BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 67, No. 1, 2019
DOI: 10.24425/bpas.2019.127337

Global stability of nonlinear feedback systems with positive descriptor linear parts

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Abstract. The global (absolute) stability of nonlinear systems with negative feedbacks and positive descriptor linear parts is addressed. Transfer matrices of positive descriptor linear systems are analyzed. The characteristics $u = f(e)$ of the nonlinear parts satisfy the condition $k_1 e \le f(e) \le k_2 e$ for some positive k_1, k_2. It is shown that the nonlinear feedback systems are globally asymptotically stable if the Nyquist plots of the positive descriptor linear parts are located in the right-hand side of the circles $\left(-1/k_1, -1/k_2\right)$.

Key words: global, stability, descriptor, nonlinear, feedback, system, positive.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1–3].

Positive linear systems with different fractional orders have been addressed in [4, 5]. Stability of standard and positive systems has been investigated in [6–11] and of fractional systems in [12–15]. Descriptor positive systems have been analyzed in [16, 17]. Linear positive electrical circuits with state feedbacks have been addressed in [7, 18]. The global stability of nonlinear feedback systems with positive linear parts has been analyzed in [19].

In this paper the global stability of nonlinear systems with negative feedbacks and positive descriptor linear parts will be addressed.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning standard positive linear systems are recalled. The decomposition of fractional descriptor linear systems into dynamical and static parts by the use of the shuffle algorithm is presented in Section 3. The transfer matrices of positive descriptor linear systems are addressed in Section 4.The main result of the paper is given in Section 5 where the sufficient conditions for the global stability of the nonlinear feedback systems with positive descriptor linear parts are established. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1. [2, 3] The system (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x(0)\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 1. [2, 3] The system (1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(s) = C[I_n s - A]^{-1}B + D. \tag{3}$$

Theorem 1. If the matrix $A\in M_n$ is Hurwitz and $B\in\Re_+^{n\times m}$, $C\in\Re_+^{p\times n}$, $D\in\Re_+^{p\times m}$ of the linear positive system (1), then all coefficients of the transfer matrix (3) are positive.

Proof. The proof is given in [19].

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Manuscript submitted 2018-07-03, revised 2018-08-23, initially accepted for publication 2018-08-30, published in February 2019.

3. Decomposition of fractional descriptor linear systems

Consider the fractional linear descriptor continuous-time system

$$E\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t)+Bu(t),\ 0<\alpha<1 \tag{4a}$$

$$y(t) = Cx(t), \tag{4b}$$

where $x(t)\in\Re^{n}$, $u(t)\in\Re^{m}$, $y(t)\in\Re^{p}$ are the state, input and output vectors and $E, A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$. In the paper the following Caputo definition of the fractional derivative of α order will be used [15]

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot f(\tau)}{(t-\tau)^{\alpha}}d\tau,\ 0<\alpha<1, \tag{4c}$$

where $\dot f(\tau)=\frac{df(\tau)}{d\tau}$ and $\Gamma(x)\int_0^{\infty} t^{x-1}e^{-t}dt$, $\mathrm{Re}(x)>0$ is the Euler gamma function.

It is assumed that $\det E=0$ and the pencil (E,A) of (4) is regular, i.e.

$$\det[Es^{\alpha}-A]\neq 0 \ \text{ for some } \ s^{\alpha}\in \mathbf{C}. \tag{5}$$

where $\mathbf{C}$ is the field of complex numbers.

The transfer matrix of the system (4) is given by

$$T(s) = C[Es^{\alpha}-A]^{-1}B\in\Re^{p\times m}(s^{\alpha}), \tag{6}$$

where $\Re^{p\times m}(s^{\alpha})$ is the set of $p\times m$ rational matrices in s^{α}.

Theorem 2. The descriptor linear system (4) with regular pencil can be decomposed into the dynamical part

$$\begin{aligned}\frac{d^{\alpha}x_1(t)}{dt} &= A_{11}x_1(t)+B_{10}u(t)\\ &+B_{11}\frac{d^{\alpha}u(t)}{dt^{\alpha}}+\dots+B_{1\mu}\frac{d^{\mu\alpha}u(t)}{dt^{\mu\alpha}}\end{aligned} \tag{7a}$$

and the static part

$$\begin{aligned}x_2(t) &= A_{21}x_1(t)+B_{20}u(t)\\ &+B_{21}\frac{d^{\alpha}u(t)}{dt^{\alpha}}+\dots+B_{2\mu}\frac{d^{\mu\alpha}u(t)}{dt^{\mu\alpha}}\end{aligned} \tag{7b}$$

where

$$\begin{aligned}x(t) &= \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},\ x_1(t)\in\Re^{n_1},\ x_2(t)\in\Re^{n-n_1},\\ n_1 &= \deg\det[Es^{\alpha}-A],\ \mu=q-1,\end{aligned} \tag{7c}$$

q is the index of E, i.e. $\operatorname{rank}E^{q}=\operatorname{rank}E^{q+1}$.

Proof. The proof follows directly from well known shuffle procedure [8] applied to (4a).

Procedure 1.

Step 1. Performing elementary row operations on the matrix

$$[E\ \ A\ \ B] \tag{8}$$

we transform it to the form

$$\begin{bmatrix}E_1 & A_1 & B_{10}\\ 0 & -A_2 & B_{20}\end{bmatrix}, \tag{9}$$

where E_1 has full row rank.

Step 2. After the shuffle in (9) we obtain

$$\begin{bmatrix}E_1 & A_1 & B_{10} & 0\\ A_2 & 0 & 0 & B_{20}\end{bmatrix}, \tag{10}$$

Step 3. Repeating μ times the Steps 1 and 2 we obtain

$$\begin{bmatrix}E_{\mu-1} & A_{1\mu} & B_{10} & \cdots & B_{1\mu}\\ 0 & A_{2\mu} & B_{20} & & B_{2\mu}\end{bmatrix} \tag{11}$$

with the nonsingular matrix

$$E_{\mu-1}. \tag{12}$$

Step 4. Performing elementary row operations on the matrix (11) we transform it to the form

$$\begin{bmatrix}I_{n_1} & 0\\ A_{21} & I_{n-n_1}\end{bmatrix}. \tag{13}$$

Step 5. From (11) and (13) we obtain

$$\begin{bmatrix}I_{n_1} & 0 & A_{11} & 0 & B_{10} & B_{11} & \cdots & B_{1\mu}\\ 0 & 0 & A_{21} & I_{n-n_1} & B_{20} & B_{21} & \cdots & B_{2\mu}\end{bmatrix} \tag{14}$$

and the equations (7).

By Weierstrass-Kronecker theorem if the condition (5) is satisfied then there exist nonsingular matrices $P, Q\in\Re^{n\times n}$ such that

$$P[Es^{\alpha}-A]Q = \begin{bmatrix}I_r s^{\alpha}-A_1 & 0\\ 0 & Ns^{\alpha}-I_{n-r}\end{bmatrix}, \tag{15}$$

where $r=\deg\det[Es^{\alpha}-A]$, $A_1\in\Re^{n\times n}$, $N\in\Re^{(n-r)\times(n-r)}$ is a nilpotent matrix with the index q, i.e. $N^{q}=0$ and $N^{q-1}\neq 0$.

Using (15) we may write the equation (4a) in the form

$$\frac{d^\alpha x_1(t)}{dt} = A_1 x_1(t) + B_1 u(t) \tag{16a}$$

and the static part

$$N x_2(t) = x_2(t) + B_2 u(t), \tag{16b}$$

and

$$y(t) = C_1 x_1(t) + C_2 x_2(t), \tag{16c}$$

where

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q^{-1} x(t),\ x_1(t) \in \Re^r,\ x_2(t) \in \Re^{n-r}, \\ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB,\ [C_1 \quad C_2] = CQ. \tag{16d}$$

The transfer matrix of the descriptor linear system (16) has the form

$$T(s) = C_1[I_r s^\alpha - A_1]^{-1} B_1 + C_2[N s^\alpha - I_{n-r}]^{-1} B_2, \tag{17}$$

where $T_{sp}(s) = C_1[I_r s^\alpha - A_1]^{-1} B_1$ is the strictly proper part and $T_p(s) = C_2[N s^\alpha - I_{r-1}]^{-1} B_2$ is the polynomial part.

Example 1. Consider the descriptor linear system (4) in the Weierstrass-Kronecker form

$$E = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & N \end{bmatrix},$$

$$A = \left[\begin{array}{cc|cc} -2 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix}, \tag{18}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ -1 \end{array}\right],\ C = [C_1 \quad C_2] = [2 \quad 0 \mid 1 \quad 0].$$

Using Procedure 1 we shall decompose the descriptor system (4) with (18) into the dynamical and static parts.

In this case using (8) and (18) we obtain

$$[E \ \ A \ \ B] = \left[\begin{array}{cccc|cccc|c} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \\ = \begin{bmatrix} E_1 & A_1 & B_{10} \\ 0 & A_2 & B_{20} \end{bmatrix} \tag{19}$$

and after the shuffle

$$\begin{bmatrix} E_1 & A_1 & B_{10} & 0 \\ A_2 & 0 & 0 & B_{20} \end{bmatrix} \\ = \left[\begin{array}{cccc|cccc|c|c} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{20}$$

Adding the third row multiplied by –1 to the fourth row of (20) we obtain

$$\begin{bmatrix} E_1 & A_{11} & B_{10} & 0 \\ 0 & A_{21} & 0 & B_{20} \end{bmatrix} \\ = \left[\begin{array}{cccc|cccc|c|c} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]. \tag{21}$$

Note that the matrix

$$\begin{bmatrix} E_1 \\ -A_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{22}$$

is nonsingular and from (21) we have the dynamical part (7a) with

$$A_{11} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ B_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{23a}$$

and the static part (7b) with

$$A_{21} = [0 \quad 0 \quad 0],\quad B_{20} = [1]. \tag{23b}$$

Consider the fractional linear system described by

$$\frac{d^\alpha x(t)}{dt} = Ax(t) + B_0 u(t) + \ldots + B_{q_1}\frac{d^{q_1\alpha}u(t)}{dt^{q_1\alpha}}, \tag{24a}$$

$$y(t) = Cx(t) + D_0 u(t) + \ldots + D_{q_2}\frac{d^{q_2\alpha}u(t)}{dt^{q_2\alpha}}, \tag{24b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B_k \in \Re^{n \times m}$, $k = 0, 1, \ldots, q_1$ $C \in \Re^{p \times n}$, $D_l \in \Re^{p \times m}$, $l = 0, 1, \ldots, q_2$.

Definition 2. The fractional system (24) is called (internally) positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs

$$\begin{gathered}\frac{d^{k\alpha}u(t)}{dt^{k\alpha}} \in \Re_+^m \text{ for } k = 0, 1, \ldots, q, \\ q = \max(q_1, q_2), \ t \geq 0.\end{gathered} \tag{25}$$

Theorem 3. The fractional system (24) is positive if and only if

$$A \in M_n, \ B_k \in \Re_+^{n \times m} \text{ for } k = 0, 1, \ldots, q_1. \tag{26}$$

and

$$C \in \Re_+^{p \times n}, \ D_l \in \Re_+^{p \times m} \text{ for } l = 0, 1, \ldots, q_2. \tag{27}$$

Proof. It is well-known that if $u(t) = 0$ then the solution $x(t)$ of (24a) for any $x(0) \in \Re_+^n$ is nonnegative for $t \geq 0$ if and only if $A \in M_n$ and $x(t)$ of (24a) is nonnegative for any $u(t)$ satisfying (25) if and only if the conditions (26) are satisfied. Note that $y(t) \in \Re_+^p$, $t \geq 0$ for $u(t)$ satisfying (25) if and only if (27) holds. □

Theorem 4. The fractional descriptor system (4) with regular pencil is positive if and only if its dynamical and static parts satisfy the conditions

$$A_{11} \in M_{n_1}, \ B_{1k} \in \Re_+^{n_1 \times m} \text{ for } k = 1, \ldots, \mu \tag{28}$$

and

$$A_{21} \in \Re_+^{(n-n_1) \times n_1}, \ B_{2k} \in \Re_+^{(n-n_1) \times m} \text{ for } k = 1, \ldots, \mu. \tag{29}$$

Proof. By Theorem 3 the dynamical system (7a) is positive if and only if the conditions (28) are satisfied. If $x_1(t) \in \Re_+^{n_1}$, $t \geq 0$ then $x_2(t) \in \Re_+^{n-n_1}$, $t \geq 0$ if and only if (29) holds. □

Example 2. (Continuation of Example 1) Consider the descriptor linear system (4) with (18). The matrices (23a) of the dynamical part of the descriptor linear system (4) with (18) satisfy the condition (28) and the matrices (23b) satisfy the conditions (29). Therefore, the descriptor system (4) with (18) is positive.

4. Transfer matrices of positive stable descriptor linear systems

Consider the descriptor linear system with regular pencil (5).

Theorem 5. The transfer matrix of the descriptor system described by (7) and (4b) has the form

$$\begin{aligned}&T(s) = C\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \\ &K_1 = [I_{n_1}s^\alpha - A_{11}]^{-1}(B_{10} + B_{11}s^\alpha + \ldots + B_{1\mu}s^{\mu\alpha}). \\ &K_2 = A_{21}[I_{n_1}s^\alpha - A_{11}]^{-1}(B_{10} + B_{11}s^\alpha + \ldots \\ &\quad + B_{1\mu}s^{\mu\alpha}) + B_{20} + B_{21}s^\alpha + \ldots + B_{2\mu}s^{\mu\alpha}.\end{aligned} \tag{30}$$

Proof. Applying the Laplace transform to (7a) and (7b) with zero initial conditions we obtain

$$\begin{aligned}s^\alpha X_1(s) = A_{11}X_1(s) + (B_{10} + B_{11}s^\alpha + \ldots \\ + B_{1\mu}s^{\mu\alpha})U(s)\end{aligned} \tag{31a}$$

and

$$\begin{aligned}X_2(s) = A_{21}X_1(s) + (B_{20} + B_{21}s^\alpha + \ldots \\ + B_{2\mu}s^{\mu\alpha})U(s),\end{aligned} \tag{31b}$$

where

$$\begin{gathered}X_1(s) = \mathcal{L}[x_1(t)] = \int_0^\infty x_1(t)e^{-st}dt, \\ X_2(s) = \mathcal{L}[x_2(t)], \ U(s) = \mathcal{L}[u(t)].\end{gathered} \tag{31c}$$

From (31a) we have

$$\begin{aligned}X_1(s) = \left[I_{n_1}s^\alpha - A_{11}\right]^{-1}(B_{10} + B_{11}s^\alpha + \ldots \\ + B_{1\mu}s^{\mu\alpha})U(s)\end{aligned} \tag{32}$$

and substituting (32) into (31b) we obtain

$$\begin{aligned}X_2(s) = \Big\{A_{21}\left[I_{n_1}s^\alpha - A_{11}\right]^{-1}(B_{10} + B_{11}s^\alpha + \ldots \\ + B_{1\mu}s^{\mu\alpha}) + B_{20} + B_{21}s^\alpha + \ldots + B_{2\mu}s^{\mu\alpha}\Big\}U(s).\end{aligned} \tag{33}$$

Substitution of (32) and (33) into

$$Y(s) = \mathcal{L}[y(t)] = C\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \tag{34}$$

yields (30). □

Example 3. Compute the transfer matrix of the descriptor system (4) with (18). Using (30) and taking into account (23) we obtain

$$T(s) = C\begin{bmatrix}[I_{n_1}s^\alpha - A_{11}]^{-1}(B_{10} + B_{11}s^\alpha + B_{12}s^{2\alpha}) \\ A_{21}[I_{n_1}s^\alpha - A_{11}]^{-1}(B_{10} + B_{11}s^\alpha + B_{12}s^{2\alpha}) + B_{20}\end{bmatrix}$$

$$= [2 \;\; 0 \;\; 1 \;\; 0]\left[\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}s^\alpha+2 & -1 & 0\\ -2 & s^\alpha+3 & 0\\ 0 & 0 & s^\alpha-2\end{bmatrix}^{-1} \cdot \tag{35}$$

$$\begin{bmatrix}1\\0\\s^{2\alpha}\end{bmatrix} + \begin{bmatrix}0\\0\\0\\-1\end{bmatrix}\right] = \frac{2s^\alpha+6}{s^{2\alpha}+5s^\alpha+4} + s^\alpha.$$

The same result we obtain using the formulas (32) and (17). In the first case we have

$$\begin{aligned} T(s) &= C[Es^\alpha - A]^{-1}B \\ &= [2 \;\; 0 \;\; 1 \;\; 0]\begin{bmatrix}s^\alpha+2 & -1 & 0 & 0\\ -2 & s^\alpha+3 & 0 & 0\\ 0 & s^\alpha+3 & -1 & s^\alpha\\ 0 & 0 & 0 & -1\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} \\ &= \frac{2s^\alpha+6}{s^{2\alpha}+5s^\alpha+4} + s^\alpha \end{aligned} \tag{36}$$

and in the second case

$$\begin{aligned} T(s) &= C_1[I_r s^\alpha - A_1]^{-1}B_1 + C_2[Ns^\alpha - I_{n-r}]^{-1}B_2 \\ &= [2 \;\; 0]\begin{bmatrix}s^\alpha+2 & -1\\ -2 & s^\alpha+3\end{bmatrix}^{-1}\begin{bmatrix}1\\0\end{bmatrix} \\ &+ [1 \;\; 0]\begin{bmatrix}-1 & s^\alpha\\ 0 & -1\end{bmatrix}^{-1}\begin{bmatrix}0\\-1\end{bmatrix} = \frac{2s^\alpha+6}{s^{2\alpha}+5s^\alpha+4} + s^\alpha. \end{aligned} \tag{37}$$

Note that

$$T_{sp}(s) = \frac{2s^\alpha+6}{s^{2\alpha}+5s^\alpha+4} \tag{38}$$

is the strictly proper part and $T_p(s) = s^\alpha$ is the polynomial part of the transfer function of the descriptor system.

Definition 3. The positive fractional descriptor linear system (4) with $u(t) = 0$ is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \;\text{ for all }\; x(0) \in \Re_+^n. \tag{39}$$

Theorem 6. The positive fractional descriptor linear system (4) is asymptotically stable if and only if $A_1 \in M_n$ and all coefficients of the characteristic polynomial

$$\det[I_{n_1}s - A_1] = s^{n_1} + a_{n_1-1}s^{n_1-1} + \ldots + a_1 s + a_0 \tag{40}$$

are positive, i.e. $a_k > 0$ for $k = 0, 1, \ldots, n_1 - 1$.

Proof. The proof is given in [15].

Lemma 1. If the matrix $A_1 \in M_n$ is asymptotically stable (is Hurwitz) then the matrix

$$[I_{n_1}s - A_1]^{-1} \in \Re^{n_1\times n_1}(s) \tag{41}$$

has positive coefficients.

Proof. The proof is given in [15].

Theorem 7. If the positive fractional descriptor linear system (4) is asymptotically stable then all coefficients of the transfer matrix (30) are positive.

Proof. By Lemma 1 if the positive system is asymptotically stable then all coefficients of the matrix (41) are positive and by Theorem 3 the system (24) is positive if and only if the conditions (26) and (27) are satisfied. In this case all coefficients of the transfer matrix (30) are positive. □

Remark 1. Note that all coefficients of the transfer matrix (30) are positive if and only if all coefficients of the transfer matrix (6) are positive.

5. Global stability of nonlinear systems

Consider the nonlinear system shown in Fig. 1 consisting of linear fractional descriptor part described by (4) and the nonlinear element with the characteristic $u = f(e)$ (Fig. 2).

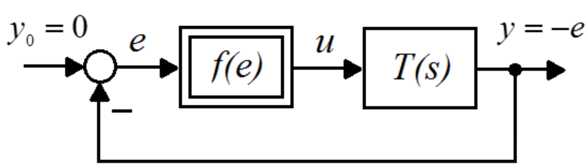

Fig. 1. Nonlinear feedback system

It is assumed that:

- the linear part with transfer function $T(s)$ is positive and asymptotically stable,
- the characteristic of the nonlinear element satisfies the condition

$$f(0) = 0,\; k_1 \le \frac{f(e)}{e} \le k_2,\; k_2 < +\infty. \tag{42}$$

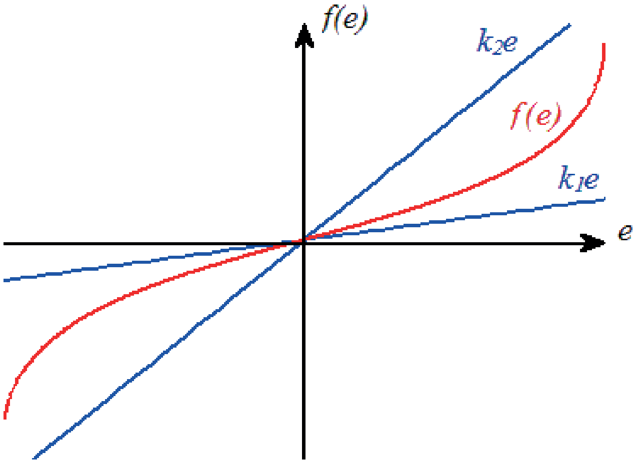

Fig. 2. Characteristic of nonlinear element

Definition 4. The nonlinear system is called globally (or absolutely) asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x(0) \in \Re_+^n. \tag{43}$$

Definition 5. The circle in the plane $(P(\omega), Q(\omega))$ with center in the point $\left(-\frac{k_1+k_2}{2k_1k_2}, 0\right)$ and radius $\frac{k_2-k_1}{2k_1k_2}$ is called the $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$ circle (see Fig. 3).

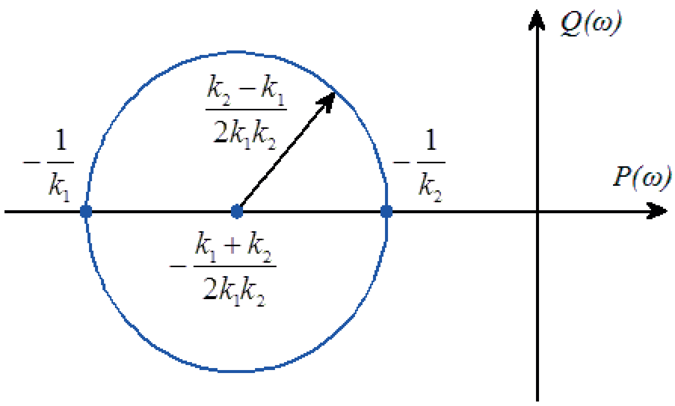

Fig. 3. Circle ilustrating Definition 5

For nonlinear systems with positive parts we have the following theorem.

Theorem 8. The nonlinear feedback system consisting of positive asymptotically stable linear part with the transfer function $T(s)$ and of nonlinear element with characteristic satisfying (42) is globally stable if the Nyquist plot of $T(j\omega) = P(\omega) + jQ(\omega)$ of the linear part is located on the right-hand side of the circle $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$.

Proof. The proof is given in [19].

It will be shown that Theorem 8 can be extended to nonlinear systems consisting of linear positive fractional descriptor globally stable part described by (4) and the nonlinear element with characteristic satisfying the condition (42).

Knowing the transfer function $T(s) = T_{sp}(s) + T_p(s)$ of the descriptor system we find its strictly proper part

$$T_{sp}(s) = C_1[I_r s^\alpha - A_1]^{-1} B_1 \tag{44}$$

and its Nyquist plot $T_{sp}(j\omega) = P(\omega) + jQ(\omega)$.

Theorem 9. The nonlinear feedback system consisting of the fractional positive descriptor asymptotically stable linear part (4) and the nonlinear element with the characteristic $u = f(e)$ satisfying the condition (42) is globally stable if the Nyquist plot of $T_{sp}(j\omega) = P(\omega) + jQ(\omega)$ of the linear part is located on the right-hand side of the circle $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$.

Proof. Note that the global stability of the descriptor nonlinear system is determined by the stability of the dynamical part (7a) with the transfer function (44). By Theorem 8 the nonlinear system with the dynamical part is globally stable if the Nyquist plot of $T_{sp}(j\omega)$ of the linear part is located on the right-hand side of the circle $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$. □

Example 4. Check the global stability of the nonlinear system shown in Fig. 1 with the linear part (18) and the nonlinear element with the characteristic $u = f(e)$ satisfying the condition (42). The strictly proper transfer function of the linear part has the form

$$T_{sp}(s) = \frac{2s^\alpha + 6}{s^{2\alpha} + 5s^\alpha + 4} \tag{45}$$

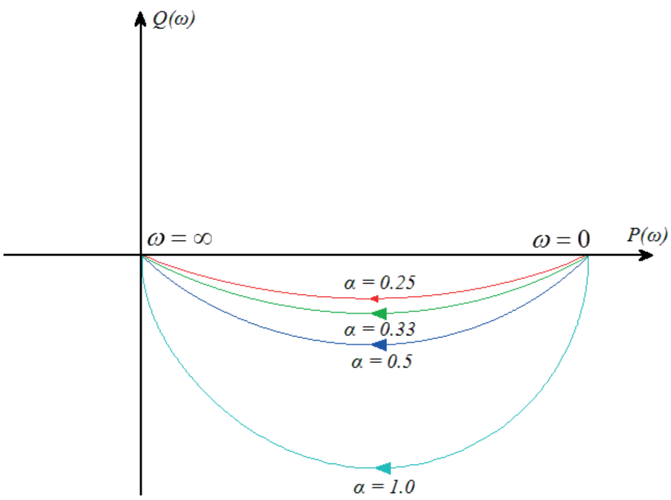

Fig. 4. Nyquist plot of (47)

and its Nyquist plot

$$T_{sp}(j\omega) = \frac{2(j\omega)^{\alpha}+6}{(j\omega)^{2\alpha}+5(j\omega)^{\alpha}+4} = P(\omega)+jQ(\omega), \quad (46)$$

for $\alpha = \{0.25, 0.33, 0.5, 1.0\}$ is shown in Fig. 3.

Note that the Nyquist plot of the linear part is located in the fourth quarter of the plane $(P(\omega), Q(\omega))$. Therefore, by Theorem 9 the nonlinear system is globally asymptotically stable for all nonlinear elements with characteristic $u = f(e)$ located in the first and third quarter (Fig. 2) since for any positive $k_2 > k_1 \geq 0$ the Nyquist plot is located on the right-hand side of the circle $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$.

6. Concluding remarks

The global stability of nonlinear systems with negative feedbacks and positive descriptor linear parts has been analyzed. The characteristics $u = f(e)$ of the nonlinear element satisfy the assumption (43) and the linear parts described by the equations (4) are asymptotically stable. It has been shown that the nonlinear systems are globally asymptotically stable if the Nyquist plots of the linear parts are located on the right-hand side of the circles $\left(-\frac{1}{k_1}, -\frac{1}{k_2}\right)$. This theorem is an extension of the Kudrewicz theorem presented in [20] for nonlinear systems with standard linear parts. The considerations have been illustrated by numerical examples.

Acknowledgment. This work was supported by National Science Centre in Poland under work No. 2017/27/B/ST7/02443.

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