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OPTIMAL FORM-FINDING OF CABLE SYSTEMS

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Tensile structures in general, achieve their load-carrying capability only after the process of initial form-finding. From the mechanical point of view, this process can be considered as a problem in statics. As cable systems are close siblings of trusses (cables, however, can carry tensile forces only), in our study we refer to equilibrium equation similar to those known from the theory of the latter. In particular, the paper regards designing pre-tensioned cable systems, with a goal to make them kinematically stable and such that the weight of so designed system is lowest possible. Unlike in typical topology optimization problems, our goal is not to optimize the structural layout against a particular applied load. However, our method uses much the same pattern. First, we formulate the variational problem of form-finding and next we describe the corresponding iterative numerical procedure for determining the optimum location of nodes of the cable system mesh. We base our study on the concept of force density which is a ratio of an axial force in cable segment to its length.

Keywords: cable systems, optimal form-finding, minimum weight, force density method

1. INTRODUCTION

The history of modern, large-span cable systems dates back to the 1950s when the Polish architect Maciej (Matthew) Nowicki designed the Dorton Arena (a.k.a. *Paraboleum*) in Raleigh, USA. This still existing construction is the first one whose roof was shaped as a net made of the high-strength steel wires. Since then, the idea of covering the buildings, especially public facilities, such as sport

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arenas, exhibition halls, or airport terminals, with cable-supported vaults has inspired many world-renown architects and civil engineers. Enumerating their works, even the ones which became iconic, lies outside the scope of the paper. Let us only mention the large-span roof covering of the Warsaw commercial hall "Supersam" – built in 1962 and destroyed in 2006. It was one of the earliest such realizations in Poland; its arch-and-cable roof had been designed under the supervision of Jerzy Hryniewiecki and Wacław Zalewski.

Frei Otto, the pioneer of modern lightweight architectural design greatly admired the freedom of formation offered by cable systems and other tensile structures, see [7]. The word "tensile" is crucial here, as these naturally flexible constructions have to be pre-tensioned to reach the required, load-carrying functionality. To visualize it, consider a single cable of length L_0, anchored at two points spanning the distance L, $L < L_0$. When subjected to self-weight only, the cable takes the form of a catenary, while with the additional force concentrated at any point along the span, the shape changes into a two-segment polygonal chain. In short, the shape of a pre-tensioned construction is tightly bonded with a load applied to it, and so understood freedom of formation is usually perceived as the main advantage of tensile structures.

It is thus clear that cable systems, and tensile structures in general, achieve the load-carrying capability only after the process of initial form-finding. From the mechanical point of view, this process can be considered as a problem in statics. As cable systems are close siblings of trusses (cables, however, can carry tensile forces only), let us refer to the equilibrium equation

$$\mathbf{B}^T\mathbf{n} = \mathbf{p} \tag{1.1}$$

where:

$\mathbf{B}$ – matrix reflecting the geometry of a truss; $\mathbf{n}$ – vector of axial forces; and $\mathbf{p}$ – vector of node loads.

Typically, $\mathbf{B}$ and $\mathbf{p}$ are fixed and the goal is to determine $\mathbf{n}$. The optimal form-finding problem calls for a different answer. Loosely speaking, it requires determining $\mathbf{B}$ for fixed $\mathbf{n}$ and $\mathbf{p}$, and by this it falls into the category of topology optimization problems formulated for constructions that are entirely in tension. Such constructions, along with those entirely in compression, are referred to as "Prager-structures", see [8], where this term was coined. These, in turn, fall into the broader context of "Michell structures", see [6] for the detailed discussion on both topics.

In this paper, we put forward a numerical procedure for designing pre-tensioned cable systems, which are kinematically stable and whose weight is lowest possible. We emphasize that, unlike in typical topology problems, our goal is not to optimize the structural layout against a particular applied load.

Main contribution of the paper consists in setting the algebraic framework for optimal form-finding problem. In the proposed approach, determining vectors belonging to the null-space of matrix **B** is of central importance. This turns out to pose non-trivial numerical challenges, not discussed in the so-far published research regarding optimal design of cable systems.

Our exposition follows the pattern which is well-established in the literature. First, we formulate the variational problem of optimum form-finding, next we describe the corresponding iterative numerical procedure, and finally we illustrate its application. In our study, we make use of the concept of *force densities* suggested in [9] by putting forward an iterative numerical scheme, but we point out that different computational algorithms, like e.g. *the ground structure method*, see [11], also seem capable of tackling the optimum problem at hand, see Sec. 7 for more details.

Optimal designing of cable structures has long been a research topic. Early theoretical works, see e.g. [4], [5], based on the concepts of differential geometry, were supplemented with computer-assisted studies over the course of recent years. Among many issues posed in this framework, optimizing with respect to steel profiles availability, see e.g. [1]; maximizing the stiffness of a net, [2]; or minimizing the values of forces in cables, [10]; are of special interest in light of our study. Various optimization problems and solution methods, including those based on force densities, are also discussed in [3].

2. THE OPTIMIZATION PROBLEM

Consider the following optimization problem:

Design a pre-tensioned cable system anchored at a rigid contour,
or a number of rigid posts, and having a minimum weight.

Let the system be composed of k cables forming a net of arbitrary mesh. Write $\bar{m}$ for the number of anchored nodes of the mesh and m for the number of the remaining ones (free nodes). The location of anchors is fixed in space; their Cartesian coordinates produce vectors $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}} \in R^{\bar{m}}$, while the position of free nodes is not given up front; suppose that they produce vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^{m}$. Each cable in the system consists of segments stretched between two nodes; in the sequel, we write s to denote the total number of segments in a cable system. The total weight of a cable mesh is

(2.1)
$$W = \gamma V, \qquad V = \sum_{S=1,\dots,s} V_S\,, \qquad V_S = \int_0^{L_S} A_S dl\,,$$

where:
$\gamma > 0$ – unit weight of cable material; $V_S > 0$ – volume of the S-th cable segment; $A_S > 0$ – cross-section area of the S-th segment; $L_S > 0$ – length of the S-th element.

From (2.1), it is immediate that weight minimization and volume minimization problems coincide. Therefore, in the remainder we focus on the latter.
Now fix $\sigma_T > 0$ for the limit tensile stress in a single segment. Thus,

$$N_S = A_S \sigma_T \,, \tag{2.2}$$

stands for the limit axial force in the S-th cable segment and it is clear that $N_S > 0$ in the entire system. Components of the vector $\mathbf{n} = (N_S)$ are *statically admissible* if the cable system is in equilibrium under the action of the pre-tensioning forces applied at the anchored nodes of the contour. Since the external load is absent, $\mathbf{p} = \mathbf{0}$ in (1.1), and the equilibrium equation of the net reads

$$\mathbf{B}^T \mathbf{n} = \mathbf{0} \,, \tag{2.3}$$

hence, the pre-tensioned net is in equilibrium if $\mathbf{n}$ lies in the null space of matrix $\mathbf{B}^T$.
We suppose that net-stretching forces are large enough to eliminate the effect of sagging due to the self-weight of cables. In other words, we tackle the case in which the values of axial forces in cables induced by stretching the entire system by forces applied at the anchors are dominating over the forces caused by the gravity load. Moreover, due to manufacturability constraints, it is reasonable to assume that the cross-section of a single cable is constant along its length; to denote this, we fix $A_S = A_0 \,,\; S = 1, \ldots, s$ in the sequel. In light of (2.2), the tensile forces are also constant and the volume of a cable net is now given by

$$V = \frac{1}{\sigma_T} \sum_{S=1,\ldots,s} N_S L_S \,,$$

where:
$L_S = L_S(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

Our conjecture is that the minimum volume, denoted by $\hat{V}$ in the sequel, is attained if the cable net is uniformly stretched to the limit tensile stress σ_T. In other words, we claim that forces in a cable system of least weight are constant and given by $N_S = A_0\sigma_T$, $S = 1, \dots, s$, and thus

$$\mathbf{n} = A_0\sigma_T\mathbf{1}\,, \tag{2.4}$$

where:
$\mathbf{1}$ – vector of ones.

The optimization problem introduced at the beginning of this section reads

$$(P_0) \quad \left| \quad \hat{V} = A_0 \min\left\{ \sum_{S=1,\dots,s} L_S(\mathbf{x},\mathbf{y},\mathbf{z}) \;\middle|\; \begin{array}{l} \mathbf{x},\mathbf{y},\mathbf{z} \in R^m, \\ \mathbf{B}^T\mathbf{1} = \mathbf{0} \end{array} \right\} \right. .$$

Justifying the conjecture in (2.4) is postponed to Section 4.

3. THE FORCE DENSITY NOTATION

Define the vectors of segment length projections at x-, y-, and z-axes of the fixed Cartesian system

$$\mathbf{h}_x = \mathbf{Cx} + \bar{\mathbf{C}}\bar{\mathbf{x}}\,, \quad \mathbf{h}_y = \mathbf{Cy} + \bar{\mathbf{C}}\bar{\mathbf{y}}\,, \quad \mathbf{h}_z = \mathbf{Cz} + \bar{\mathbf{C}}\bar{\mathbf{z}}\,.$$

Matrices $\mathbf{C}, \bar{\mathbf{C}}$ form the block matrix $[\mathbf{C}\ \bar{\mathbf{C}}]$, which is an *incidence matrix* of a cable net understood as the directed graph with s edges (net segments) and $m + \bar{m}$ vertices (net nodes.) Free nodes are numbered from 1 to m and the anchored ones from $m + 1$ to $m + \bar{m}$. Figure 1 shows a part of such a net.

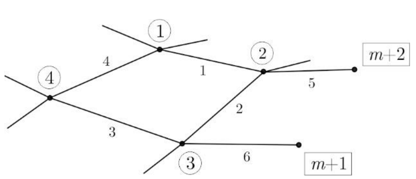

Fig. 1. A part of the cable net. Boxed numbers pertain to anchored nodes; circled numbers pertain to free nodes; plain numbers pertain to net segments.

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Assuming that the boxed numbers pertain to the anchored nodes and those circled to the free ones, and that each segment in the net is directed from the node with the lower number towards the node with the larger one, we get

$$\mathbf{C} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{\substack{s\text{ rows} \\ m\text{ columns}}}, \quad \bar{\mathbf{C}} = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}_{\substack{s\text{ rows} \\ \bar{m}\text{ columns}}}, \tag{3.1}$$

for the part of a net shown in Fig. 1. Rows of matrices in (3.1) show the connections between nodes. Here, "-1" corresponds to the beginning and "1" corresponds to the end of a given cable segment. For example, first row corresponds to segment 1 linking node 1 (free) with node 2 (free), the sixth one corresponds to segment 6 linking node 3 (free) with node $m+1$ (anchored), etc.
Now, use the notation in which $\mathbf{v} = [v_J]$, $J = 1, \ldots, j$, denotes a vector and $\mathbf{V} = \lceil v_J \rfloor$ is a corresponding diagonal matrix of dimensions $j \times j$. With this, the matrix of cable segment lengths is given by

$$\mathbf{L} = \left(\mathbf{H}_x \circ \mathbf{H}_x + \mathbf{H}_y \circ \mathbf{H}_y + \mathbf{H}_z \circ \mathbf{H}_z\right)^{\frac{1}{2}}, \tag{3.2}$$

where:
"$\circ$" symbol – the Hadamard product of matrices.

Namely, if $\mathbf{A}, \mathbf{B}$ are matrices of equal dimensions then the components of $\mathbf{C} = \mathbf{A} \circ \mathbf{B}$ are calculated through $C_{IJ} = A_{IJ} B_{IJ}$.

For further discussion, it is convenient to split (2.3) into *x*-, *y*-, and *z*-directions according to

$$(\mathbf{C}^T \mathbf{H}_x \mathbf{L}^{-1})\mathbf{n} = \mathbf{0}, \quad (\mathbf{C}^T \mathbf{H}_y \mathbf{L}^{-1})\mathbf{n} = \mathbf{0}, \quad (\mathbf{C}^T \mathbf{H}_z \mathbf{L}^{-1})\mathbf{n} = \mathbf{0}, \tag{3.3}$$

where:
matrix products in brackets represent matrices of direction cosines, i.e. the cosines of angles between segment axes and coordinate axes.

We also introduce the *force density vector*

$$\mathbf{q} = \mathbf{L}^{-1}\mathbf{n}\,,$$

and we fix

$$\mathbf{D} = \mathbf{C}^T\mathbf{Q}\mathbf{C}, \quad \bar{\mathbf{D}} = \mathbf{C}^T\mathbf{Q}\bar{\mathbf{C}}\,,$$

in (3.3), which allows for calculating the position of free nodes of the least weight cable net through

(3.4) $$\mathbf{x} = -\mathbf{D}^{-1}\bar{\mathbf{D}}\bar{\mathbf{x}}\,, \quad \mathbf{y} = -\mathbf{D}^{-1}\bar{\mathbf{D}}\bar{\mathbf{y}}\,, \quad \mathbf{z} = -\mathbf{D}^{-1}\bar{\mathbf{D}}\bar{\mathbf{z}}\,.$$

4. JUSTIFYING THE UNIFORM STRETCH CONJECTURE

Now, we turn back to the uniform stretch conjecture from Section 2. To justify it, we first define the functional

(4.1) $$\wp(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathrm{Tr}(\mathbf{Q} \circ \mathbf{L}^2(\mathbf{x}, \mathbf{y}, \mathbf{z}))\,,$$

where:
"Tr" – trace operator.

With (2.4) and (3.2) we get

$$\hat{\mathbf{q}} = \sigma_T A_0 \mathbf{1}\mathbf{L}^{-1}\,,$$

and thus

$$\wp(\hat{\mathbf{q}}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \sigma_T A_0 \mathrm{Tr}\big(\mathbf{L}(\mathbf{x}, \mathbf{y}, \mathbf{z})\big) = \sigma_T A_0 \sum_{S=1,\ldots,s} L_S(\mathbf{x}, \mathbf{y}, \mathbf{z})\,,$$

which shows that the functional in (4.1) attains the value of $\hat{V}$ for the uniform limit stretch in the entire cable system. This, in turn, confirms the conjecture set in Section 2.
Hence, (P_0) now reads

$$(P) \quad \left| \quad \hat{V} = \frac{1}{\sigma_T} \min \left\{ \wp(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \;\middle|\; \begin{array}{l} \mathbf{q} - \text{arbitrary} \\ \text{and} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in R^m, \text{ where} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ satisfy (3.7).} \end{array} \right\} \right. .$$

The optimization problem (P) serves as a basis for a numerical procedure put forward in Section 5.

5. NUMERICAL PROCEDURE FOR (*P*)

Formulae (3.4) determine the coordinates $\mathbf{x}, \mathbf{y}, \mathbf{z}$, of free nodes in the cable net. Due to the fact that the right hand sides of these formulae also depend on $\mathbf{x}, \mathbf{y}, \mathbf{z}$, the optimization problem (P) is nonlinear. We solve it through the iterative procedure generally described in [9]. Our purpose is to specify this procedure in light of the minimum weight problem.

Let,

$$\mathbf{g}(\mathbf{q}, \mathbf{x}(\mathbf{q}), \mathbf{y}(\mathbf{q}), \mathbf{z}(\mathbf{q})) = \mathbf{0}, \tag{5.1}$$

stand for the nonlinear equation representing the general form of constraints imposed on the cable system. Due to the assumed dependence of the free node coordinate vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ on the force density vector $\mathbf{q}$, in the sequel we write $\mathbf{g} = \mathbf{g}(\mathbf{q})$ for short. For the description of the iteration process, assume $\mathbf{q}_{n-1}$ for the vector of force densities at the beginning of the n-th iteration step, $n = 1,2, \ldots$, and $\mathbf{q}_0$ for the initial forces densities.

Next, introduce

$$\mathbf{b} = -\mathbf{g}\left(\mathbf{q}_{n-1}\right),$$

and

$$\mathbf{G} = \left.\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\mathbf{q}}\right|_{\mathbf{q}=\mathbf{q}_{n-1}}.$$

Consequently,

$$\mathbf{G}\, \Delta\mathbf{q} = \mathbf{b}, \tag{5.2}$$

where:

$\Delta\mathbf{q}$ – vector of increments of force densities.

Equation (5.2) allows for calculating the values of force densities at the end of the n-th iteration step through

(5.3) $$\mathbf{q}_n = \mathbf{q}_{n-1} + \Delta\mathbf{q}.$$

The matrix $\mathbf{G}$ in (5.2) is calculated with the help of the chain rule, i.e.

(5.4) $$\mathbf{G} = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\mathbf{q}} = \frac{\partial\mathbf{g}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{y}}\frac{\partial\mathbf{y}}{\partial\mathbf{q}} + \frac{\partial\mathbf{g}}{\partial\mathbf{z}}\frac{\partial\mathbf{z}}{\partial\mathbf{q}}\,.$$

Here, see [9]:

$$\frac{\partial\mathbf{x}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\,\mathbf{C}^T\,\mathbf{H}_x\,,\quad \frac{\partial\mathbf{y}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\,\mathbf{C}^T\,\mathbf{H}_y\,,\quad \frac{\partial\mathbf{z}}{\partial\mathbf{q}} = -\mathbf{D}^{-1}\,\mathbf{C}^T\,\mathbf{H}_z\,,$$

while calculating the gradients $\partial\mathbf{g}/\partial\mathbf{q}$, $\partial\mathbf{g}/\partial\mathbf{x}$, $\partial\mathbf{g}/\partial\mathbf{y}$, $\partial\mathbf{g}/\partial\mathbf{z}$, requires additional assumptions, so that the numerical procedure suits the purpose of solving the minimum weight problem in (P). These assumptions regard the construction of a contour bounding an optimized cable system. The systems are categorized as:

i) "closed" if the entire contour is rigid;

ii) "open" if the contour is made of cables stretched between rigid posts.

Boundary cables in open systems usually serve for supporting the net, and hence are often excluded from the optimization process. To deal with such a case, we use (2.4) in rewriting (5.1) in the form

(5.5) $$\mathbf{g}(\mathbf{q}^*) = \mathbf{L}^*\mathbf{q}^* - A_0\sigma_T\mathbf{1} = \mathbf{0}\,,$$

expressing the optimality of forces in selected cable segments. Here we assume s^* for the number of the optimized segments and $\mathcal{S}^*$ for the set containing their numbers. Consequently, $\mathcal{S}^* \subseteq \mathcal{S}$, where $\mathcal{S} = \{1,2,\dots,s\}$ stands for the set of all segment numbers, and $s^* \le s$. Also, $\dim\mathbf{l}^* = \dim\mathbf{q}^* = \dim\mathbf{1} = s^*$ in (5.5).

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Next, we calculate

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}} = \mathbf{L}^* \ , \ \ \frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \mathbf{Q}^* \, (\mathbf{L}^*)^{-1} \, \mathbf{H}_x^* \, \mathbf{C}^* \ , \ \ \frac{\partial \mathbf{g}}{\partial \mathbf{y}} = \mathbf{Q}^* \, (\mathbf{L}^*)^{-1} \, \mathbf{H}_y^* \, \mathbf{C}^* \ , \ \ \frac{\partial \mathbf{g}}{\partial \mathbf{z}} = \mathbf{Q}^* \, (\mathbf{L}^*)^{-1} \, \mathbf{H}_z^* \, \mathbf{C}^* \ .$$

Since $\dim \partial\mathbf{g}/\partial\mathbf{q} = s^* \times s^*$ and the dimensions of the remaining matrices in (5.4) are $s^* \times s$, then it is necessary to define an extended matrix $\mathbf{L}^{**}$ by introducing the zero columns in $\mathbf{L}^*$. Indices of these columns belong to the set $\mathcal{S} \setminus \mathcal{S}^*$. Finally, (5.4) becomes

$$\mathbf{G} = \mathbf{L}^{**} \ - \mathbf{Q}^* \, (\mathbf{L}^*)^{-1} \left(\mathbf{H}_x^* \, \mathbf{C}^* \mathbf{D}^{-1} \, \mathbf{C}^T \, \mathbf{H}_x + \mathbf{H}_y^* \, \mathbf{C}^* \mathbf{D}^{-1} \, \mathbf{C}^T \, \mathbf{H}_y + \mathbf{H}_z^* \, \mathbf{C}^* \mathbf{D}^{-1} \, \mathbf{C}^T \, \mathbf{H}_z \right) ,$$

with $\dim \mathbf{G} = s^* \times s$. From $s^* \le s$, it follows that the system of equations in (5.2) is underdetermined; the sought components in $\Delta\mathbf{q}$ outnumbers the equations. Assuming that (5.2) is consistent in the sense of the Kronecker-Capelli theorem, there is an infinitude of solutions determining $\Delta\mathbf{q}$. For calculating the "best fitting" solution, we use the formula

$$\Delta\mathbf{q} = \mathbf{G}^+ \, \mathbf{b} \, , \ \ \mathbf{G}^+ = \mathbf{G}^T (\mathbf{G} \, \mathbf{G}^T)^{-1} \ , \tag{5.6}$$

where:

$\mathbf{G}^+$ – the Moore-Penrose inverse of matrix $\mathbf{G}$.

Thus calculated $\Delta\mathbf{q}$ coincides with the Least Square Approximation of the solution to (5.2). Proving this fact falls out of scope of this paper. Note that in case of a closed cable system, or in case of an open one with all cable segments optimized, we have $\mathcal{S}^* = \mathcal{S}$, thus $\mathbf{Q}^* = \mathbf{Q}$, etc., hence $\mathbf{G}^+ = \mathbf{G}^{-1}$. Reassuming, the n-th iteration step of the procedure for determining the optimal configuration of a cable system consists in the following substeps:

1) determining the vector $\Delta\mathbf{q}$ of force density increments from (5.6);
2) updating the vector $\mathbf{q}$ of force densities through (5.3) and subtracting vector $\mathbf{q}^*$ if necessary;
3) updating the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of free node coordinates through (3.4);
4) updating the vector $\mathbf{l}$ of segment lengths through (3.3) and subtracting vector $\mathbf{l}^*$ if necessary;
5) checking the stop condition from (5.5).

6. EXAMPLES OF OPTIMAL DESIGN

The following examples show an open cable net anchored at several points located at two parabolic arches; the anchors are marked with stars in Fig. 2.

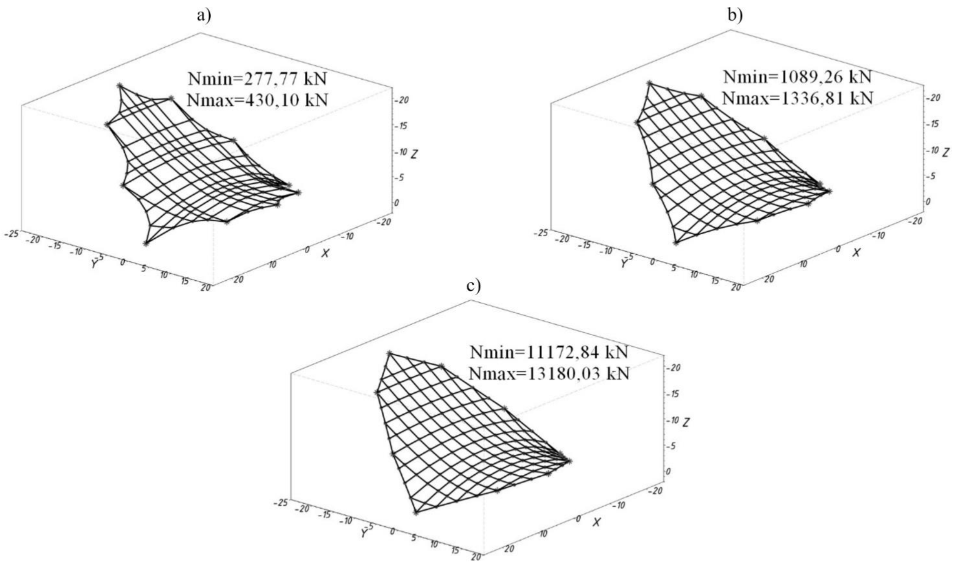

Fig. 2. Optimum open cable nets with optimized inner element layout; forces in the inner net cables are equal to 100 kN. Outer cables are excluded from the optimization process; their role is limited to supporting the inner net. Forces in outer cables fall within the range (Nmin, Nmax).

Three configurations, with various force density values in outer cables were assumed. Table 1 shows these values, as well as force density values taken as initial in the iteration procedure for inner cables. The goal of (P) was to find $\mathbf{x}, \mathbf{y}, \mathbf{z}$ determining the net whose inner elements are uniformly stretched by a force equal to 100 kN.

Table 1. Force density values for open cable nets in Fig. 2

Variant	Force density values in elements [kN/m]		Sum of inner elements lengths [m]
	Outer (assumed)	Inner (initial for (P))	
a)	35	35	687,43
b)	350	35	723,50
c)	3500	35	737,42

Figure 2 shows optimal configurations with extremal forces in outer cables. The rightmost column in Tab. 1 provides the summed up inner cable lengths for different force densities in outer cables. Together with the increase of the stretching force in outer elements, the sum of inner cable lengths also increases towards a value corresponding to a closed net, i.e. such whose all outer nodes are anchored.

7. DISCUSSION OF THE SOLUTION METHOD

The main result of this study is rather straightforward. It says that the pre-tensioned, minimum volume cable net is also a *minimum way net*. In such a net, the sum of lengths of cable segments is minimal, see (P_0). This statement may seem trivial, but is not so due to the complexity of the computational part of the optimization problem. Its variant denoted by (P) is nonlinear in the design variables and as such requires an iterative solution procedure. Its robustness was shown in Section 6.

An alternative approach is to call the *ground structure method*, typically used in solving the topology optimization problems for trusses. Loosely speaking, the method starts with fixing the $\mathbf{B}$ matrix consisting of billions of rows, i.e. describing a "ground structure" consisting of billions of bars. The topology of a ground structure, in particular the position of all nodes, is also fixed. Next, efficient techniques of the Linear Programming are used to select optimal matrix $\widehat{\mathbf{B}}$ consisting of only those rows from $\mathbf{B}$, which determine the optimal, least volume truss. It is concluded in e.g. [11] that the ground structure method is capable of solving the large-scale problems, also those formulated in three dimensions.

Applying the ground structure method is possible in solving (P_0). It requires the same procedure as for the truss problem with one important handicap: the matrix $\mathbf{B}$ should have the $\mathbf{1}$ vector in its null space. This, however, is not guaranteed up front for a general choice of the ground structure. Therefore, fulfilling $\mathbf{B}^T\mathbf{1} = \mathbf{0}$ in (P_0) needs a special subroutine providing the "best fit" approximation to this constraint.

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SPIS RYSUNKÓW I TABEL:

OPTIMAL FORM-FINDING OF CABLE SYSTEMS

Keywords: *cable systems, optimal form-finding, minimum weight, force density method*

SUMMARY:

This paper regards designing pre-tensioned cable systems, with a goal to make them kinematically stable and such that the weight of so designed system is lowest possible. Unlike in typical topology optimization problems, our goal is not to optimize the structural layout against a particular applied load. However, our method is similar. First, we formulate the variational problem of form-finding and next we describe the corresponding iterative numerical procedure for determining the optimum location of nodes of the cable system mesh.

From the mechanical point of view, the process of form-finding can be considered as a problem in statics. As cable systems are close siblings of trusses (cables, however, can carry tensile forces only), let us refer to the equilibrium equation $\mathbf{B}^T\mathbf{n} = \mathbf{p}$, where: $\mathbf{B}$ denotes the matrix reflecting the geometry of a truss; $\mathbf{n}$ stands for the vector of axial forces; and $\mathbf{p}$ is a vector of node loads. Typically, $\mathbf{B}$ and $\mathbf{p}$ are fixed and the goal is to determine $\mathbf{n}$. The optimal form-finding problem calls for a different answer. Loosely speaking, it requires determining $\mathbf{B}$ for fixed $\mathbf{n}$ and $\mathbf{p}$, and by this it falls into the category of topology optimization problems formulated for constructions that are entirely in tension. Such constructions, along with those entirely in compression, are referred to as "Prager-structures". These, in turn, fall into the broader context of "Michell structures".

In our study, we suppose that net-stretching forces are large enough to eliminate the effect of sagging due to the self-weight of cables. In other words, we tackle the case in which the values of axial forces in cables induced by stretching the entire system by forces applied at the anchors are dominating over the forces caused by the gravity load. Moreover, due to manufacturability constraints, it is reasonable to assume that a single cable has a constant cross-section A_0 along its length. Our conjecture is that the minimum volume of a cable net is attained if the entire system is uniformly stretched to the limit tensile stress σ_T. In other words, we claim that forces in a cable system of least weight are constant and given by $N_S = A_0\sigma_T$, $S = 1, \dots, s$, where s denotes the number of segments.

In the search for solution to thus posed optimization problem, we make use of the concept of *force density* which is a ratio of an axial force in cable segment to its length. Namely, we set $q_S = N_S/L_S$ for the force density in the S-th segment and $\mathbf{q} \in R^s$ for the vector of force densities. Assuming that the designed, pre-tensioned cable system is anchored at a rigid contour, or a number of rigid posts, we write $\bar{m}$ for the number of these anchored nodes and m for the number of the remaining nodes (free nodes) in the cable mesh. The location of anchors is fixed in space; their Cartesian coordinates produce vectors $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}} \in R^{\bar{m}}$, while the position of free nodes is not given up front; suppose that they produce vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^m$. Reassuming, vectors $\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ are the design variables in the following optimization problem:

$$
(P) \quad \left| \quad \hat{V} = \frac{1}{\sigma_T} \min \left\{ \wp(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \;\middle|\; \begin{array}{l} \mathbf{q} - \text{arbitrary} \\ \text{and} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in R^m \text{, where} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ determine the equilibrium configuration of a cable system.} \end{array} \right\} \right. .
$$

The functional $\wp$ in (P) can be expressed in the form

$$
\wp(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{S=1,\dots,s} q_S L_S^2(\mathbf{x}, \mathbf{y}, \mathbf{z}) .
$$

The numerical procedure for (P) is iterative due to the nonlinear dependence of $\wp$ on the design variables. It requires repetitive solutions of the linear equation for force density increments $\Delta\mathbf{q}$ and subsequent updating of the free node position vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$. The scheme of the iteration step of the procedure for determining the optimal configuration of a cable system consists in the following substeps:

1) determining the vector $\Delta\mathbf{q}$ of force density increments;
2) updating the vector $\mathbf{q}$ of force densities;
3) updating the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of free node coordinates;
4) updating the vector $\mathbf{l}$ of segment lengths;
5) checking the stop condition from the constraint $\mathbf{L}^*\mathbf{q}^* - A_0\sigma_T\mathbf{1} = \mathbf{0}$, where $\mathbf{L}^* = \operatorname{diag}\lceil \mathbf{l}^* \rfloor$, expressing the equality of forces in selected cable segments. Here we assume s^* for the number of the optimized segments, and hence $\dim \mathbf{l}^* = \dim \mathbf{q}^* = \dim \mathbf{1} = s^*$.

Since the number of optimized segments is such that $s^* \leq s$, it follows that the system of equations for the force density increments $\Delta\mathbf{q}$ is underdetermined; the sought components in $\Delta\mathbf{q}$ outnumbers the equations. Assuming that this system is consistent in the sense of the Kronecker-Capelli theorem, there is an infinitude of solutions determining $\Delta\mathbf{q}$. For calculating the “best fitting” solution, we use the Moore-Penrose generalized inverse for rectangular matrices. Thus calculated $\Delta\mathbf{q}$ coincides with the Least Square Approximation of the solution to the underdetermined system of equations.

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OPTYMALNE KSZTAŁTOWANIE KONSTRUKCJI CIĘGNOWYCH

Słowa kluczowe: konstrukcje cięgnowe, optymalne kształtowanie, minimum ciężaru, metoda gęstości sił

STRESZCZENIE:

W artykule rozważane są zagadnienia związane z projektowaniem wstępnie napiętych konstrukcji cięgnowych. Głównym celem jest znalezienie konfiguracji kinematycznie niezmiennej o minimalnym ciężarze. W odróżnieniu od typowych zagadnień optymalizacji topologicznej celem zadania nie jest optymalizacja konstrukcji pod działaniem konkretnego obciążenia, chociaż zastosowana metoda rozwiązania jest podobna. Na początku formułujemy problem wariacyjny poszukiwania kształtu konstrukcji, a następnie przedstawiamy numeryczną procedurę iteracyjną służącą do znajdowania optymalnego położenia węzłów siatki cięgnowej.

Z punktu widzenia mechaniki proces poszukiwania kształtu może być rozpatrywany jako problem statyki. Siatki cięgnowe są pokrewne kratownicom (w każdym cięgnie występuje tylko siła osiowa), więc rozważania można oprzeć na równaniu równowagi $\mathbf{B}^T\mathbf{n} = \mathbf{p}$, gdzie: $\mathbf{B}$ jest macierzą odzwierciedlającą geometrię kratownicy; $\mathbf{n}$ jest wektorem sił osiowych; $\mathbf{p}$ jest wektorem zewnętrznych sił węzłowych. W typowym zadaniu $\mathbf{B}$ i $\mathbf{p}$ są znane, a poszukiwany jest wektor $\mathbf{n}$. W zadaniu optymalnego kształtu siatki należy odpowiedzieć na inne pytanie. W uproszczeniu można powiedzieć, że poszukuje się macierzy $\mathbf{B}$ przy założonych wektorach $\mathbf{n}$ i $\mathbf{p}$, a zatem jest to zadanie z kategorii optymalizacji topologicznej sformułowane dla konstrukcji poddanej jedynie rozciąganiu. Tego typu konstrukcje, wraz z układami poddanymi jedynie ściskaniu, tworzą grupę tzw. konstrukcji Pragera, które można zaliczyć do szerszej kategorii konstrukcji Michella.

W naszych rozważaniach zakładamy, że siły rozciągające cięgna są na tyle duże, że eliminują efekt luźnego zwisu pod ciężarem własnym. Zajmujemy się zatem przypadkiem, w którym osiowe siły wywołane przez wstępne sprężenie konstrukcji dominują nad siłami grawitacji. Co więcej, uwzględniając ograniczenia techniczne związane z produkcją lin stalowych, zakłada się stały przekrój A_0 na długości cięgna. Nasze rozważania opierają się na przypuszczeniu, że siatka o minimalnej objętości materiału jest równomiernie napięta do wartości granicznej naprężeń σ_T. Inaczej mówiąc, zakłada się, że siły w siatce cięgnowej o minimalnym ciężarze są stałe i równe: $N_S = A_0\sigma_T$, $S = 1, \dots, s$, gdzie s oznacza liczbę elementów siatki.

Poszukując rozwiązania tak postawionego zadania optymalizacji, korzystamy z pojęcia gęstości sił czyli stosunku siły w elemencie do jego długości. Gęstość siły w elemencie o numerze S definiuje się jako: $q_S = N_S/L_S$, a zatem wektor gęstości sił $\mathbf{q} \in R^s$. Zakładając, że projektowana, wstępnie napięta siatka cięgnowa jest zamocowana na obwodzie do sztywnego elementu lub punktowo w wybranych węzłach, można przyjąć oznaczenia: $\overline{m}$ dla liczby węzłów zamocowanych oraz m dla liczby pozostałych (wolnych) węzłów siatki. Położenie zakotwionych węzłów jest ustalone i opisane wektorami współrzędnych kartezjańskich $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}} \in R^{\overline{m}}$, natomiast wektory współrzędnych węzłów wolnych są poszukiwane i oznaczone jako: $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^m$. Podsumowując, wektory $\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ są zmiennymi projektowymi w następującym zadaniu optymalizacji:

$$(P) \quad \left| \quad \hat{V} = \frac{1}{\sigma_T} \min \left\{ \wp(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \;\middle|\; \begin{array}{l} \mathbf{q} - \text{dowolne} \\ \text{oraz} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in R^m \text{, gdzie} \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \text{ określają położenie równowagi konstrukcji cięgnowej.} \end{array} \right\} \right. .$$

Funkcjonał $\wp$ w zadaniu (P) można zapisać w formie:

$$\wp(\mathbf{q},\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{S=1,\dots,s} q_S L_S^2(\mathbf{x},\mathbf{y},\mathbf{z})\ .$$

Ze względu na nieliniową zależność funkcjonału $\wp$ od zmiennych projektowych, rozwiązanie zadania (P) wymaga zastosowania iteracyjnej procedury numerycznej. W każdej iteracji należy rozwiązać liniowe równanie na przyrost wektora gęstości sił $\Delta\mathbf{q}$ oraz znaleźć aktualne wektory położenia węzłów wolnych $\mathbf{x},\mathbf{y},\mathbf{z}$. Algorytm postępowania w każdym kroku procedury znajdowania optymalnego kształtu siatki cięgnowej składa się z następujących punktów:

1) znalezienie wektora przyrostów gęstości sił $\Delta\mathbf{q}$;
2) znalezienie aktualnego wektora gęstości sił $\mathbf{q}$;
3) wyznaczenie aktualnych wektorów współrzędnych węzłów wolnych $\mathbf{x},\mathbf{y},\mathbf{z}$;
4) obliczenie aktualnych długości elementów zebranych w wektorze $\mathbf{l}$;
5) sprawdzenie warunku stopu wynikającego z narzucenia równych wartości sił w wybranych elementach siatki: $\mathbf{L}^*\mathbf{q}^* - A_0\sigma_T\mathbf{1} = \mathbf{0}$, gdzie $\mathbf{L}^* = \mathrm{diag}\lceil \mathbf{l}^* \rfloor$. Założono, że s^* jest liczbą elementów włączonych w proces optymalizacji, a zatem: $\dim \mathbf{l}^* = \dim \mathbf{q}^* = \dim \mathbf{1} = s^*$.

Ponieważ liczba elementów włączonych do procesu optymalizacji spełnia warunek $s^* \leq s$, układ równań na przyrosty gęstości sił $\Delta\mathbf{q}$ może być niedookreślony, co oznacza, że liczba poszukiwanych składników wektora $\Delta\mathbf{q}$ jest większa niż liczba równań. Przy założeniu niesprzeczności układu równań w rozumieniu twierdzenia Kroneckera-Capellego, istnieje nieskończenie wiele rozwiązań $\Delta\mathbf{q}$. W celu znalezienia rozwiązania „najlepiej dopasowanego" można zastosować operację uogólnionego odwracania macierzy Moore'a-Penrose'a, która ma zastosowanie w przypadku macierzy prostokątnych. Znaleziony w ten sposób wektor $\Delta\mathbf{q}$ odpowiada rozwiązaniu niedookreślonego układu równań uzyskanemu Metodą Najmniejszych Kwadratów.