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Non integer order, state space model of heat transfer process using Caputo-Fabrizio operator

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Abstract. The paper is intended to show a new state space, non integer order model of an one-dimensional heat transfer process. The proposed model derives directly from time continuous, state space semigroup model. The fractional order derivative with respect to time is described by a new operator proposed by Caputo and Fabrizio, the non integer order spatial derivative is expressed by Riesz operator. The Caputo-Fabrizio operator can be directly implementated using MATLAB, because it does not require us to apply any approximation. Analytical formulae of step response are given, the system decomposition was discussed also. Main results from the paper show that the use of Caputo Fabrizio operator allows us to obtain the simple in implementation and analysis model of the considered heat transfer process. The accuracy of the proposed model in the sense of a MSE cost function is satisfying.

Key words: non-integer order systems, heat transfer equation, infinite dimensional systems, fractional order state equation, Riesz operator, Caputo operator, Caputo-Fabrizio operator.

1. Introduction

Mathematical models of distributed parameter systems obtained on the basis of partial differential equations can be described in an infinite-dimensional state space, usually in a Hilbert space, but Sobolev space can also be applied. This problem has been analyzed by many authors. The fundamentals have been determined in [26], they are given also in [18], analysis of a hyperbolic system in Hilbert space was presented in [4]. This paper also provides a broad overview of literature.

The modeling of processes and phenomena hard to analyse with the use of other tools is one of main areas of application non-integer order calculus. Non-integer models for many physical phenomena were presented by many Authors, for example in [5, 7, 9, 10, 20, 27, 32]. Analysis of anomalous diffusion problem with the use of fractional order approach and semigroup theory was presented for example in [28]. An observability problem for fractional order systems was presented for example in [13]. Minimal energy control for FO descriptor systems was analysed for example in [30].

Heat transfer processes can also be modeled with the use of non-integer order approach. This problem has been investigated for example in [1, 3, 8, 17, 19, 20, 29]. The use of Caputo-Fabrizio operator in modeling of heat transfer processes was discussed in [31], the use operators with non singular kernel to modeling of thermal processes was deeply analysed in paper [2].

This paper is intented to propose and analyse a new state-space model for heat transfer process in one dimensional plant. The considered model is derived directly from time-continuous, semigroup model given in [24, 25] after replacing the Caputo operator by Caputo-Fabrizio operator. This operator was proposed in [6], it was presented also in [15], its use to modeling of heat transfer was considered for example in [31].

The paper is organized as follows: at the beginning elementary ideas and definitions are mentioned. Next the proposed by author, state space model using CF operator is given and its elementary properties are discussed. Finally, the experimental verification of the proposed model is presented.

2. Preliminaries

A presentation of elementary ideas is started with a definition of a noninteger-order, integro-differential operator. It was given for example in [7, 12, 16, 27]:

Definition 1. (The elementary non integer order operator). The noninteger-order integro-differential operator is defined as follows:

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \frac{d^{\alpha} f(t)}{dt^{\alpha}} & \alpha > 0 \\ f(t) & \alpha = 0 \\ \int\limits_a^t f(\tau)(d\tau)^{\alpha} & \alpha < 0. \end{cases} \tag{1} $$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the noninteger order of the operation.

Next an idea of Gamma Euler function will be applied (see for example [16]).

Definition 2. (The Gamma function).

$$ \Gamma(x) = \int\limits_0^{\infty} t^{x-1} e^{-t} dt, \tag{2} $$

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An idea of Mittag-Leffler function needs to be given next. It is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in the solution of fractional order (FO) state equation. The one parameter Mittag-Leffler function is defined as underneath.

Definition 3. (The one parameter Mittag-Leffler function).

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}. \tag{3}$$

and the two parameter Mittag-Leffler function is defined below.

Definition 4. (The two parameters Mittag-Leffler function).

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}. \tag{4}$$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3). The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL), Riemann and Liouville (RL) and Caputo (C). With respect to particular additional assuptions these definitions are equivalent. In this paper the Caputo definition is applied (see for example [7, 12, 16, 27]).

Definition 5. (The Caputo definition of the FO operator).

$${}^{C}_{0}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^\infty \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}} d\tau. \tag{5}$$

where $N-1 < \alpha < N$ denotes the non-integer order of operation and $\Gamma(..)$ is the Gamma function expressed by (2).

The Laplace transform can be defined for this operator (see for example [11]) as follows.

Definition 6. (The Laplace transform for Caputo operator).

$$\begin{gathered}\mathscr{L}\left({}^{C}_{0}D^{\alpha}_{t} f(t)\right) = s^\alpha F(s), \ \ \alpha < 0 \\ \mathscr{L}\left({}^{C}_{0}D^{\alpha}_{t} f(t)\right) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}{}_0D^k_t f(0), \\ \alpha > 0, \ \ n-1 < \alpha \leq n \in N.\end{gathered} \tag{6}$$

Consequently, the inverse Laplace transform for non-integer order function is expressed as follows (see for example [16]):

$$\begin{gathered}\mathscr{L}^{-1}[s^\alpha F(s)] = {}_0D^\alpha_t f(t) + \sum_{k=0}^{n-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) \\ n-1 < \alpha < n, \ \ n \in \mathbb{Z}.\end{gathered} \tag{7}$$

A fractional-order linear state space system is described as:

$$\begin{gathered}{}_0D^\alpha_t x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t).\end{gathered} \tag{8}$$

where $\alpha \in (0, 1)$ denotes the fractional order of the state equation, $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^L$, $y(t) \in \mathbb{R}^P$ are the state, control and output vectors, respectively, A, B, C are the state, control and output matrices, respectively.

An interesting proposition of an alternative FO operator is the Caputo-Fabrizio derivative proposed in [6], presented also in [14, 15]. It derives directly from Caputo definition (5). After replacing the kernel $(t-\tau)^{-\alpha}$ by the exponential function in the form $exp\left(\frac{-\alpha}{1-\alpha}t\right)$ we obtain [6] the following definition.

Definition 7. (The Caputo-Fabrizio definition of the FO operator).

$${}^{CF}_{0}D^{\alpha}_{t} f(t) = \frac{1}{1-\alpha} \int_0^\infty \dot{f}(\tau) exp\left(\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{9}$$

For the operator defined by (9) the Laplace transform is also defined (see [6]). For $0 \leq \alpha < 1$ it takes the following form.

Definition 8. (The Laplace transform of Caputo-Fabrizio operator).

$$\mathscr{L}\{f(t)\} = \frac{s\mathscr{L}(f(t) - f(0))}{s + \alpha(1-s)}. \tag{10}$$

The above Laplace transform of CF operator (10) allows us to define a transfer function also. It takes the following form:

$$G(s) = \frac{s}{(1-\alpha)s + \alpha}. \tag{11}$$

Notice that the above transfer function needs not to be approximated to use at MATLAB platform. On the other hand it has the form of a known transfer function for real derivative plant. The CF operator can be also employed to define a FO state equation.

Definition 9. (FO state equation using CF operator).

$$\begin{cases} {}^{CF}_{0}D^{\alpha}_{t} x(t) = Ax(t) + Bu(t). \\ y(t) = Cx(t). \end{cases} \tag{12}$$

Where $x(t)$, $u(t)$ and $y(t)$ are state, control and output of the system, A, B and C are the state, control and output matrices respectively.The solution of the state equation (12) takes the form:

$$\hat{x}(t) = e^{\hat{A}t}(\hat{x}_0 + \hat{B}u_0) + \int_0^t e^{\hat{A}(t-\tau)} \hat{B}[\gamma u(\tau)] d\tau. \tag{13}$$

where:

$$\begin{aligned} \hat{A} &= \alpha\left[I - (1-\alpha)A\right]^{-1} A, \\ \hat{B} &= (1-\alpha)\left[I - (1-\alpha)A\right]^{-1} B, \\ \hat{x}_0 &= \left[I - (1-\alpha)A\right]^{-1} x_0, \\ e^{\hat{A}t} &= \mathscr{L}^{-1}\left\{\left[sI - \hat{A}\right]^{-1}\right\}, \\ \gamma &= \frac{\alpha}{1-\alpha}. \end{aligned} \tag{14}$$

The convolution in (13) is the sum of two factors associated to control $u(t)$ and its derivative $\dot{u}(t)$:

$$\hat{x}(t) = e^{\hat{A}t}(\hat{x}_0 + \hat{B}u_0) + \int_0^t e^{\hat{A}(t-\tau)}\tilde{B}u(\tau)d\tau + \\ + \int_0^t e^{\hat{A}(t-\tau)}\hat{B}\dot{u}(\tau)d\tau. \tag{15}$$

where:

$$\tilde{B} = \gamma\hat{B} = \alpha\big[I - (1-\alpha)A\big]^{-1}B. \tag{16}$$

If we assume homogenenous inital condition: $x_0 = 0$ and the control signal in the form of the Heaviside function $u(t) = 1(t)$ and $u(0) = 0$, the formula for step response takes the form as follows:

$$\hat{y}_u(t) = C\int_0^t e^{\hat{A}(t-\tau)}\tilde{B}1(\tau)d\tau + \\ + C\int_0^t e^{\hat{A}(t-\tau)}\hat{B}\delta(\tau)d\tau. \tag{17}$$

and the free response ($u(t) = 0$ and $x_0 \neq 0$) of the system (13) is as underneath:

$$\hat{x}_f(t) = e^{\hat{A}t}\hat{x}_0. \\ \hat{y}_f(t) = C\hat{x}_f(t). \tag{18}$$

The non-integer order spatial derivative was given by Riesz and it has the following form (see for example [33]):

Definition 10. (The Riesz definition of FO spatial derivative).

$$\frac{\partial^\beta \Theta(x,t)}{\partial x^\beta} = -r_\beta\Big({}_0D_x^\beta\Theta(x,t) + {}_xD_1^\beta\Theta(x,t)\Big). \tag{19}$$

where:

$$r_\beta = \frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}. \tag{20}$$

In (19) ${}_0D_x^\beta$ and ${}_xD_1^\beta$ denote left- and right-side Riemann-Liouville derivatives, defined as underneath:

$${}_0D_x^\beta\Theta(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial}{\partial x}\int_0^x \frac{\Theta(\xi,t)d\xi}{(x-\xi)^{\beta-1}}. \tag{21}$$

$${}_xD_1^\beta\Theta(x,t) = \frac{1}{\Gamma(2-\beta)}\frac{\partial}{\partial x}\int_x^1 \frac{\Theta(\xi,t)d\xi}{(\xi-x)^{\beta-1}}. \tag{22}$$

3. The non-integer order, state space models using Caputo and Caputo-Fabrizio operator

The simplified scheme of the considered heat plant is shown in Fig. 1. It has the form of a thin copper rod heated with an electric heater of the length Δx_u localized at one end of rod. An output temperature is measured using Pt-100 RTD sensors Δx long located in points: 0.29, 0.50 and 0.73 of rod length. More details of the construction are given in Section 4.

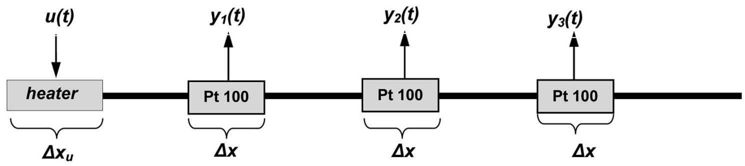

Fig. 1. The simplified scheme of the experimental system

The fundamental mathematical model describing the heat conduction in the plant is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange along the length of rod and distributed control and observation. This equation with integer orders of both differentiations has been considered in many papers, for example in [21–23].

The non integer order model with respect to both time and space coordinates, employing Caputo and Riesz operators was given in [25]. It takes the following form:

$$\begin{cases} {}^CD_t^\alpha Q(x,t) = a_w\dfrac{\partial^\beta Q(x,t)}{\partial x^\beta} - R_aQ(x,t) + b(x)u(t) \\ \dfrac{\partial Q(0,t)}{dx} = 0, t \geq 0 \\ \dfrac{\partial Q(1,t)}{dx} = 0, t \geq 0 \\ Q(x,0) = Q_0, 0 \leq x \leq 1 \\ y(t) = k_0\int_0^1 Q(x,t)c(x)dx. \end{cases} \tag{23}$$

Where α, $\beta > 0$ denote non integer orders of the system, a_w, R_a denote coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model. Now we need to express (23) as an inifinte dimensional state equation in the Hilbert space. The proposed state equation is written as follows:

$$\begin{cases} {}^CD_t^\alpha Q(t) = AQ(t) + Bu(t) \\ Q(0) = Q_0 \\ y(t) = k_0CQ(t) \end{cases} \tag{24}$$

where:

$$\begin{cases} AQ = a_w \dfrac{\partial^\beta Q(x)}{\partial x^\beta} - R_a Q, \\ D(A) = \{Q \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0\}, \\ a_w, R_a > 0, \\ H^2(0,1) = \{u \in L^2(0,1) : u', u'' \in L^2(0,1)\}, \\ CQ(t) = \langle c, Q(t)\rangle, Bu(t) = \langle bu(t)\rangle \\ Q(t) = [q_1(t), q_2(t)..]^T. \end{cases} \tag{25}$$

The following set of the eigenvectors for the state operator A creates the orthonormal basis of the state space:

$$h_n = \begin{cases} 1, n = 0 \\ \sqrt{2}cos(n\pi x), n = 1, 2, \ldots \end{cases} \tag{26}$$

Eigenvalues of the state operator are expressed as underneath:

$$\lambda_{\beta_n} = -a_w \pi^\beta n^\beta - R_a, n = 0, 1, 2, \ldots \tag{27}$$

and consequently the state operator has the form:

$$A = diag\{\lambda_{\beta_0}, \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots\}. \tag{28}$$

Next, the spectrum σ of the state operator A is expressed as underneath:

$$\sigma(A) = \{\lambda_{\beta_0}, \lambda_{\beta_1}, \lambda_{\beta_2}, \ldots\}. \tag{29}$$

From (27) it follows at once that $\lambda_{\beta_0} > \lambda_{\beta_1} > \lambda_{\beta_2} \ldots$

The input operator B has the following form:

$$B = [b_0, b_1, b_2, \ldots]^T. \tag{30}$$

Where $b_n = \langle b, h_n\rangle$, $b(x)$ denotes the control function:

$$b(x) = \begin{cases} 1, x \in [0, x_0] \\ 0, x \notin [0, x_0] \end{cases}. \tag{31}$$

The output operator C is expressed as follows:

$$C = \begin{bmatrix} C_{s1} \\ C_{s2} \\ C_{s3} \end{bmatrix}. \tag{32}$$

Rows of output operator C are as underneath:

$$C_{sj} = [c_{sj,0}, c_{sj,1}, c_{sj,2}, \ldots] \quad j = 1, 2, 3 \ldots \tag{33}$$

where $c_{sj,n} = \langle c, h_n\rangle$, $c(x)$ denotes the output sensor function:

$$c(x) = \begin{cases} 1, x \in [x_1, x_2] \\ 0, x \notin [x_1, x_2] \end{cases}. \tag{34}$$

Coordinates x_1 and x_2 depend on sensor location on the rod and they are equal:

$$\begin{cases} x = 0.29: \quad x_1 = 0.26, \ x_2 = 0.32 \\ x = 0.50: \quad x_1 = 0.47, \ x_2 = 0.53 \\ x = 0.73: \quad x_1 = 0.70, \ x_2 = 0.76 \end{cases}.$$

From (31) and (34) it turns out that the control function $b(x)$ and output function $c(x)$ are the interval constant functions. If the control $u(t) = 1(1)$ then the step response formula can be given:

$$y_u(t) = \sum_{n=0}^{\infty} \frac{\left(E_\alpha(\lambda_{\beta_n} t^\alpha) - 1(t)\right)}{\lambda_{\beta_n}} \langle b, h_n\rangle \langle c, h_n\rangle, \tag{35}$$

and the free response takes the form as underneath:

$$y_f(t) = y_{0_j} \sum_{n=1}^{\infty} E_\alpha(\lambda_{\beta_n} t^\alpha) Q_0 \langle c, h_n\rangle. \tag{36}$$

The proposed in this paper Caputo-Fabrizio model is obtained after replacing the Caputo operator in [24] by Caputo-Fabrizio operator [9]:

$$\begin{cases} {}^{CF}D_t^\alpha Q(t) = AQ(t) + Bu(t). \\ Q(0) = Q_0. \\ y(t) = k_0 CQ(t). \end{cases} \tag{37}$$

The solution of state equation (37) takes the form (13, 15) with the operators (14) in the form:

$$\begin{aligned} &\hat{A} = diag\left\{\hat{\lambda}_0, \hat{\lambda}_1, \ldots\right\}, \\ &\hat{B} = \left[\hat{b}_0, \hat{b}_1, \ldots\right]^T, \\ &\tilde{B} = \left[\tilde{b}_0, \tilde{b}_1, \ldots\right]^T, \\ &\hat{C} = C, \\ &\hat{Q}_0 = diag\left\{\frac{1}{1-(\alpha-1)\lambda_{\beta_0}}, \frac{1}{1-(\alpha-1)\lambda_{\beta_1}}, \ldots\right\} Q_0, \\ &e^{\hat{A}t} = diag\left\{e^{\hat{\lambda}_0}, e^{\hat{\lambda}_1}, \ldots\right\}, \\ &\gamma = \frac{\alpha}{1-\alpha}. \end{aligned} \tag{38}$$

where:

$$\hat{\lambda}_n = \frac{\alpha\lambda_{\beta_n}}{1-(\alpha-1)\lambda_{\beta_n}}. \quad \hat{b}_n = \frac{(\alpha-1)b_n}{1-(\alpha-1)\lambda_{\beta_n}}. \quad \tilde{b}_n = \frac{\alpha b_n}{1-(\alpha-1)\lambda_{\beta_n}}. \tag{39}$$

The spectrum of the state operator $\hat{A}$ has the form:

$$\sigma(\hat{A}) = \{\hat{\lambda}_0, \hat{\lambda}_1, \ldots\}. \tag{40}$$

The spectrum $\sigma(\hat{A})$ contains single, real, negative, separated eigenvalues ordered in decreasing order also: $\hat{\lambda}_0 > \hat{\lambda}_1, \ldots$ This means that the spectrum decomposition property is kept for FO system described with the use of the CF operator.

With respect to (17, 38) and (39) the step response of the considered heat plant is expressed as:

$$\hat{y}_u(t) = \sum_{n=0}^{\infty} c_n\left(\tilde{b}_n\frac{e^{\hat{\lambda}_n t}-1(t)}{\hat{\lambda}_n} + \hat{b}_n e^{\hat{\lambda}_n t}\right). \tag{41}$$

and the free response with respect to (18) takes the form:

$$\hat{Q}_f(t) = diag\left\{\frac{e^{\hat{\lambda}_0 t}}{1-(1-\alpha)\lambda_{\beta_0}}, \frac{e^{\hat{\lambda}_1 t}}{1-(1-\alpha)\lambda_{\beta_1}}, \ldots\right\}\hat{Q}_0. \quad \hat{y}_f(t) = C\hat{Q}_f(t). \tag{42}$$

Notice that both step and free responses of the system we deal with can be decomposed into infinite number of modes associated to certain eingenvalues $\hat{\lambda}_n$ of the state operator $\hat{A}$. This follows directly from spectrum decomposition. The n-th mode of solution is expressed as follows:

$$\hat{y}_n(t) = \hat{y}_{un} + \hat{y}_{fn}, \; n = 0, 1, 2, \ldots \tag{43}$$

where:

$$\hat{y}_{un} = c_n\left(\tilde{b}_n\frac{e^{\hat{\lambda}_n t}-1(t)}{\hat{\lambda}_n} + \hat{b}_n e^{\hat{\lambda}_n t}\right). \quad \hat{y}_{fn} = c_n\left(\frac{e^{\hat{\lambda}_0 t}}{1-(1-\alpha)\lambda_{\beta_0}}\hat{q}_{0n}\right). \tag{44}$$

The model (37–44) is infinite-dimensional. Its implementation in MATLAB or another platform requires us to use its finite dimensional approximant. It can be obtained by truncating further modes in solutions (41, 42). Consequently the operators $\hat{A}$, $\hat{B}$, $\tilde{B}$ and $\hat{C}$ can be interpreted as matrices and solution of state equation takes the form of finite sum:

$$\hat{y}_n(t) = \hat{y}_{un} + \hat{y}_{fn}, \; n = 0, 1, 2, \ldots, N. \tag{45}$$

where:

$$\hat{y}_u(t) = \sum_{n=0}^{N} c_n\left(\tilde{b}_n\frac{e^{\hat{\lambda}_n t}-1(t)}{\hat{\lambda}_n} + \hat{b}_n e^{\hat{\lambda}_n t}\right). \quad \hat{Q}_f(t) = diag\left\{\frac{e^{\hat{\lambda}_0 t}}{1-(1-\alpha)\lambda_{\beta_0}}, \ldots, \frac{e^{\hat{\lambda}_N t}}{1-(1-\alpha)\lambda_{\beta_N}}\right\}\hat{Q}_0. \quad \hat{y}_f(t) = C\hat{Q}_f(t). \tag{46}$$

The order N of the above finite approximation can be estimated with the use of numerical experiments, analogically as it was presented in [24]. The comparison of step responses for the proposed model and real plant is given in the next section.

4. Experimental results

Experiments were done with the use of an experimental system shown with details in Fig. 2. The length of rod is equal 260 mm. The control signal in the system is the standard current signal 0–20 mA given from analog output of the PLC. This signal is amplified to the range 0–1.5 A and it is the input signal for the heater. The temperature distribution along length of the rod is measured with the use of standard RTD sensors of Pt-100 type. Signals from the sensors are directly read by analog inputs of the PLC in Celsius degrees. Data from PLC are read and archivised with the use of SCADA. The whole system is connected via PROFINET industrial network. The temperature distribution with respect to time and length is shown in Fig. 3. The step response of the model was tested in time range from 0 to $T_f = 300$ s with sample time 1 s, parameters were calculated via minimization of the MSE (Medium Square Error) cost function (47) with the use of MATLAB *fminsearch* function.

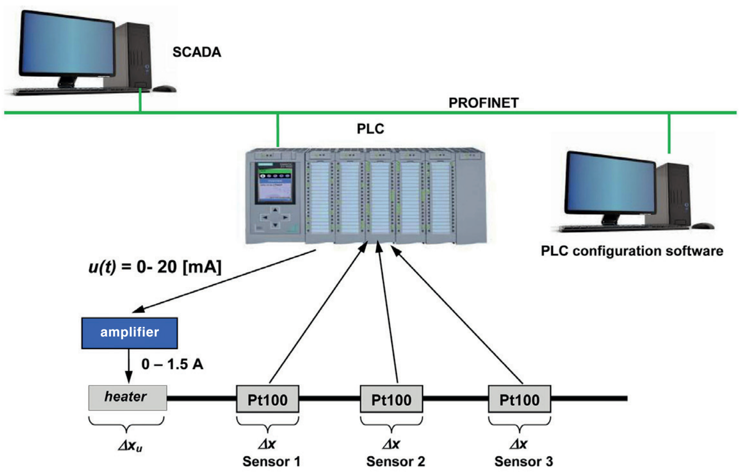

Fig. 2. The construction of the experimental system

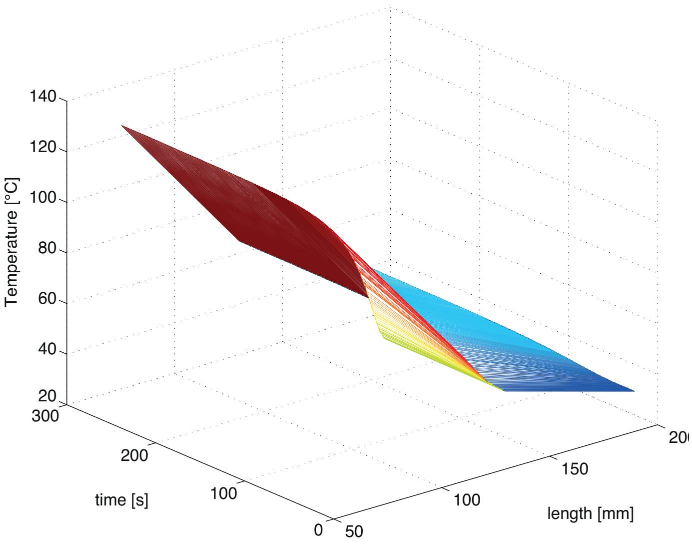

Fig. 3. The spatial-time temperature distribution in the plant

To estimate the accuracy, the typical MSE cost function was employed:

$$MSE = \frac{1}{3K_s}\sum_{j=1}^{3}\sum_{k=1}^{K_s} e_j^+(k). \tag{47}$$

In (47) K_s denotes the number of collected samples for one sensor, $e_j^+(k)$ is the difference between responses of model and plant in k-th time moment and at j-th output:

$$e_j^+(k) = y_{p_j}^+(k) - y_j^+(k). \tag{48}$$

An example of step response for the proposed model compared to step response of the plant is given in Fig. 4. From Fig. 4 it can be concluded that the proposed model is constructed properly and it assures good accuracy in the sense of cost function (47) while keeping the resonable size of the model, expressed by its size N.

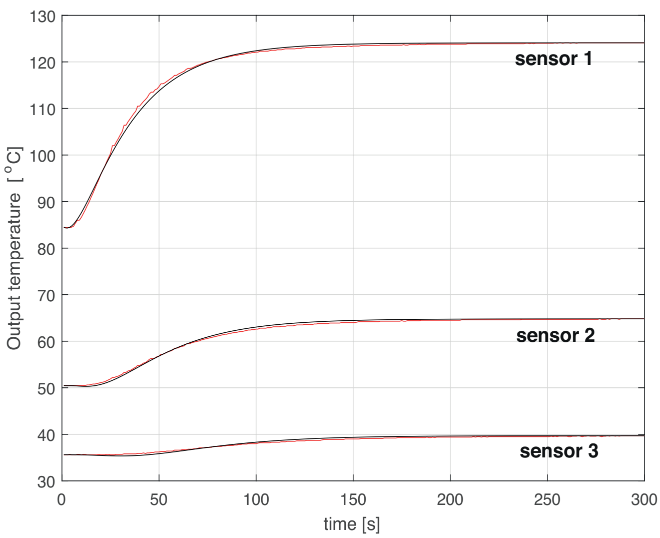

Fig. 4. The comparison experiment on the model. Parameters of the model: $\alpha = 0.622$, $\beta = 2.0671$, $a_w = 0.0010$, $R_a = 0.0536$, $N = 15$, $MSE = 0.1098$, experiment: red, proposed CF model: black

5. Final conclusions

The final conclusions from paper can be formulated as follows:

- The use of Caputo-Fabrizio operator allows us to obtain the fractional order model of real heat transfer process possible to implement without the use of any approximation. The accuracy of the proposed model in the sense of MSE cost function is satisfying,
- An interesting problem is to analyze the exponential stability of the proposed model, instead of the Mittag-Leffler stability. This is caused by the fact that the solution of FO state equation using CF operator is expressed by exponential function. The association of the exponential stability with the Mittag-Leffler stability is planned to be analyzed,
- The presented results can be generalized into class of fractional order, linear systems described by a diagonal state operator (for example in the Jordan canonical form).

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