,

$$u(t) = -\mu_1(x_1(t))K_1 x(t) - \mu_2(x_1(t))K_2 x(t).$$

This nonlinear control law guarantees the stability of the fuzzy control system (fuzzy model + PDC control). Fig. 1 shows the response of the system using fuzzy model with the PDC control for initial condition $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$.

Now, we introduce the external disturbances and we approximate the system by the following two-rule fuzzy model:
Rule 1 : If x_1 is M_{11} then

$$\dot{x}(t) = A_1 x(t) + B_1 u(t) + f_1(t, x(t))$$

Rule 2 : If x_1 is M_{21} then

$$\dot{x}(t) = A_2 x(t) + B_2 u(t) + f_2(t, x(t)),$$

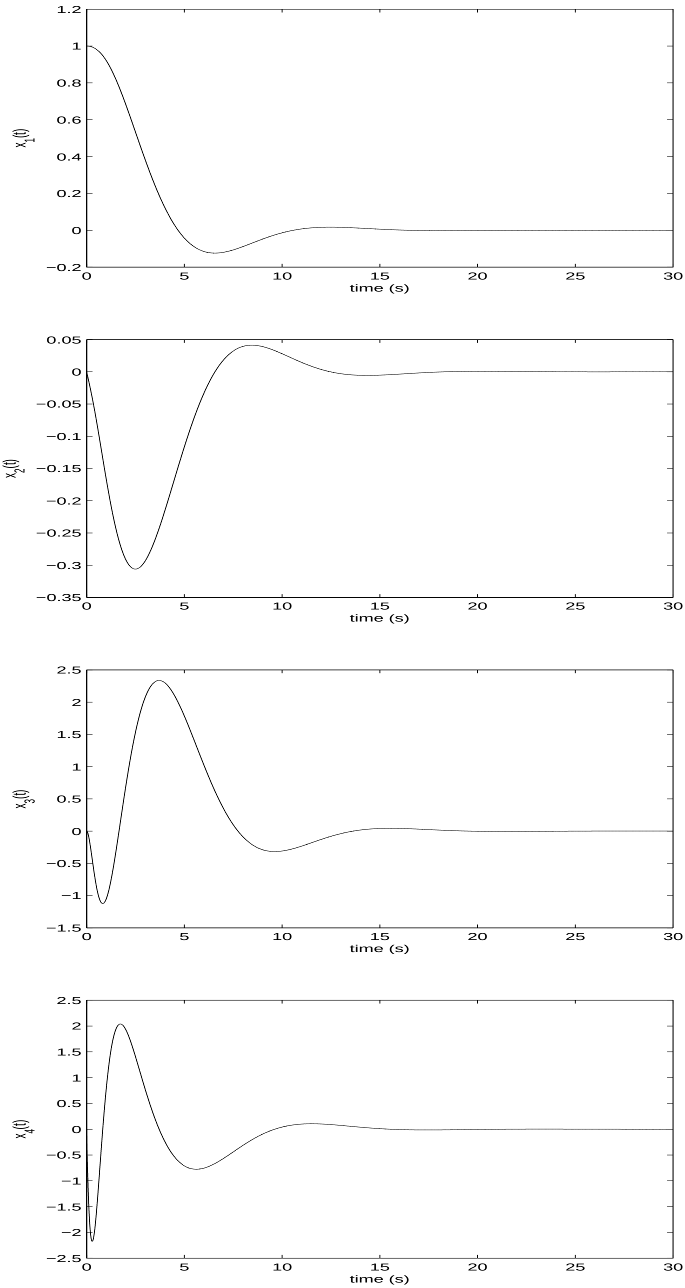

Figure 1: *The state of the controlled system*

where

$$f_1(t,x(t)) = f_2(t,x(t)) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\lambda_0}{4\lambda_{max}(P)(1+t^2)}x_2 + \dfrac{1}{2-\sin(t)^2} \\ 0 \end{bmatrix}.$$

We can see that

$$\|f_1(t,x)\| = \|f_2(t,x)\| \leqslant \frac{\lambda_0}{4\lambda_{max}(P)(1+t^2)}\|x\| + \frac{1}{2-\sin(t)^2}, \text{ for all } t \geqslant 0.$$

Therefore, we can choose

$$\alpha_1(t) = \alpha_2(t) = \frac{\lambda_0}{4\lambda_{max}(P)(1+t^2)},$$

and

$$\beta_1(t) = \beta_2(t) = \frac{1}{2-\sin(t)^2}.$$

It follows that,

$$\alpha(t) = (2)^{\frac{1}{2}}\alpha_1(t), \text{ and } \delta(t) = \beta_1(t).$$

Since

$$\int_0^{+\infty} \alpha(s)ds = \frac{\lambda_0\pi}{2\lambda_{max}(P)}, \text{ and } \delta(t) \leqslant 1,$$

then we can choose $M_\alpha = \dfrac{\lambda_0\pi}{2\lambda_{max}(P)}$ and $M_\delta = 1$. Thus, by using theorem 1 the trajectories of the system are globally uniformly exponentially convergent to the following ball,

$$\mathcal{B}_\eta = \{x \in \mathbb{R}^2 \ / \ \|x\| \leqslant \eta = 2M_\delta e^{M_\alpha}\frac{\lambda^2_{max}(P)}{\lambda_{min}(P)\lambda_0} = 56479\}.$$

Fig. 2 shows the response of the flexible-joint robot arm system for initial condition $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$. Also, it shows that the trajectories of the system are globally uniformly ultimately bounded and they converge toward a neighborhood of the origin, under external disturbances.

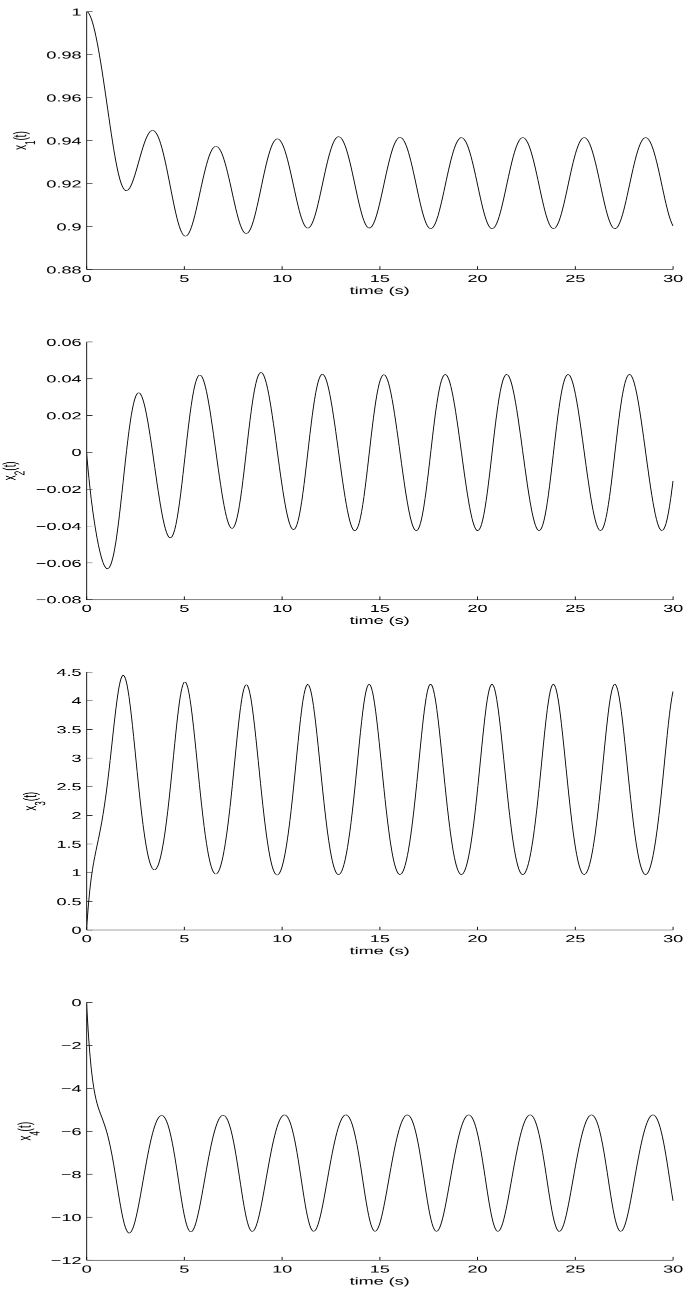

Figure 2: *The state of the controlled system under external disturbances*

Example 2 Consider the following nonlinear fuzzy planar system,

$$\dot{x}_1 = -2x_1 + \sin(x_1)u \tag{29}$$

$$\dot{x}_2 = x_1 \sin(x_1) + u, \tag{30}$$

where $x(t) = [x_1(t) \quad x_2(t)]^T \in \mathbb{R}^2$ is the state vector and $u(t)$ is the input vector. One can represent exactly the system by the following two-rule fuzzy model:

Rule 1 : If x_1 is M_{11} then

$$\dot{x}(t) = A_1 x(t) + B_1 u(t)$$

Rule 2 : If x_1 is M_{21} then

$$\dot{x}(t) = A_2 x(t) + B_2 u(t)$$

where

$$A_1 = \begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

We define the membership functions as

$$\mu_1(x_1(t)) = \frac{1 - \sin(x_1(t))}{2} \quad and \quad \mu_2(x_1(t)) = \frac{\sin(x_1(t)) + 1}{2}.$$

Using an LMI optimisation algorithm, yields

$$P = \begin{bmatrix} 0.0377 & 0.0000 \\ 0.0000 & 0.0183 \end{bmatrix},$$

the following feedback gains:

$$K_1 = \begin{bmatrix} -0.0452 & 0.7962 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 0.0452 & 0.7962 \end{bmatrix},$$

and the matrices:

$$Q_1 = \begin{bmatrix} 0.0771 & -0.0063 \\ -0.0063 & 0.0145 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.0771 & 0.0063 \\ 0.0063 & 0.0145 \end{bmatrix} \text{ and } Q_{12} = \begin{bmatrix} 0.1024 & 0.0000 \\ 0.0000 & 0.0196 \end{bmatrix}.$$

Then, we have

$$\lambda_{min}(P) = 0.0183, \ \lambda_{max}(P) = \|P\| = 0.0377$$

and

$$\lambda_0 = \inf\{(\lambda_{min}(Q_i); i = 1,2), (\lambda_{min}(Q_{12}))\} = 0.0139.$$

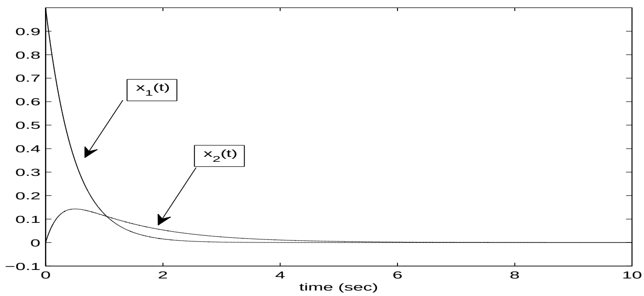

Figure 3: *The state responses of the system*

Fig. 3 shows the stability of the fuzzy control system (4.1) and (4.2) (fuzzy model + PDC control) with $x_1 = 1$ and $x_2 = 0$ as initial condition.

Now, we introduce parametric uncertainties and external disturbances and we approximate the system by the following two-rule fuzzy models:

Rule 1 : If x_1 is M_{11} then

$$\dot{x}(t) = (A_1 + \Delta A_1)x(t) + B_1 u(t) + f_1(t, x(t))$$

Rule 2 : If x_1 is M_{21} then

$$\dot{x}(t) = (A_2 + \Delta A_2)x(t) + B_2 u(t) + f_2(t, x(t))$$

where

$$\Delta A_1 = \rho_1(t) F_1^T E_1(t) F_1,$$
$$\Delta A_2 = \rho_2(t) F_2^T E_2(t) F_2,$$

with

$$F_1 = F_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix} \quad \text{and} \quad \rho_1(t) = \rho_2(t) = \frac{\lambda_0}{4(2)^{\frac{1}{2}}(\|F\|^2 \lambda_{max}^2(P) + \|F\|^2)(1+t^2)},$$

and

$$f_1(t, x(t)) = f_2(t, x(t)) = \begin{bmatrix} \mu_1 + \mu_2 \\ 0 \end{bmatrix}$$

On the one hand, we can see that

$$\rho(t) = (\sum_{i=1}^{r} \rho_i(t)^2)^{\frac{1}{2}} = \frac{\lambda_0}{4(\|F_1\|^2 \lambda_{max}^2(P) + \|F_1\|^2)(1+t^2)},$$

also

$$\int_0^{+\infty} \rho(s)ds = \frac{\lambda_0}{4(\|F_1\|^2\lambda_{max}^2(P)+\|F_1\|^2)} \int_0^{+\infty} \frac{1}{(1+s^2)}ds$$
$$= \frac{\pi\lambda_0}{8(\|F_1\|^2\lambda_{max}^2(P)+\|F_1\|^2)},$$

therefore, we can get

$$M_\rho = \frac{\pi\lambda_0}{8(\|F_1\|^2\lambda_{max}^2(P)+\|F_1\|^2)}.$$

On the other hand, we have

$$\|f_1(t,x(t))\| = \|f_2(t,x(t))\| \leqslant 1,$$

then we can get

$$\alpha_1(t)=\alpha_2(t)=0, \quad \beta_1(t)=\beta_2(t)=1 \quad \text{and} \quad \delta(t)=\sum_{i=1}^{2}\mu_i\beta_i(t)=1,$$

therefore, we can choose $M_\alpha = 0$ and $M_\delta = 1$. Thus, by using Corollary (3.4), it follows that, the system is uniformly globally exponentially converge to the following ball,

$$\mathcal{B}_{\eta_{\rho\alpha}} = \{x \in \mathbb{R}^2 \;/\; \|x\| \leqslant \eta = 2M_\delta e^{\rho M_\rho} e^{M_\alpha} \frac{\lambda_{max}^2(P)}{\lambda_{min}(P)\lambda_0} = 11.2365\}.$$

where $\rho = \|F_1\|^2\lambda_{max}^2(P)+\|F_1\|^2$. The simulation results with initial conditions $x_1 = 1$ and $x_2 = 0$ are shown in figure 4. It shows that the trajectories of the system converge toward a neighborhood of the origin, under parametric uncertainties and external disturbances.

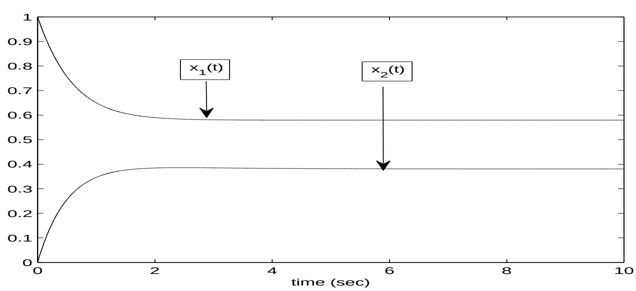

Figure 4: *The state responses of the system under parametric uncertainties and external disturbances*

5. Conclusion

In this paper, we have studied the global uniform practical exponential stability for a class of uncertain T-S fuzzy systems in term of convergence toward a neighborhood of the origin. The uncertainties are supposed uniformly to be bounded by known integrable functions. We have used quadratic Lyapunov function and parallel distributed compensation (PDC) controller techniques to show the global uniform practical exponential stability of the closed-loop system. Therefore, new LMIs are obtained for the controller in order to handel the uncertainties. Then, systems' performance is proved by adjusting the practical stability conditions. The effectiveness of the proposed theory is illustrated by computer simulation of a flexible-joint robot arm and a planar systems.

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