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Synchronous motors linear, cylindrical and spherical with permanent magnets or excited

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Abstract. The paper deals with the problem of electromagnetic field analysis for linear, cylindrical and spherical electromechanical converters at synchronous state of work. There are considered synchronous motor with windings on moving part (rotor, carriage) and with permanent magnets thereon. The electromagnetic field is determined analytically with the help of separation method proposed for each problem. The boundary conditions are formulated for electromechanical converters linear, cylindrical and spherically shaped. The results obtained can be used as benchmark for electromagnetic field numerical analysis and force/torque calculations.

1. Introduction

The determination of forces or torques produced by electromagnetic field in synchronous motor is important problem due the fact that linear and spherical motors are still of interest [1-5]. Modern technologies enable to construct electromechanical converters with parts occurring wide range of suitable properties such as permanent magnets and superconductors. Especially, synchronous motors – linear, cylindrical and spherical present wide range of properties that can be useful for railway transport (linear motor) and mechatronics devices (spherical and cylindrical motors).

The intention of this paper is to present the analytical calculations for electromagnetic forces and torques basing on analytical solutions of electromagnetic field equations for linear, cylindrical and spherical synchronous motors. The analytical analysis plays important role in electromechanical converters technology [5-7].

2. Electromagnetic field equations

For force/torque calculation the distribution of electromagnetic field in the synchronous motor region is needed.

The first pair of Maxwell equations [8-10] take the well-known form

$$\operatorname{curl}\vec{E} = -\dot{\vec{B}} \quad \wedge \quad \operatorname{div}\vec{B} = 0 \cdot \tag{1}$$

The second pair of Maxwell equations can be presented in vector notation as follows

$$\operatorname{div}\vec{D} = \rho \quad \wedge \quad \operatorname{curl}\vec{H} = \vec{j} + \dot{\vec{D}} \cdot \tag{2}$$

Constitutive relations for electromagnetic field vectors for non-hysteresis, isotropic medium [8] are

$$H_u = \nu B_v, \tag{3}$$

$$D_u = \varepsilon E_v, \tag{4}$$

where e denote dielectric permittivity, ν are magnetic reluctivity, u, v mean particular co-ordinate of curvilinear system co-ordinate (summation due to twice appearing indices is accepted). For the three considered cases of synchronous motors only one component of magnetic vector potential does not vanish, it is denoted as third component i.e.

- for linear problem (Cartesian co-ordinate system 1-x, 2-y, 3-z, $\vec{i}_x \times \vec{i}_y = \vec{i}_z$, $L_x=L_y=L_z=1$ – Lame's coefficient Table 1)

$$\vec{A} = \vec{A}_z = A_z\vec{i}_z = A\vec{i}_z, \tag{5}$$

- for cylindrical problem (cylindrical co-ordinate system 1-r, 2-a, 3-z, $\vec{i}_r \times \vec{i}_\alpha = \vec{i}_z$, $L_r=1$, $L_a=r$, $L_z=1$)

$$\vec{A} = \vec{A}_z = A_z\vec{i}_z = A\vec{i}_z, \tag{6}$$

- for spherical problem (spherical co-ordinate system 1-r, 2-φ, 3-θ, $\vec{i}_r \times \vec{i}_\varphi = \vec{i}_\theta$, $L_r=1$, $L_\varphi = r\sin\theta$, $L_\theta=r$)

$$\vec{A} = \vec{A}_\theta = A_\theta\vec{i}_\theta = A\vec{i}_\theta. \tag{7}$$

Table 1
Lame coefficients for Cartesian, cylindrical and spherical co-ordinate systems

Co-ordinate system	L_1	L_2	L_3
Cartesian ($x_1=x$, $x_2=y$, $x_3=z$)	1	1	1
cylindrical ($x_1=r$, $x_2=\alpha$, $x_3=z$)	1	r	1
spherical ($x_1=r$, $x_2=\varphi$, $x_3=\theta$)	1	rsinθ	r

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The assumed magnetic vector potential functions (5)-(7) are due to the shape of stator magnetomotive force pattern and adequate co-ordinate system placement. The accuracy of such assumption for magnetic vector potential results from the technological construction of each electromechanical converter i.e. linear, cylindrical and spherical. However, these assumption are not exactly correct (e.g. end-effects), there are acceptable for analytical analysis. The magnetic flux density of vector magnetic potential can be presented as follows [6,8]

$$\vec{B} = \mathrm{curl}\vec{A}\,. \qquad (8)$$

The curl differential operator for curvilinear orthogonal co-ordinate system is as follows

$$\mathrm{div}\vec{A}=\frac{1}{L_1L_2L_3}\left[\frac{\partial}{\partial u_1}(L_2L_3B_1)+\frac{\partial}{\partial u_2}(L_3L_1B_2)+\frac{\partial}{\partial u_3}(L_1L_2B_3)\right] \qquad (9)$$

hence the Eq. (8) can be rewritten in unified form with Lame coefficients as given below

$$\vec{B} = \frac{\vec{i}_1}{L_2L_3}\frac{\partial(L_3A)}{\partial x_2} - \frac{\vec{i}_2}{L_1L_3}\frac{\partial(L_3A)}{\partial x_1}\,. \qquad (10)$$

The third component of magnetic flux density vanishes

$$B_3 = 0\,. \qquad (11)$$

The above presented notation simplifies the analysis for three cases of synchronous motors considered. It must be underlined that the numbers do not denote tensors components [2,11]. The magnetic field strength components due to equation (3) for the isotropic region can be shown in the form of

$$H_1 = \nu B_1\,, \qquad (12)$$

$$H_2 = \nu B_2\,. \qquad (13)$$

The Maxwell equation for conducting region if electric displacement current vanishes (small field frequency) takes the form of

$$\mathrm{curl}(\vec{H}) = \vec{j} = \gamma\vec{E} = -\gamma\dot{\vec{A}}\,, \qquad 14)$$

hence for third components

$$\frac{1}{L_1L_2}\left(\frac{\partial L_2H_2}{\partial x_1} - \frac{\partial L_1H_1}{\partial x_2}\right) = -\gamma\dot{A}_3 = -\gamma\dot{A}\,. \qquad (15)$$

Combining Eqs. (12), (13) and (15) it is obtained equation for vector magnetic component

$$\frac{\nu}{L_1L_2}\frac{\partial}{\partial x_1}(L_2B_2) - \frac{\nu}{L_1L_2}\frac{\partial}{\partial x_2}(L_1B_1) = -\gamma\dot{A}\,. \qquad (16)$$

hence

$$\frac{\partial}{\partial x_1}\left(-\frac{L_2\nu}{L_3}\frac{\partial AL_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\nu}{L_2}\frac{\partial A}{\partial x_2}\right) = -L_2\gamma\dot{A}\,, \qquad (17)$$

where it was taken into account that for all three cases (see Table 1) it is satisfied

$$\frac{\partial L_3}{\partial x_2} = 0 \quad \text{and} \quad L_1 = 1. \qquad (18)$$

For homogeneous region magnetic reluctivities are spatially constant, thus

$$\nu\frac{\partial}{\partial x_1}\frac{L_2}{L_3}\frac{\partial AL_3}{\partial x_1} + \nu\frac{\partial}{\partial x_2}\frac{\partial A}{L_2\partial x_2} = L_2\gamma\dot{A}\,. \qquad (19)$$

Subsequently, the Eq. (19) leads to relation for the linear converter (Cartesian co-ordinate system)

$$\nu\frac{\partial^2 A}{\partial x^2} + \nu\frac{\partial^2 A}{\partial y^2} = \gamma\dot{A}\,, \qquad (20)$$

for the cylindrical converter (cylindrical co-ordinate system)

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{\nu}{r^2}\frac{\partial^2 A}{\partial \alpha^2} = \gamma\dot{A}\,, \qquad (21)$$

for the spherical converter (spherical co-ordinate system)

$$\frac{\nu}{r}\frac{\partial^2(rA)}{\partial r^2} + \frac{\nu}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial \varphi^2} = \gamma\dot{A}\,. \qquad (22)$$

At complex analysis the time-partial derivative of A as presented as multiplication of the operand iw (i means imaginary unit) and the complex magnetic potential A at the steady state for time-sinusoidal varying fields [8] as follows

$$\dot{A} \rightarrow i\omega A\,, \qquad (23)$$

where w means rotor field pulsation. The magnetic vector potential is represented by complex vector. The real part of magnetic vector potential complex representation means the time-varying solution

$$A(t) = \mathrm{Re}\{Ae^{i\omega t}\}\,. \qquad (24)$$

For synchronous motors the rotor currents and field pulsation $\omega = 0$.

Table 2
Separation method chosen for Eqs. (20), (21) and (22)

Co-ordinate system	**A=A**(x_1, x_2, x_3)	Function
Cartesian $(x_1=x, x_2=y, x_3=z)$	$A=X(x)Y(y)$	$Y(y)=\exp(-iky)$
Cylindrical $(x_1=r, x_2=\alpha, x_3=z)$	$A=R(r)S(\alpha)$	$S(\alpha)=\exp(-ip\alpha)$
Spherical $(x_1=r, x_2=\varphi, x_3=\theta)$	$A=R(r,\theta)F(\varphi)$	$F(\varphi)=\exp(-ip\varphi)$

The Eqs. (20), (21) and (22) will be solved with the help of separation method [1,12,13]. The separated functions for all problems are collected in Table 2. These equations take the forms as given below.

- for the linear converter (Cartesian co-ordinate system)

$$\frac{d^2X}{dx^2} - k^2X = 0. \tag{25}$$

with the solutions in the form of (Table 4)

$$X(x) = a\exp(kx) + b\exp(-kx). \tag{26}$$

- for the cylindrical converter (cylindrical co-ordinate system)

$$\frac{d^2R}{dr^2} + \frac{dR}{rdr} - \frac{p^2}{r^2}R = 0, \tag{27}$$

with the solution in the form [14] (Table 5)

$$R(r) = ar^p + br^{-p}. \tag{28}$$

- for the spherical co-ordinate system

$$\frac{d^2rR}{dr^2} - (\kappa+1)\kappa\frac{R}{r} = 0, \tag{29}$$

with the solution (Table 6)

$$R(r,\theta) = ar^{\kappa 1} + br^{\kappa 2}v, \tag{30}$$

where κ_1, κ_2 denote the solutions of square equation

$$(\kappa+1)\kappa = p^2/\sin^2\theta, \tag{31}$$

that depend on the angle θ.

The solutions presented should be combined with the boundary conditions, then the unknown constants can be calculated.

3. Boundary conditions for electromagnetic field problems

There are defined four conditions for electromagnetic field vectors [8,10,14-16,], that enable to calculate the four unknown constants a_s, b_s, a_r, b_r – two for field exerted by stator currents (index s), and two for field of rotor/rail (index r).

The magnetic field strength exerted by stator disappears on the inner rotor surface (x=0 for linear motor, r=R for cylindrical and spherical motors)

$$H_2 = \nu B_2 = 0. \tag{32}$$

For rotor magnetic field strength Eq. (32) is satisfied on the stator surface (at x=g for linear motor, r=R+g for cylindrical and spherical motors). These condition results form the assumption that magnetic reluctivity of rotor and stator core are infinitely.

The stator magnetomotive force induced by moving part (carriage, rotor) currents leads to the condition for tangential component of magnetic field strength on rail/stator surface (either x=g or r=R+g) as follows

$$\nu B_2 = -\frac{\partial\Theta_s}{L_2\partial x_2}, \tag{33}$$

which is derived under the assumption that the magnetic field strength vanishes on the outer side of winding surface (magnetic reluctivity for stator core is infinite) [10,17,18]. The condition (33) is satisfied for rotor magnetic field strength at either x=0 or r=R. The boundary conditions defined are physically motivated and are grouped in Table 3.

For carriage/rotor with permanent magnets the the boundary condition is written in Table 3.

Table 3

Boundary	Stator field	Rotor field
x =g / r=R+g	$\nu B_2 = -\frac{\partial\Theta_s}{L_2\partial x_2}$	$\nu B_2 = 0$
x=0 / r=R	$\nu B_2 = 0$	$\nu B_2 = -\frac{\partial\Theta_r}{L_2\partial x_2}$ - for mmf $\nu B_2 = I_r$ - for magnets

Table 4
Solution of the differential equations

	Rail field (index s)	Carriage field (index r)
Solution constants	$X_s(x) = a_s\exp(kx) + b_s\exp(-kx)$ $a_s = \frac{\Theta_s}{\nu}\{e^{kx_g} - e^{2kx}e^{-kx_g}\}^{-1}$ $b_s = a_s\exp(2kx)$	$X_r(x) = a_r\exp(kx) + b_r\exp(-kx)$ $a_r = \frac{\Theta_r}{\nu}\{e^{kx} - e^{2kx_g}e^{-kx}\}^{-1}$ - for mmf $a_r = \frac{I_r}{\nu k}\{e^{kx} - e^{2kx_g}e^{-kx}\}^{-1}$ - for magnet: $b_r = a_r\exp(2kx_g)$
Magnetic flux densi ty components	$B_x = \frac{\partial A}{\partial y} = -ikX(x)e^{-iky}$	$B_y = -\frac{\partial A}{\partial x} = -dX(x)e^{-iky}$
Auxiliary functions	$dX_s(x) = ka_s\exp(kx) - kb_s\exp(-kx)$	$dX_r(x) = ka_r\exp(kx) - kb_r\exp(-kx$

Table 5
Solution of the differential equations for cylindrical co-ordinate system

	Stator field (index s)	Rotor field (index r)
Solutions	$R_s(r) = a_s r^p + b_s r^{-p}$	$R_r(r) = a_s r^p + b_s r^{-p}$
constants	$a_s = \Theta_s \nu^{-1}\{R_g^p - R^{2p}R_g^{-p}\}^{-1}$ $b_s = a_s R^{2p}$	$a_r = \Theta_r \nu^{-1}\{R^p - R_g^{2p}R^{-p}\}^{-1}$ for mmf $a_r = \frac{I_r R}{\nu p}\{R^p - R_g^{2p}R^{-p}\}^{-1}$ - for magnets $b_r = a_r R_g^{2p}$
Magnetic flux density components	$B_r = \frac{\partial A}{r\partial\alpha} = -ipR(r)e^{-ip\alpha}$	$B_\alpha = -\frac{\partial A}{\partial r} = -dR(r)e^{ip\alpha}$
Auxiliary functions	$dR_s(r) = pa_s r^{p-1} - pb_s r^{-p-1}$	$dR_r(r) = pa_r r^{p-1} - pb_r r^{-p-1}$

4. Solutions for electromagnetic field

The analysis of electromagnetic field of relations presented above and boundary conditions formulated in Section 3 lead to analytical solution for electromagnetic field problem. For the three geometrical problems chosen above the unknown constants are calculated, easily. The solutions are grouped in Tables 4-6.

4.1. Cartesian co-ordinate system – linear problem. Solutions of Eq. (20) for stator and rail at boundary conditions defined in Table 3 lead to the four unknown constants a_s, b_s, a_r, b_r values. The solutions presented in Table 4 enable to finish electromagnetic field analysis.

4.2. Cylindrical co-ordinate system. The solutions of Eq. (21) at boundary conditions given in Table 3 enable to calculate the four unknown constants a_s, b_s, a_r, b_r – see Table 5.

The solutions presented in Table 5 enable to evaluate magnetic field vector potential and magnetic field flux density.

Table 6
Solution of the differential equations for spherical co-ordinate system

	Stator field (index s)	Rotor field (index r)
Solutions	$R_s(r,\theta) = a_s r^{\kappa_1} + b_s r^{\kappa_2}$	$R_r(r,\theta) = a_r r^{\kappa_1} + b_r r^{\kappa_2}$
constants	$a_s = \frac{p\Theta_s}{\nu(\kappa_1+1)(R_g^{\kappa_1} - R^{\kappa_1-\kappa_2}R_g^{\kappa_1})}$ $b_s = -a_s\frac{\kappa_1+1}{\kappa_2+1}R^{\kappa_1-\kappa_2}$	$a_r = \frac{p\Theta_r}{\nu(\kappa_1+1)(R^{\kappa_1} - R_g^{\kappa_1-\kappa_2}R^{\kappa_1})}$ - for mm $a_r = \frac{I_r R\sin\theta}{\nu p(R^{\kappa_1} - R_g^{\kappa_1-\kappa_2}R^{\kappa_1})}$ - for magnets $b_r = -a_r\frac{\kappa_1+1}{\kappa_2+1}R_g^{\kappa_1-\kappa_2}$
Magnetic flux density components	$B_r = \frac{\partial A}{r\sin(\theta)\partial\varphi} = \frac{-ip}{r\sin(\theta)}R(r,\theta)e^{-ip\varphi}$, $B_\varphi = -\frac{\partial(rA)}{r\partial r} = \frac{-1}{r}dR(r,\theta)e^{-ip\varphi}$	
Auxiliary functions	$dR_s(r,\theta) = (\kappa_1+1)a_s r^{\kappa_1-1} + (\kappa_2+1)b_s r^{\kappa_2-1}$	$dR_r(r,\theta) = (\kappa_1+1)a_r r^{\kappa_1-1} + (\kappa_2+1)b_r r^{\kappa_2-1}$

4.3. Spherically shaped spherical electromechanical converter – spherical co-ordinate system. The solutions of Eq. (22) at boundary given in Table 3 enable to derive the four unknown constants a_a, b_a, a_r, b_r for stator and spherical rotor field. The analytical solution for the spherical problem can be presented in terms of separated function R(r,q) and F(j) obtained with the help of separation proposed. The solutions are presented in Table 6.

The partial differential equations are transformed into ordinary differential Eqs. (25), (27) and (31). The solutions for ordinary differential equations for separated functions R( ) are verified. There are calculated left hand LD and right hand LR sides of ordinary differential Eqs. (25), (27) and (29) for $r \in [R, R+g]$. Exemplary, the ratio Acc(r)=LD/RD for ordinary differential equation (29) for spherical problem is presented in Fig. 1. For exact solutions it should be satisfied Acc(r)=1).

electromagnetic force/torque for linear, cylindrical and spherical induction motor problems. Basing on the main equation for electromagnetic forces density there is presented and discussed force/torque value acting in electromechanical converters. The analytical relations for force/torque are the fundamental for material parameters influence analysis. The presented below solutions could be used as a benchmark for numerical algorithms, too.

6. Electromagnetic field force equations

The electromagnetic force density is given by Eq. (A.11) which is valid for each orthogonal curvilinear co-ordinate system. The electric displacement can be neglected, and Maxwell's stress tensor consists only of magnetic field vector components due to considered range of the frequency for electromechanical converter. Hence, the total force density is given by the following relation

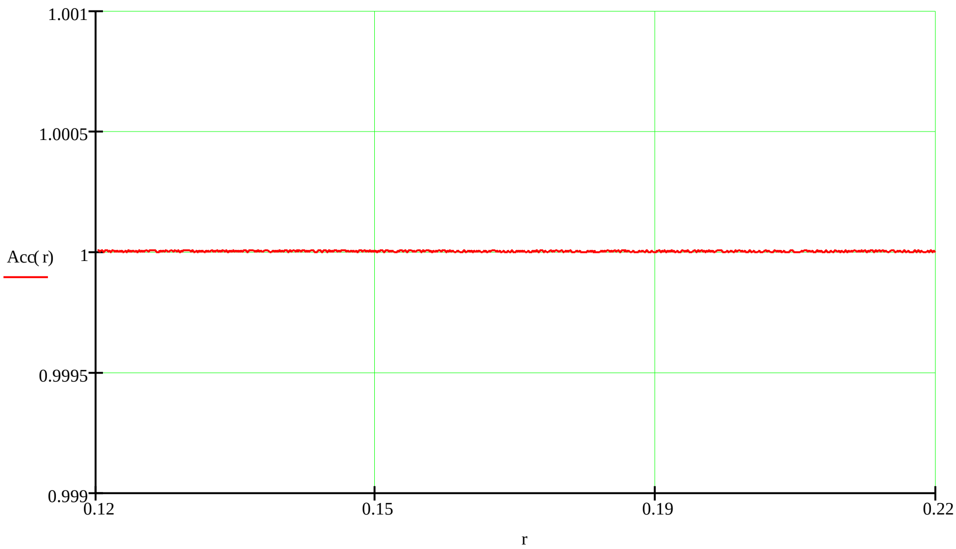


Fig. 1. Accuracy for differential equation solution – spherical problem
(θs=120 A, θr=300 A, R=0.1 m, g=0.05m, p=2)

Also, the boundary conditions fulfillments are checked for boundary conditions (Table 3) for linear, cylindrical and spherical problems, too. The calculated constants a_s, b_s, a_r, b_r which appears in provided solutions (26), (28) and (30) are evaluated correctly.

5. Electromechanical converters

The determination of force/torque induced by electromagnetic field in electromechanical converters is important problem [1-4,11,19]. For electromechanical converters the acting force/torque plays often deciding role for users. Modern technologies enable to construct electromechanical converters with parts having wide range of properties. Especially, the synchronous motors – linear, cylindrical and spherically shaped play important role for mechatronics and robotics purposes. The intention of this paragraph is to present the analytical solutions for

$$\vec{f} = \vec{f}_L + \vec{Q} = -\vec{i}_u \operatorname{div}_u(\vec{\sigma}_u) - \vec{\Delta}, \tag{34}$$

where all functions are defined in Appendix 1. It should be emphasized that the operator div( ) in Eq. (2) differs from the operator div_u( ) in (A.11).

For synchronous motors with coiled moving part (carriage, rotor) the total force/torque results only from Lorentz force, thus

$$\vec{f} = \vec{f}_L = -\vec{i}_u \operatorname{div}_u(\vec{\sigma}_u) - \vec{\Delta}. \tag{35}$$

The total electromagnetic force/torque for synchronous motor with permanents magnets on moving part (carriage, rotor) is exerted by magnets and is described only by hysteresis component

$$\vec{f} = \vec{Q} = -\vec{i}_u \operatorname{div}_u(\vec{\sigma}_u) - \vec{\Delta}. \tag{36}$$

Let us physically interpret the hysteresis component features. For Cartesian co-ordinate system ($L_x = L_y = L_z$ = 1, L = 1, k=0) force component for u=x of Eq. (36a) describes the electromagnetic force density

$$f_x = -\mathrm{div}(\vec{\sigma}_x) = Q_x, \tag{37a}$$

Taking into account Eq. (A.6f) it can be written

$$f_x = Q_x = -B_u \frac{\partial I_u}{\partial x}. \tag{37b}$$

For depicted below simple example the physical representation of hysteresis component force is presented. Namely, the permanents magnets only for u = y magnetization does not vanish $I_y \neq 0$.

The forced magnetic field is oriented along y-axis as shown in Fig. 2. The permanent magnet is put into this field so as magnetization vector is parallel to y-axis. The electromagnetic force is given by hysteresis component and appears at x=a where the spatial derivative of I_y does not vanish: $\frac{\partial I_y}{\partial x} \neq 0$.

According to Eq. (37b) force density $f_x > 0$ appears at x=a. The orientation of the force is rather evident due to the placement of S-poles and N-poles as shown in Fig. 2.

Equations (35) and (36) are useful for force/torque evaluation and they enable for surface representation of

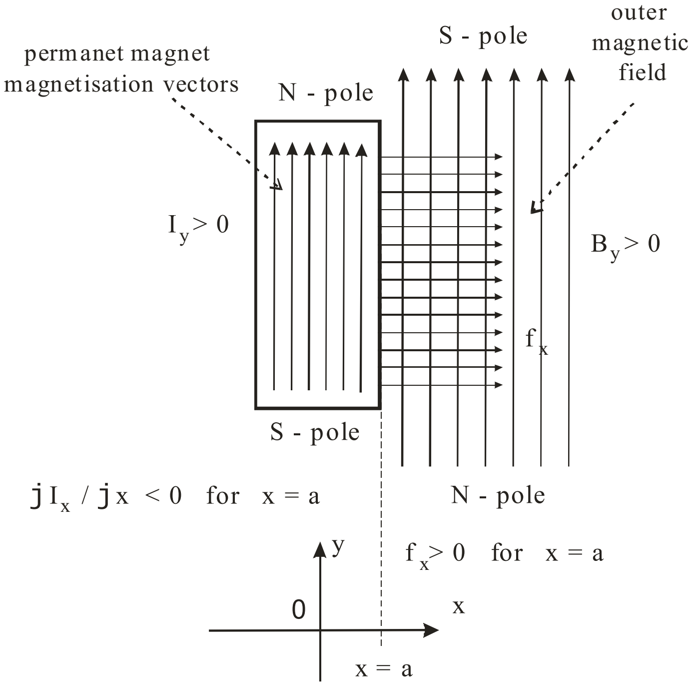


Fig. 2. Magnets of the exemplary electromechanical converter

total electromagnetic torque (see Section 7) i.e. total force/torque can be calculated by means of surface integral, not only by volume integral.

7. Electromechanical synchronous converters – linear, cylindrical and spherical

The analysis of electromagnetic field is the basis for electromechanical converter force/torque analysis. The obtained solution for magnetic field vector potential will be used for force/torque calculation.

7.1. Linear synchronous motor. Linear motors are used in electrical traction and in robotics technologies. The simplified model and dimensions thereof are presented in Fig. 2 taking into account that only first mmf harmonic for rail is important the magnetic vector field potential distribution both the magnetic flux density components and the electromagnetic force can be evaluated, analytically. The Lorentz's force describes the force for synchronous linear motor with coiled rail and carriage. Otherwise, the second component – the called hysteresis component – describes total force for electromechanical converter with permanent magnets on carriage. In both cases the Maxwell stress tensor leads to the total force values with the help of well-known formula

$$F_e = \nu \int_{\partial V} B_x B_y dS. \tag{38}$$

Firstly, the electromagnetic force for synchronous motor with coils on carriage can be evaluated by Lorentz's force density as follows

$$F_L = \int_V j_z B_x dV. \tag{39}$$

Basing on the magnetic flux density complex representations given in Table 4 the electromagnetic forces can be calculated analytically (Appendix 2 and 5).

The difference between them – often called material force – is equal to

$$F_{Fe} = F_e - F_L, \tag{40}$$

and disappears according to Eq. (51) when region is homogeneous and isotropic

$$F_{Fe} = 0. \tag{41}$$

Secondly, for the synchronous motor with permanent magnets - Fig. 3 the total force is given by Eq. (37a) and can be evaluated by means of hysteresis component Eq. (36) (see Appendix 6)

$$Q_y = -B_u \frac{\partial(\Delta I_u)}{L_y \partial y} = f_y = -\frac{1}{L_y} \mathrm{div}(-L_y H_y \vec{B} + \vec{i}_y e), \tag{42}$$

where all functions are defined in Appendix 1.

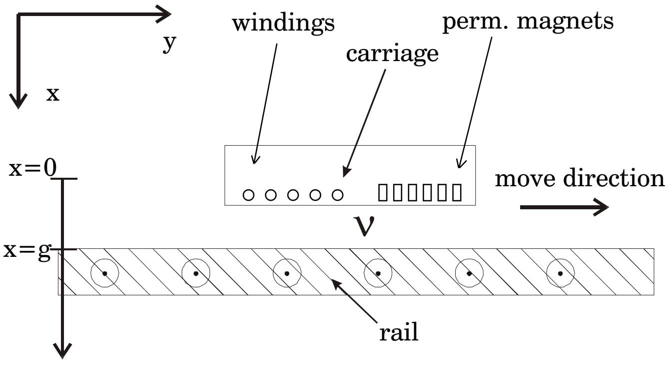


Fig. 3. Linear motor with either windings or permanent magnets on carriage

All forces are calculated for chosen linear motor (Mathcadä accuracy 10^{-15}) - Fig. 4.

The electromagnetic force is calculated twice due to both Maxwell and Lorentz's methods – Fig. 4. Analogously, the electromagnetic force is calculated twice due to both Maxwell and hysteresis component for linear synchronous motor wit permanent magnets. The results obtained enable to discuss influence of parameters on electromagnetic force developed by electromechanical converter versus air-gap width Fig. 5.

Constants from boundary conditions for Laplace's equation:

$$as := \frac{\theta s}{v y \delta}\cdot\left(e^{k\cdot xg} - e^{2\cdot k\cdot x}\cdot e^{-k\cdot xg}\right)^{-1} \quad bs := as\cdot e^{2k\cdot x} \quad Ds(x) := as\cdot e^{k\cdot x} + bs\cdot e^{-k\cdot x} \quad dDs(x) := k\cdot as\cdot e^{k\cdot x} - k\cdot bs\cdot e^{-k\cdot x}$$

$$ar := \frac{Ir}{v y \delta\, k}\cdot\left(e^{k\cdot x} - e^{2\cdot k\cdot xg}\cdot e^{-k\cdot x}\right)^{-1} \quad br := ar\cdot e^{2k\cdot xg} \quad Dr(x) := ar\cdot e^{k\cdot x} + br\cdot e^{-k\cdot x} \quad dDr(x) := k\cdot ar\cdot e^{k\cdot x} - k\cdot br\cdot e^{-k\cdot x}$$

Checking for boundary conditions fulfillment (1, 0, 1, 0):

$$\frac{v y\delta\, dDs(xg)}{\theta s\cdot k} = 1.000000000 \quad dDs(x) = 0.000000000 \quad \frac{v y\delta\, dDr(x)}{Ir} = 1.000000000 \quad dDr(xg) = 0.000000000$$

Electromagnetic force:

by the Lorentz force density

$$FeL := \pi\cdot h\cdot Ds(x)\cdot Ir\cdot \sin(k\cdot \Delta y)$$

$$FeL = 242.16495$$

by Maxwell stress tensor:

$$Fe := \pi\cdot h\cdot v y\delta\,(dDr(x)\cdot Ds(x) - dDs(x)\cdot Dr(x))\cdot \sin(k\cdot \Delta y)$$

$$Fe = 242.16495$$

$$\frac{FeL}{Fe} = 1.0000000$$

Fig. 4. Force calculations for synchronous linear motor with permanent magnets - Mathcad™ program ($q=3000$ A, $I_r=1000$ A/m, $h=0.5$ m, $g=0.01$ m, $k=2.5$ m^{-1})

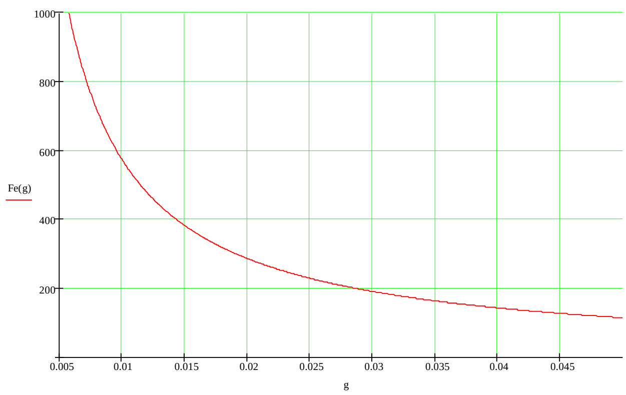


Fig. 5. Linear motor coiled force vs. air-gap width [N vs. m] ($q_s=3000$ A, $q_r=1000$ A, $h=0.5$ m, $g=0.01$ m, $k=3.5$ m^{-1})

The analytical model presented enables to present solution for electromagnetic filed distribution and force values for linear synchronous motors both coiled and with permanent magnets.

7.2. Cylindrical motor. The rotating - cylindrically shaped synchronous motors are commonly used in industry. Often, the permanent magnets are put on rotor surface for exciting the magnetic field. Synchronous motor with coiled rotor is considered, too. The simplified model and dimensions thereof are presented in Fig. 6.

Let us consider the model of the synchronous motor presented in Fig. 6. The motor is either coiled (supplied by DC current) or with permanent magnets that induce radial magnetic field. The magnetic rotor does not exhibit hysteresis phenomenon. The machine rotor is cylindrical - and its outer radius is R. The air-gap width equals to

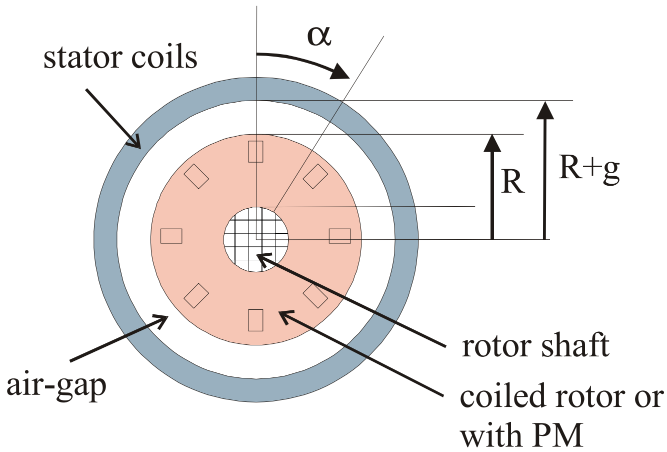


Fig. 6. Synchronous cylindrical motor model

g=const. The stator windings (thin layer on inner stator surface) exerts the sinusoidal 2p-pole magneto motive force

$$\Theta_s(\alpha) = \Theta_s \cos(p\alpha - 2\pi f t), \tag{43}$$

where Θ_s stands for the magnitude of mmf, a is the position angle, f means the stator supply frequency. For such a simplified model the vector magnetic field potential, magnetic flux density components and the electromagnetic torque are evaluated, analytically. The Maxwell stress tensor leads to the total electromagnetic torque by means of the formula

$$T_e = \nu r \int_{\partial V} B_\alpha B_r dS\,. \tag{44}$$

For synchronous motor with coiled rotor the electromagnetic torque can be evaluated by Lorentz force density as follows

$$T_{eL} = \int_{V} r j_z B_r dV\,. \tag{45}$$

Torques are calculated by particular relation developed in Appendix 3 and 5. As an example the synchronous motor with coiled rotor is taken into account. The electromagnetic torque curves versus power angle d for two stator mmf are presented in Fig. 7.

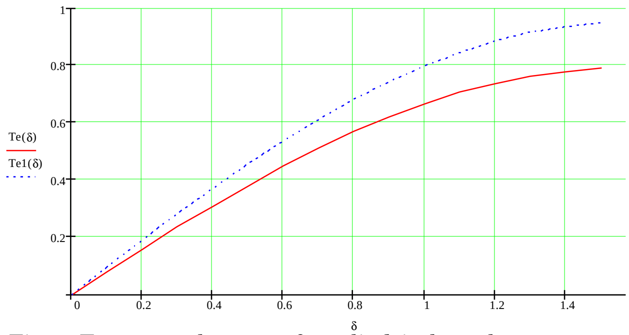


Fig. 7. Torque-angle curves for cylindrical synchronous motor: Te(δ) [Nm] for stator mmf θ_s = 120 [A], and Te1(δ) for stator mmf θ_s = 145 [A] (θ_r=90 A, p=2, R=0.15 m, g=0.01 m, l=0.4 m)

For the synchronous motor with permanent magnets the total torque can be calculated with the Maxwell stress tensor method and by means of hysteresis component - Eq. (36), too. The hysteresis component describes the total torque density as follows

$$f_\alpha = Q_\alpha = -B_u \frac{\partial(\Delta I_u)}{L_\alpha \partial\alpha} = -B_r \frac{\partial \Delta I_r}{r\partial\alpha}\,, \tag{46}$$

and it enables to calculate the total torque (Appendix 6). The both methods give the same electromagnetic torque values as shown in Fig. 6. For the synchronous cylindrical motor the torque-angle curves are presented in Fig. 8.

The analyses carried for cylindrically shaped electromechanical converter have brought out to the issue that for the synchronous motor the hysteresis component method can be used successfully.

7.3. Spherical motor. Spherically shaped motors are used in robotics technologies and as actuators [1,2,9,14]. The simplified model and dimensions thereof are presented in Fig. 9 and its dimensions in Fig. 6 (analogously as for the cylindrical motor)

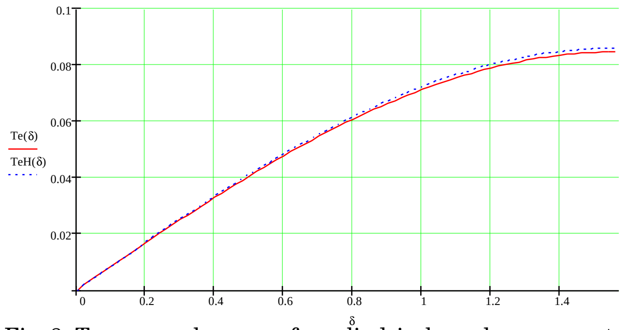


Fig. 8. Torque-angle curves for cylindrical synchronous motor with permanent magnets: Te(δ) [Nm] for Maxwell method, and TeH(δ) for hysteresis component method
(θ_s = 150 A, I_r=1000 A/m, p=2, R=0.15 m, g=0.025 m, l=0.5 m)

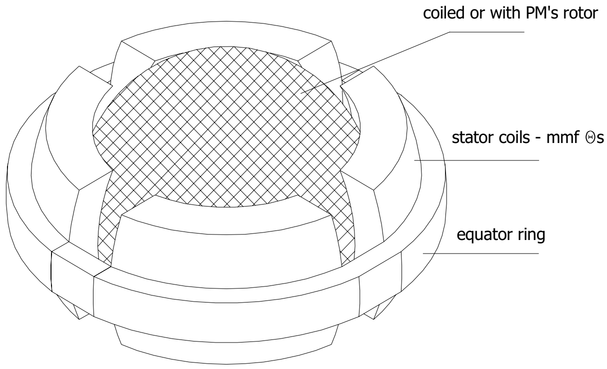


Fig. 9. Spherically shaped electromechanical converter

The analytical solution for magnetic field for the spherical motor is given in terms of separated function R(r,q) and F(j) obtained with the help of non-standard separation proposed – Table 2. The exemplary spherical induction motor which rotor contains permanent magnets is considered. The monoharmonic magnetomotive force of stator is spread in longitudinal direction. The permanents magnets exert magnetic field which together with stator field determine the magnetic field distribution and electromagnetic torque. There is only synchronous motors with permanent magnets on rotor surface discussed (the coiled rotor could be considered theoretically only).

For spherical co-ordinate system (L_r = 1, L_φ = rsinq, L_θ = r) for horizontal force component u = j, the Eq. (36) describes the electromagnetic force density as follows

$$f_\varphi = -\frac{1}{L_\varphi} \mathrm{div}(-L_\varphi H_\varphi \vec{B} + \vec{i}_\varphi e) = Q_\varphi = -B_u \frac{\partial(\Delta I_u)}{L_\varphi \partial\varphi}\,, \tag{47}$$

For the permanent magnets placed in the rotor due to the motor construction practically only magnetization vector for u = r is dominant, so $\Delta I_r \neq 0$. Hence, electromagnetic torque density equals to

$$f_\varphi r\sin\theta = -\mathrm{div}(-r\sin\theta H_\varphi \vec{B} + \vec{i}_\varphi e) = Q_\varphi r\sin\theta = -B_r \frac{\partial \Delta I_r}{\partial \varphi}, \quad (48)$$

hence

$$Q_\varphi = -B_r \frac{\partial \Delta I_r}{r\sin\theta\partial\varphi}. \quad (49)$$

For the synchronous motor with permanent magnets and coiled the electromagnetic torque is calculated twice by Maxwell stress tensor T_e and hysteresis component T_{eH} (Appendix 4 and 6).

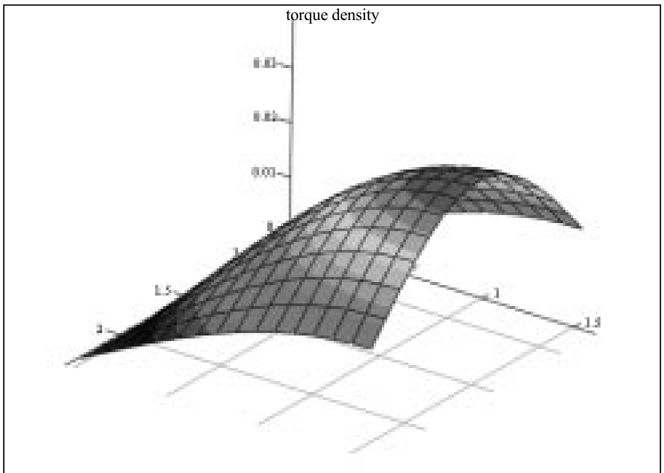
The both torque values are equal

$$T_e = T_{eH} = \int_V r\sin(\theta) Q_\varphi dV. \quad (50)$$

Exemplary, for spherical motor with permanent magnets the angular torque density

$$t_e = \frac{\partial T_e}{\partial \theta}, \quad (51)$$

versus altitudinal angle θ and power angle d is presented in Fig. 8.



(θ,δ,te)

Fig. 10. Electromagnetic torque density t_e [N/rad] versus altitudinal angle q [rad] (left-hand axis) and power angle d [rad] (right-hand line) for motor with PM?s for q_s = 350 A, I_r=1000 A/m, p=2, R=0.12 m, g=0.015 m, q=q_1¸q_2, q_1=p/2-p/4, q_2=p/2+p/4, d=0¸p/2

8. Conclusions

- The electromagnetic field analytical analyses for synchronous motors linear, cylindrical and spherical are obtained. However the analyses are carried out for simplified models of electromechanical converters they could improve design process for electromechanical converters in spite of the fact its approximate form.
- The separation method has been used for electromagnetic field analysis – Table 2. For spherical co-ordinate system the non-standard separation has been proposed. The solutions for vector magnetic field potential are obtained and shown in Tables 4-6 for the boundary conditions are defined - Table 3.
- The paper presents electromagnetic force/torque calculations described in analytical way that could be treated as benchmark tasks for numerical algorithms.
- There are approached analytically the electromagnetic torque: Lorentz T_{eL} for synchronous motors with coiled moving part (carriage, rotor) and hysteresis force component Q for synchronous motor with permanent magnets. The electromagnetic torque for each motor has been calculated for second times with the help of Maxwell stress tensor method that confirms the accuracy of the calculations carried out.
- The force/torque components indicate the physical reason for theirs induction i.e. Lorentz force indicates for the currents and the hysteresis component for the permanents magnets.
- The mathematical prove for the main Eq. (34) for force density has been provided (Appendix 1). The operator defined in Eq. (A.6b) could be convenient due to its mathematical form for analytical calculations.

Appendix 1. Electromagnetic field forces

Electromagnetic field force volume density in curvilinear co-ordinate system can be presented with the help of Maxwell equation

$$\mathrm{curl}\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \quad (A.1)$$

and Lorentz's force density in the form of

$$\vec{f}_L = \rho\vec{E} + \vec{j}\times\vec{B}, \quad (A.2)$$

that constitute one component of total electromagnetic field force.

$$\vec{f}_L = \rho\vec{E} + \vec{j}\times\vec{B}. \quad (A.3)$$

Furthermore,

$$\vec{f}_L = \vec{j}\times\vec{B} + \rho\vec{E} = \mathrm{rot}\vec{H}\times\vec{B} + \vec{H}\mathrm{div}\vec{B} - \frac{\partial(\vec{D}\times\vec{B})}{\partial t} + \vec{D}\times\mathrm{rot}\vec{E} + \vec{E}\mathrm{div}\vec{D}, \quad (A.4)$$

where it was added $\vec{H}\cdot\mathrm{div}\vec{B} = 0$.

The constitutive relation takes the following form for hysteresis regions

$$H_u = \nu_{uw} B_w - \Delta I_u, \quad (A.5)$$

where reluctivities are symmetrical.

The second and third component on the right-hand (4) can be written in the form of

$$\mathrm{rot}\vec{H}\times\vec{B} + \vec{H}\,\mathrm{div}\,\vec{B} = \vec{i}_u\,\mathrm{div}_u(\vec{\sigma}_{\mu u}) - \vec{\Delta}_\mu - \vec{N}_\mu - \vec{Q}_\mu, \quad (A.6a)$$

where it was denoted

$$\vec{i}_u \mathrm{div}_u(\{*\}) = \vec{i}_u L_u^{-1} \mathrm{div}_u(L_u \cdot \{*\}), \tag{A.6b}$$

$$\vec{\sigma}_{\mu u} = -H_u \vec{B} + \vec{i}_u e_\mu, \tag{A.6c}$$

$$e_\mu = \tfrac{1}{2}\vec{H}\vec{B}, \tag{A.6d}$$

$$\vec{N}_\mu = \tfrac{1}{2} B_u B_w \,\mathrm{grad}(\nu_{uw}), \tag{A.6e}$$

is called nonhomogenous force component

$$\vec{Q}_\mu = \frac{1}{2}\mathrm{grad}(B_j \Delta I_j^*) - B_j \mathrm{grad}(\Delta I_j^*), \tag{A.6f}$$

is called hysteresis component, and an auxiliary vector

$$\vec{\Delta}_\mu = \tfrac{1}{2} B_v H_v^* \vec{i}_u \frac{\partial \ln(L_v^2/|L|)}{L_u \partial x_u}, L=L_u L_v L_w. \tag{A.6g}$$

The fourth and fifth component on the right-hand side of (A.4) can be rearranged analogously

$$\mathrm{curl}\vec{E}\times\vec{D} + \vec{E}\,\mathrm{div}\vec{D} = \vec{i}_u \,\mathrm{div}_u(\vec{\sigma}_{\varepsilon u}) - \vec{\Delta}_\varepsilon - \vec{N}_\varepsilon - \vec{Q}_\varepsilon, \tag{A.7}$$

where the constitutive relation (dielectric permittvities are symmetrical)

$$D_u = \varepsilon_{uw} E_w - P_u, \tag{A.8}$$

and

$$\vec{\sigma}_{\varepsilon u} = -E_u \vec{D} + \vec{i}_u e_\varepsilon. \tag{A.9a}$$

$$\vec{N}_\varepsilon = -\tfrac{1}{2} E_u E_w \,\mathrm{grad}(\varepsilon_{uw}), \tag{A.9b}$$

$$\vec{Q}_\varepsilon = -\tfrac{1}{2}\mathrm{grad}(\Delta P_j E_j^*) + E_j^* \mathrm{grad}(\Delta P_j), \tag{A.9c}$$

$$\vec{\Delta}_\varepsilon = \tfrac{1}{2} D_v E_v^* \vec{i}_u \frac{\partial \ln(L_v^2/|L|)}{L_u \partial x_u}, \tag{A.9d}$$

and an auxiliary vector

$$\vec{f}_L = -\frac{\partial(\vec{D}\times\vec{B})}{\partial t} - \vec{i}_u \mathrm{div}_u(\vec{\sigma}_u) - \vec{\Delta} - \vec{N} - \vec{Q} - \vec{M}. \tag{A.9d}$$

Hence, Eq. (A.4) takes the form of
,(A.9e)

where

$$\vec{\sigma}_u = -E_u \vec{D} - H_u \vec{B} + \vec{i}_u e, \tag{A.10a}$$

$$e = e_\varepsilon + e_\mu = \tfrac{1}{2}\vec{E}\vec{D} + \tfrac{1}{2}\vec{H}\vec{B}, \tag{A.10b}$$

$$\vec{N} = \vec{N}_\varepsilon + \vec{N}_\mu = -\tfrac{1}{2} E_u E_w \,\mathrm{grad}(\varepsilon_{uw}) + \tfrac{1}{2} B_u B_w \,\mathrm{grad}(\nu_{uw}), \tag{A.10c}$$

$$\vec{Q} = \vec{Q}_\varepsilon + \vec{Q}_\mu. \tag{A.10d}$$

The total force density
is equal to

$$\vec{f} = \vec{f}_L + \vec{f}_P + \vec{N} + \vec{Q} + \vec{M} = -\vec{i}_u \mathrm{div}_u(\vec{\sigma}_u) - \vec{\Delta}, \tag{A.11}$$

where Maxwell stress tensor equals to

$$\vec{\sigma}_u = -H_u \vec{B} - E_u \vec{D} + \vec{i}_u e, \tag{A.12}$$

and Poynting force (of electromagnetic field momentum)

$$\vec{f}_P = \frac{\partial(\vec{D}\times\vec{B})}{\partial t}. \tag{A.13}$$

Appendix 2. Electromagnetic force calculations - linear converter

The electromagnetic force over axial period for $\psi \in [0, l]$ equals to

$$F_e(t) = \nu \int_{\partial V} B_x(t) B_y(t) dS = \nu l \int_0^{2\pi/k} B_x(t) B_y(t) dy, \tag{A.14}$$

where

$$B_x(t) B_y(t) = \mathrm{Re}\{B_x e^{i\omega t}\}\mathrm{Re}\{B_y e^{-i\omega t}\} = $$
$$= \tfrac{1}{2}\mathrm{Re}\{B_x e^{i\omega t}(B_y e^{i\omega t} + B_y^* e^{-i\omega t})\} \tag{A.15}$$

hence

$$B_x(t) B_y(t) = \tfrac{1}{2}\mathrm{Re}\{B_x B_y e^{i2\omega t} + B_x B_y^*\}. \tag{A.16}$$

Putting in the presented in Table 2 solutions there is obtained

$$B_x(t) B_y(t) = \tfrac{1}{2}\mathrm{Re}\{ik(X_s e^{-iky} + X_r e^{-ik(y-\Delta y)})(dX_s e^{-iky} + $$
$$+ dX_r e^{-ik(y-\Delta y)})e^{i2\omega t} + ik(X_s e^{-iky} + X_r e^{-ik(y-\Delta y)})$$
$$(dX_s^* e^{iky} + dX_r^* e^{ik(y-\Delta y)})\}, \tag{A.17}$$

where Dy denotes spatial displacement between rail and carriage fields. Furthermore

$$B_x(t) B_y(t) = -\tfrac{k}{2}\mathrm{Im}\{(X_s + X_r e^{ik\Delta y})(dX_s + dX_r e^{ik\Delta y})e^{i2(\omega t - ky)}$$
$$+ (X_s + X_r e^{ik\Delta y})(dX_s^* + dX_r^* e^{-ik\Delta y})\}. \tag{A.18}$$

It should be pointed out that any multiplication of function and its derivative is real function; hence they can be omitted in Eq. (A.5), thus ($\delta = k\Delta y$)

$$B_x(t) B_y(t) = -\tfrac{k}{2}\mathrm{Im}\{(X_s + X_r e^{i\delta})(dX_s + dX_r e^{i\delta})e^{i2(\omega t - ky)}$$
$$+ (dX_s^* X_r e^{i\delta} + X_s dX_r^* e^{-i\delta})\}. \tag{A.19}$$

Integration over the interval $[0, 2\pi/k]$ according to (A.1) leads to relation

$$F_e = -\pi\nu l\,\mathrm{Im}(dX_s^* X_r e^{i\delta} + X_s dX_r^* e^{-i\delta}) = $$
$$\pi\nu l(X_s dX_r^* - dX_s^* X_r)\sin(\delta). \tag{A.20}$$

Appendix 3. Electromagnetic torque calculations – cylindrical converter

The electromagnetic force over axial period for $\alpha \in [0, 2\pi]$, $z \in [0, l]$ equals to

$$T_e(t) = \nu \int_{\partial V} B_r(t)B_\alpha(t)dS = \nu r^2 l \int_0^{2\pi} B_r(t)B_\alpha(t)d\alpha, \quad (A.21)$$

where

$$B_r(t)B_\alpha(t) = \mathrm{Re}\{B_r e^{i\omega t}\}\mathrm{Re}\{B_\alpha e^{-i\omega t}\} = \tfrac{1}{2}\mathrm{Re}\{B_r e^{i\omega t} (B_\alpha e^{i\omega t} + B_\alpha^* e^{-i\omega t})\}, \quad (A.22)$$

hence

$$B_r(t)B_\alpha(t) = \tfrac{1}{2}\mathrm{Re}\{B_r B_\alpha e^{i2\omega t} + B_r B_\alpha^*\}. \quad (A.23)$$

Putting in the presented in Table 3 solutions there is satisfied

$$B_r(t)B_\alpha(t) = \tfrac{1}{2}\mathrm{Re}\{\frac{ip}{r}(R_s e^{-ip\alpha} + R_r e^{-ip(\alpha-\delta)})\frac{1}{r}(dR_s e^{-ip\alpha} + dR_r e^{-ip(\alpha-\delta)})e^{i2\omega t} + \frac{ip}{r}(R_s e^{-ip\alpha} + R_r e^{-ip(\alpha-\delta)})\frac{1}{r}(dR_s^* e^{ip\alpha} + dR_r^* e^{ip(\alpha-\delta)})\}, \quad (A.24)$$

where d denotes spatial displacement between rail and carriage fields. Furthermore

$$r^2 B_r(t)B_\alpha(t) = -\tfrac{p}{2}\mathrm{Im}\{(R_s + R_r e^{ip\delta})(dR_s + dR_r e^{ip\delta}) e^{i2(\omega t - p\alpha)} + (R_s + R_r e^{ip\delta})(dR_s^* + dR_r^* e^{-ip\delta})\}. \quad (A.25)$$

It should be pointed out that any multiplication of function and its derivative is real function; hence they can be omitted in Eq. (A.5), thus

$$r^2 B_r(t)B_\alpha(t) = -\tfrac{p}{2}\mathrm{Im}\{(R_s + R_r e^{ip\delta})(dR_s + dR_r e^{ip\delta})e^{i2(\omega t - p\alpha)} + (dR_s^* R_r e^{ip\delta} + R_s dR_r^* e^{-ip\delta})\}. \quad (A.26)$$

Integration over the interval [0, 2p] according to (A.1) leads to relation

$$T_e = -\pi p \nu l\, \mathrm{Im}(dR_s^* R_r e^{ip\delta} + R_s dR_r^* e^{-ip\delta}) = \pi p \nu l (R_s dR_r^* - dR_s^* R_r)\sin(p\delta). \quad (A.27)$$

Appendix 4. Electromagnetic torque calculations – spherical converter

The electromagnetic force over axial period for $j \in [0, 2\pi]$, $\theta \in [\theta_1, \theta_2]$ equals to

$$T_e(t) = \nu \int_{\partial V} r\sin(\theta) B_r(t)B_\varphi(t)dS = \nu r^3 \int_{\theta_1}^{\theta_2} \sin^2(\theta) \int_0^{2\pi} B_r(t)B_\varphi(t)d\varphi d\theta, \quad (A.28)$$

where

$$B_r(t)B_\varphi(t) = \mathrm{Re}\{B_r e^{i\omega t}\}\mathrm{Re}\{B_\varphi e^{-i\omega t}\} = \tfrac{1}{2}\mathrm{Re}\{B_r e^{i\omega t}(B_\varphi e^{i\omega t} + B_\varphi^* e^{-i\omega t})\}, \quad (A.29)$$

hence

$$B_r(t)B_\varphi(t) = \tfrac{1}{2}\mathrm{Re}\{B_r B_\varphi e^{i2\omega t} + B_r B_\varphi^*\}. \quad (A.30)$$

Putting in the presented in Table 3 solutions there is satisfied

$$B_r(t)B_\varphi(t) = \tfrac{1}{2}\mathrm{Re}\{\frac{ip}{r\sin(\theta)}(R_s e^{-ip\varphi} + R_r e^{-ip(\varphi-\delta)})\frac{1}{r}(dR_s e^{-ip\varphi} + dR_r e^{-ip(\varphi-\delta)})e^{i2\omega t} + \frac{ip}{r\sin(\theta)}(R_s e^{-ip\varphi} + R_r e^{-ip(\varphi-\delta)})\frac{1}{r}(dR_s^* e^{ip\varphi} + dR_r^* e^{ip(\varphi-\delta)})\} \quad (A.31)$$

where d denotes spatial displacement between rail and carriage fields (the functions $R_s()$ and $R_r()$ depend on r and θ that is not specified). Furthermore

$$r^2\sin(\theta)B_r(t)B_\varphi(t) = -\tfrac{p}{2}\mathrm{Im}\{(R_s + R_r e^{ip\delta})(dR_s + dR_r e^{ip\delta})e^{i2(\omega t - p\alpha)} + (R_s + R_r e^{ip\delta})(dR_s^* + dR_r^* e^{-ip\delta})\}. \quad (A.32)$$

It should be pointed out that any multiplication of function and its derivative is real function; hence they can be omitted in Eq. (A.5), thus

$$r^2\sin(\theta)B_r(t)B_\varphi(t) = -\tfrac{p}{2}\mathrm{Im}\{(R_s + R_r e^{ip\delta})(dR_s + dR_r e^{ip\delta})e^{i2(\omega t - p\alpha)} + (dR_s^* R_r e^{ip\delta} + R_s dR_r^* e^{-ip\delta})\}. \quad (A.33)$$

Integration over the interval j Î [0, 2p] according to (A.1) leads to relation

$$T_e = -\pi p \nu \int_{\theta_1}^{\theta_2} \mathrm{Im}(dR_s^* R_r e^{ip\delta} + R_s dR_r^* e^{-ip\delta})d\theta. \quad (A.34)$$

and finally

$$T_e = \pi p r \nu \left(\int_{\theta_1}^{\theta_2} (R_s(r,\theta)dR_r^*(r,\theta) - dR_s^*(r,\theta)R_r(r,\theta))\sin(\theta)d\theta \right)\sin(p\delta). \quad (A.35)$$

Appendix 5. Force/torque by Lorentz's force calculations – linear, cylindrical and spherical converters

The electromagnetic force/torque equals to

$$F_e / T_e(t) = \int_V L_2 j_3(t) B_1(t) dV = \tfrac{1}{2}\mathrm{Re}\{\int_{X_3}\int_{X_3}\int_{X_3} L_2 j_3 B_1^* L_1 dx_1 L_2 dx_2 L_3 dx_3\}, \quad (A.36)$$

Table A.1
Parameters for Cartesian, cylindrical and spherical co-ordinate systems

Co-ordinat e system	Cartesian $(x_1=x,\ x_2=y,\ x_3=z)$	cylindrical $(x_1=r,\ x_2=\alpha,\ x_3=z)$	spherical $(x_1=r,\ x_2=\varphi,\ x_3=\theta)$
X_1	$[0,g]$	$[0,R+g]$	$[0,R+g]$
X	0	R	R
X_2	$[0,2\pi/k]$	$[0,2\pi]$	$[0,2\pi]$
a	k	p	p
A	$2\pi/k$	2π	2π
X_3	$[0,h]$	$[0,l]$	$[\theta_1,\theta_2]$
F/T	$F_{eL} = \pi k h \Theta_r D_1^*\big\vert_g \sin(\delta)$	$T_{eL} = \pi p^2 l \Theta_r D_1^*\big\vert_R \sin(\delta)$	$T_{eL} = \pi p^2 R \int_{\theta_1}^{\theta_2} \Theta_r D_1^*\big\vert_R \sin(\theta) d\theta \sin(\delta)$

where the current equals to

$$j_3 = \frac{\partial\Theta}{L_2 \partial x_2}\delta(x_1 - X) = \frac{\partial\left(i\Theta_r e^{iax_2 - i\delta}\right)}{L_2\partial x_2}\delta_D(x_1-X) = -\frac{a\Theta_r}{L_2} e^{iax_2-i\delta}\delta_D(x_1-X). \qquad (A.37)$$

First integral (the inner one) equals to

$$\int_{X_1} L_2 j_3 B_1^* L_1 dx_1 = -\int_{X_1} L_2 \frac{a\Theta_r}{L_2} e^{iax_2 - i\delta}\delta_D(x_1-X)B_1^* dx_1 = -a\Theta_r e^{iax_2-i\delta} B_1^*\Big|_{x_1=X}, \qquad (A.38)$$

The second integral

$$-\int_{X_2} a\Theta_r e^{iax_2-i\delta} B_1^*\Big|_{x_1=X} L_2 dx_2 = -a\Theta_r \int_A e^{iax_2-i\delta}\frac{ia}{L_2} D_1^*\Big|_{x_1=X} e^{-iax_2} L_2 dx_2 = -ia^2 A\Theta_r D_1^*\Big|_{x_1=X} e^{-i\delta}. \qquad (A.39)$$

The third integral equals

$$F_e/T_e(t) = \tfrac12 \mathrm{Re}\{\int_{x_3}^{x_3} -ia^2 A\Theta_r D_1^*\Big|_{x_1=X} e^{-i\delta} L_3 dx_3\} = \tfrac12 A a^2 \int_{x_3}^{x_3} \Theta_r D_1^*\Big|_{x_1=X} L_3 dx_3 \sin(\delta). \qquad (A.40)$$

Appendix 6. Force/torque calculations by means hysteresis component

$$F_e/T_e = \int_V \tfrac12 L_2\, \mathrm{grad}\ (B_1 \Delta I_1)\, dV - \int_V L_2 B_1\, \mathrm{grad}\ (\Delta I_1)\, dV, \qquad (A.41)$$

The first integral over the volume vanishes due to the fact that ....

$$F_e/T_e = -\int_V L_2 B_1 \frac{\partial(\Delta I_1)}{L_2\partial x_2} dV = -\int_V B_1 \frac{\partial \Delta I_1}{\partial x_2} dV, \qquad (A.42)$$

for permanent magnets can be denoted

$$\Delta I_1 = \nu B_{1,mag}. \qquad (A.43)$$

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