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Eigenvalue assignment in fractional descriptor discrete-time linear systems

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The problem of eigenvalue assignment in fractional descriptor discrete-time linear systems is considered. Necessary and sufficient conditions for the existence of a solution to the problem are established. A procedure for computation of the gain matrices is given and illustrated by a numerical example.

Key words: eigenvalue assignment, fractional, descriptor, discrete-time linear system, gain matrix.

1. Introduction

A dynamical system is called a fractional-order system if its state equations are given by fractional-order derivative of state vector. Mathematical fundamentals of the fractional calculus are given in the [23, 25, 26]. The standard and positive fractional linear systems have been investigated in [18, 24] and the positive fractional linear electrical circuits in [20]. Some recent interesting results in the fractional systems theory and its applications can be found in [8, 27, 28, 30].

Descriptor (singular) linear systems were considered in many papers and books [1-7, 9-11, 17, 18, 22, 29, 31]. The positive standard and descriptor systems and their stability have been analyzed in [13-16, 28]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [10] and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in [14]. New stability tests of positive standard and fractional linear systems have been investigated in [12]. The controllability of dynamical systems has been investigated in [21].

In this paper the eigenvalue assignment problem for fractional descriptor discrete-time linear systems will be investigated and procedure for computation of the state-feedback gain matrices will be presented.

The paper is organized as follows. In section 2 the problem of eigenvalue assignment in fractional descriptor discrete-time linear systems is formulated. In section 3 the

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problem is solved and procedure for computation of the state-feedback gain matrices is presented. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ — the set of real numbers, $\Re^{n\times m}$ — the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, I_n — the $n\times n$ identity matrix, Z_+ — the set of nonnegative integers.

2. Problem formulation

Consider the descriptor discrete-time linear system

$$E\Delta^{\alpha}x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ = \{0,1,\ldots\} \tag{1}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. The fractional difference of the order α is defined by

$$\Delta^{\alpha}x_k = \sum_{i=0}^{k}(-1)^k\begin{pmatrix}\alpha\\ i\end{pmatrix}x_{k-i}, \quad \begin{pmatrix}\alpha\\ i\end{pmatrix} = \begin{cases} 1 & \text{for} \quad i=0,\\ \frac{\alpha(\alpha-1)\ldots(\alpha-i+1)}{i!} & \text{for} \quad i=1,2,\ldots\end{cases} \tag{2}$$

Substituting (2) into (1) yields

$$Ex_{k+1} = A_{\alpha}x_k + \sum_{i=1}^{k+1} c_i Ex_{k-i+1} + Bu_k \tag{3}$$

where

$$A_{\alpha} = A + \alpha E, \quad c_i = (-1)^i\begin{pmatrix}\alpha\\ i+1\end{pmatrix}, \quad i=1,2,\ldots\,. \tag{4}$$

It is assumed that $\operatorname{rank} E = r < n$ and $\operatorname{rank} B = m$. In practical problems it is also assumed that i is bounded by natural number $h = k+1 > n$. We may write the equation (3) in the form

$$\bar{E}\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}u_k, \tag{5}$$

where

$$\bar{A}=\begin{bmatrix} A_\alpha & c_1E & c_2E & \cdots & c_{h-1}E & c_hE \\ I_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n & 0 \end{bmatrix}\in\Re^{\bar{n}\times\bar{n}},\quad \bar{B}=\begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\in\Re^{\bar{n}\times m},$$

$$\bar{E}=\begin{bmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \end{bmatrix}\in\Re^{\bar{n}\times\bar{n}},\quad \bar{x}_k=\begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \\ x_{k-h} \end{bmatrix}\in\Re^{\bar{n}},\quad k\in Z_+,\ \bar{n}=n(h+1). \tag{6}$$

Let us consider the system (1) with the state-feedback

$$\bar{u}_k = K_1\bar{x}_{k+1} + K_2\bar{x}_k \tag{7}$$

where $\bar{u}_k \in \Re^m$ is a new input vector and $K_1, K_2 \in \Re^{m\times\bar{n}}$ are gain matrices. Substitution of (7) into (5) yields

$$(\bar{E}-\bar{B}K_1)\bar{x}_{k+1} = (\bar{A}+\bar{B}K_2)\bar{x}_k. \tag{8}$$

The problem can be stated as follows. Given E, A, B, $\alpha\in(0,1)$ find K_1, K_2 such that the closed-loop system has desired eigenvalues $z_1, z_2, \ldots, z_n$, $|z_k|<1$, $k=1,\ldots n$.

3. Problem solution

The problem will be solved by the use of the following two steps procedure.

Step 1. (Subproblem 1) Find K_1 such that $\bar{E}-\bar{B}K_1 = I_{\bar{n}}$.

Step 2. (Subproblem 2) Find K_2 such that $\bar{A}+\bar{B}K_2$ has desired eigenvalues.

The first subproblem has a solution if and only if [3]

$$\operatorname{rank}\begin{bmatrix} \bar{E} & \bar{B} \end{bmatrix} = \bar{n},\quad \operatorname{rank}\bar{B} = m. \tag{9}$$

Theorem 8 *If the conditions* (9) *are satisfied then the equation*

$$\bar{E}-\bar{B}K_1 = I_{\bar{n}} \tag{10}$$

has the solution

$$K_1 = \{[\bar{B}^T\bar{B}]^{-1}\bar{B}^T + K[I_{\bar{n}} - \bar{B}[\bar{B}^T\bar{B}]^{-1}\bar{B}^T]\}(\bar{E} - I_{\bar{n}}), \tag{11}$$

where K is an arbitrary matrix.

Proof From (10) we have

$$\bar{B}K_1 = \bar{E} - I_{\bar{n}}. \tag{12}$$

If conditions (9) are met then there exists the left pseudoinverse of the matrix $\bar{B}$ given by the formula [19]

$$\bar{B}_L = [\bar{B}^T\bar{B}]^{-1}\bar{B}^T + K[I_{\bar{n}} - \bar{B}[\bar{B}^T\bar{B}]^{-1}\bar{B}^T] \tag{13}$$

and

$$K_1 = \bar{B}_L(\bar{E} - I_{\bar{n}}) = \{[\bar{B}^T\bar{B}]^{-1}\bar{B}^T + K[I_{\bar{n}} - \bar{B}[\bar{B}^T\bar{B}]^{-1}\bar{B}^T]\}(\bar{E} - I_{\bar{n}}), \tag{14}$$

which is equivalent to (11). □

Remark 1 In particular case when $K = 0$ we have

$$K_1 = [\bar{B}^T\bar{B}]^{-1}\bar{B}^T(\bar{E} - I_{\bar{n}}) = \begin{bmatrix} [B^TB]^{-1}B^T(E - I_n) & 0 & \cdots & 0 \end{bmatrix} \tag{15}$$

and then

$$K_1\bar{x}_{k+1} = [B^TB]^{-1}B^T(E - I_n)x_{k+1}. \tag{16}$$

The second subproblem will be solved substituting (10) into (8). Thus we have

$$\bar{x}_{k+1} = (\bar{A} + \bar{B}K_2)\bar{x}_k. \tag{17}$$

Theorem 9 *There exists a matrix K_2 such that the matrix $\bar{A} + \bar{B}K_2$ has the desired eigenvalues λ_k, $k = 1,\ldots,\bar{n}$ if and only if the pair $(\bar{A},\bar{B})$ is controllable.*

Proof The proof is given in [11].

To solve the problem one of the well-known methods [11] can be applied. To simplify the notation we consider the single-input system (17) with a controllable pair $(\bar{A},\bar{B})$. Following [11] there exists a matrix

$$P = \begin{bmatrix} p_1 \\ p_1\bar{A} \\ \vdots \\ p_1\bar{A}^{\bar{n}-1} \end{bmatrix} \tag{18}$$

that transforms every controllable pair $(\bar{A},\bar{B})$ to the canonical form

$$\widetilde{A} = P\bar{A}P^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\widetilde{a}_0 & -\widetilde{a}_1 & -\widetilde{a}_2 & \cdots & -\widetilde{a}_{\bar{n}-1} \end{bmatrix}, \quad \widetilde{B} = P\bar{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{19}$$

The vector p_1 in (18) is the $\bar{n}$-th row of the matrix

$$[\ \bar{B} \quad \bar{A}\bar{B} \quad \cdots \quad \bar{A}^{\bar{n}-1}\bar{B}]^{-1}. \tag{20}$$

The characteristic polynomial of the matrix $\widetilde{A}$ has the form

$$\det[I_{\bar{n}}z - \widetilde{A}] = z^{\bar{n}} + \widetilde{a}_{\bar{n}-1}z^{\bar{n}-1} + \ldots + \widetilde{a}_1 z + \widetilde{a}_0 \tag{21}$$

and the characteristic polynomial of the closed-loop system matrix $\widetilde{A} + \widetilde{B}K_2$ has the form

$$\det[I_{\bar{n}}z - \widetilde{A} - \widetilde{B}K_2] = z^{\bar{n}} + \widetilde{d}_{\bar{n}-1}z^{\bar{n}-1} + \ldots + \widetilde{d}_1 z + \widetilde{d}_0. \tag{22}$$

The matrix satisfying (22) is given by

$$K_2 = [\ \widetilde{d}_0 - \widetilde{a}_0 \quad \widetilde{d}_1 - \widetilde{a}_1 \quad \cdots \quad \widetilde{d}_{\bar{n}-1} - \widetilde{a}_{\bar{n}-1}\ ]. \tag{23}$$

The considerations can be easily extended to multi-input systems [11].

From the above we have the following procedure.

Procedure 1.

Step 1. Knowing A, B, E, α choose $h > n$ and compute the matrices $\bar{A}$, $\bar{B}$, $\bar{E}$ defined by (6).

Step 2. Check the conditions (9), then using $\bar{E}$ and $\bar{B}$ compute K_1 defined by (11). In particular case when $K = 0$ we can use matrices E and B (see (15)).

Step 3. Applying one of the well-known methods [11] and using $\bar{A}$, $\bar{B}$ compute K_2 such that the matrix $\bar{A} + \bar{B}K_2$ has the desired eigenvalues λ_k, $k = 1,\ldots,\bar{n}$, $\mathrm{Re}\,\lambda_k < 0$. The method for single-input systems presented above can be used.

Example 1 Consider the fractional descriptor discrete-time linear system (1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{24}$$

and $\alpha = 0.5$. Find K_1 and K_2 such that the closed-loop system has the eigenvalues $\lambda_k = 0$, $k = 1, \dots, 9$. Using the Procedure 1 we obtain the following.

Step 1. Step 1. We choose $h = 2$. From (6) we have

$$\bar{A} = \begin{bmatrix} 0.5 & 1 & 0 & 0.125 & 0 & 0 & 0.0625 & 0 & 0 \\ 0 & 0.5 & 1 & 0 & 0.125 & 0 & 0 & 0.0625 & 0 \\ 1 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$
$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

Step 2. The conditions (9) are satisfied. Using (25) with (11) for
$K = [\,1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$ we obtain the first gain matrix

$$K_1 = [\,0 \quad 0 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]. \tag{26}$$

It is easy to check that $\bar{E} - \bar{B}K_1 = I_9$.

Step 3. Step 3. Using the presented algorithm for single-input systems we compute the matrix

$$[\ \bar{B} \quad \bar{A}\bar{B} \quad \cdots \quad \bar{A}^{\bar{n}-1}\bar{B}]^{-1} = \tag{27}$$

$$= \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -0.5 \\ -13.5 & 0.5 & 0 & 5.5 & 10.5 & -0.5 & 2.4 & 3.1 & 5.4 \\ 54 & 52 & 0 & -48 & -96 & -52 & 3.3 & 22.7 & 9.2 \\ 82 & 370 & 0 & -2 & -248 & -370 & -49.3 & -104.8 & -29.2 \\ -688 & -1656 & 0 & 376 & 1776 & 1656 & 114 & 84 & -174 \\ 384 & 1056 & 0 & -160 & -984 & -1056 & -104 & -156 & 24 \\ -1024 & -2368 & 0 & 576 & 2560 & 2368 & 160 & 112 & 224 \\ 640 & 1408 & 0 & -384 & -1600 & -1408 & -80 & -16 & 176 \end{bmatrix}$$

The vector has the form

$$p_1 = [\ 640 \quad 1408 \quad 0 \quad -384 \quad -1600 \quad -1408 \quad -80 \quad -16 \quad 176\ ]. \tag{28}$$

Using (18) we compute the matrix

$$P = \begin{bmatrix} 640 & 1408 & 0 & -384 & -1600 & -1408 & -80 & -16 & 176 \\ -64 & -256 & 0 & 0 & 160 & 256 & 40 & -88 & 40 \\ -32 & -32 & 0 & 32 & 56 & 32 & -4 & -16 & -4 \\ 16 & 8 & 0 & -8 & -20 & -8 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0.5 & 1 \\ 0 & -1 & 0 & 1 & 0.5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -0.1 & 0 & 0 & -0.1 & 0 \\ 0.5 & 0.9 & 0 & 0.1 & -0.1 & 0.1 & 0.1 & 0 & 0.1 \\ 0.4 & 0.9 & 1 & 0.1 & 0.1 & 0.1 & 0 & 0.1 & 0 \end{bmatrix} \tag{29}$$

which transforms the pair $(\bar{A}, \bar{B})$ to the canonical form (see (19))

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.002 & -0.0117 & -0.0234 & -0.0781 & 1.125 & -0.5 & 1.5 \end{bmatrix},$$

$$\widetilde{B} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1]^T. \tag{30}$$

Using (23) we have the second gain matrix

$$K_2 = [\ 0 \quad 0 \quad -0.002 \quad -0.0117 \quad -0.0234 \quad -0.0781 \quad 1.125 \quad -0.5 \quad 1.5\ ]. \tag{31}$$

The closed-loop system matrix is given by

$$\widetilde{A} + \widetilde{B}K_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

and has desired eigenvalues $\lambda_k = 0$, $k = 1, \dots, 9$.

4. Concluding remarks

The problem of eigenvalue assignment in fractional descriptor discrete-time linear systems has been considered. Necessary and sufficient conditions for the existence of a solution to the problem have been established. A procedure for computation of the gain matrices has been given and illustrated by a numerical example.

The considerations can be extended to fractional descriptor continuous-time linear systems.

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