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The Lepschy stability test and its application to fractional-order systems

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It is shown how a stability test, alternative to the classical Routh test, can profitably be applied to check the presence of polynomial roots inside half-planes or even sectors of the complex plane. This result is obtained by exploiting the peculiar symmetries of the root locus in which the basic recursion of the test can be embedded. As is expected, the suggested approach proves useful for testing the stability of fractional-order systems. A pair of examples show how the method operates. It is believed that the suggested geometric approach can also be of some didactic value in introducing basic control-system tools to engineering students.

Key words: fractional-order systems, $\mathcal{D}$-stability, recursive algorithms, complex polynomials, root locus, symmetries, control-theory didactics

1. Introduction

In the late 1980s and early 1990s, motivated by the renewed interest in the robust stability analysis of uncertain linear systems, Lepschy and his coworkers showed how the classic recursive algorithms for polynomial root location and signal processing could be given a common framework [18, 19, 39] from which new efficient stability-test and model-reduction procedures could be derived [23, 25]. Their essentially geometric approach [24, 41] classifies the aforementioned algorithms according to the shape of the root loci associated with the basic recurrence relations that generates a sequence of polynomials of descending degree from an original characteristic polynomial to be tested. For instance, the classical Routh stability criterion [36], of which a numerically ef-

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ficient version has been proposed quite recently [5], recursively generates the even and odd parts of the polynomials in the sequence starting from the even and odd parts of the original polynomial, each polynomial having one part in common with the preceding polynomial. As is well known, the rows of the standard Routh table are formed from the coefficients of the even or odd powers of these polynomials. For didactic reasons almost all control textbooks still present the Routh algorithm by referring only to the aforementioned component parts of the polynomials and not to the complete polynomials. To use the terminology introduced in [9, 10] and adopted in [26, 28] in a more general context, such presentation corresponds to the *split* or *three-term* form of the algorithm (which plays a role similar to that played by the simpler Liénard and Chipart test [32] with respect to the computationally more demanding Hurwitz criterion [14]).

However, a better insight into the mechanism of stability-test algorithms can be gained by referring to their so-called *two-term* form [29, 30] by which every complete polynomial in the sequence is related to the preceding complete polynomial. For example, in the case of Routh's algorithm, by denoting the ith degree real polynomial in the sequence by

$$p_i(s) = q_{i,i}(s) + q_{i,i-1}(s), \tag{1}$$

where $q_{i,i}(s)$ and $q_{i,i-1}(s)$ are the even and odd parts of $p_i(s)$ if i is even, and vice versa if i is odd, the *two-term* step-down recursion (on which a simple proof of the test is based [13]) is

$$p_{i-1}(s) = p_i(s) - r_i\, s\, q_{i,i-1}(s), \tag{2}$$

where r_i is the ratio between the leading coefficients of $q_{i,i}(s)$ and $q_{i,i-1}(s)$. From (2) it follows that $q_{i-1,i-1}(s) = q_{i,i-1}(s)$ and $q_{i-1,i-2}(s) = q_{i,i}(s) - r_i\, s\, q_{i,i-1}(s)$ (so that the first subscript can be dropped in this case, which leads directly to the standard *three-term* form of the test: $q_{i-2}(s) = q_i(s) - r_i\, s\, q_{i-1}(s)$).

The right-hand side of the recurrence relation (2) can be imbedded into the locus described by the solutions of

$$p_i(s) + k\, s\, q_{i,i-1}(s) = 0 \tag{3}$$

as parameter k varies over the reals ($-\infty < k < +\infty$). Rewriting (3) as

$$q_{i,i}(s) + q_{i,i-1}(s) + k\, s\, q_{i,i-1}(s) = 0, \tag{4}$$

shows that, for an s having zero real part, either $q_{i,i-1}(s)$ is real and $q_{i,i}(s) + k\, s\, q_{i,i-1}(s)$ is imaginary, or vice versa; in both cases, the only possibility for (3) to hold is that both $q_{i,i}(s)$ and $q_{i,i-1}(s)$ are zero. As a consequence, if $q_{i,i}(s)$ and $q_{i,i-1}(s)$ are coprime the root locus (3) cannot cross the imaginary axis, except for one branch that follows the real axis and crosses the imaginary axis at infinity for $k_i = -r_i$ (see, as an example, the bold line in Figure 1). Note that for $k = 0$ the solutions of (3) coincide with the roots of $p_i(s)$, while for $k = -r_i$ they coincide with the roots of $p_{i-1}(s)$ whose location with respect to the imaginary axis is therefore the same as the location of $i - 1$ roots of $p_i(s)$. The remaining root of $p_i(s)$ corresponds to the point from which departs, for $k = 0$, the only root locus branch that crosses the point at infinity for $k = r_i$. This root of $p_i(s)$ is located in the same half-plane as the root approaching the point at infinity along the aforementioned branch as $k \to r_i$ from below for $r_i > 0$ or from above for $r_i < 0$. Now, the position with respect to the imaginary axis of this (real) root of large magnitude, say $z_{i,M}$, is determined by the sign of the ratio of the coefficients of the two highest powers in the polynomial (3) with $k \simeq r_i$ and $k < r_i$ if $r_i > 0$ and $k > r_i$ if $r_i < 0$. Precisely, by continuity, for k close to r_i and, consequently, $z_{i,M}$ close to the point at infinity

$$z_{i,M} \simeq -\frac{1}{r_i - k}, \tag{5}$$

so that $z_{i,M} < 0$, i.e., $z_{i,M}$ is in the left half-plane (LHP), if $r_i > 0$ and $z_{i,M} > 0$, i.e., $z_{i,M}$ is in the right half-plane (RHP), if $r_i < 0$. Therefore, by assuming that $p_i(s)$ has $n_{i,LHP}$ LHP roots and $n_{i,RHP}$ RHP roots, for $r_i > 0$ polynomial $p_{i-1}(s)$ has $n_{i-1,LHP} = n_{i,LHP} - 1$ LHP roots and $n_{i-i,RHP} = n_{i,HHP}$ RHP roots, while for $r_i < 0$ $n_{i-1,LHP} = n_{i,LHP}$ and $n_{i-1,RHP} = n_{i,HHP} - 1$.

By successively applying this criterion to all pairs of consecutive polynomials generated from an original polynomial $p_n(s)$ according to (2), it turns out that the root distribution of $p_n(s)$ depends only on the sign of the n ratios r_i, $i = 1, 2, \ldots, n$ (Routh's theorem [40, 41]). Since the $p_i(s)$, $i = 1, \ldots, n$, are real polynomials whose complex roots, if any, are in conjugate pairs, the root locus of (3) is necessarily symmetric with respect to the real axis, but it is not so with respect to the imaginary axis, as Fig. 1 shows. Instead, the Lepschy algorithm [23] can be embedded into a root locus that is symmetric with respect to both axes [39, 41]. This characteristics makes it more suitable for determining the root distribution of a polynomial with respect to a sector and, thus, checking the stability of a fractional-order system, as we shall see in a while.

The rest of the paper is organised as follows. Section 2 briefly recalls the main features of the Lepschy test. Section 3 shows how the test can be adapted to determine the root distribution of a polynomial with respect to a sector straddling the real axis and, in particular, to check the stability of a fractional-order system (see, e.g., [17, 34]). Section 4 applies the Lepschy criterion to a pair of examples taken from the literature. Section 5 indicates possible directions of future research on this evergreen subject.

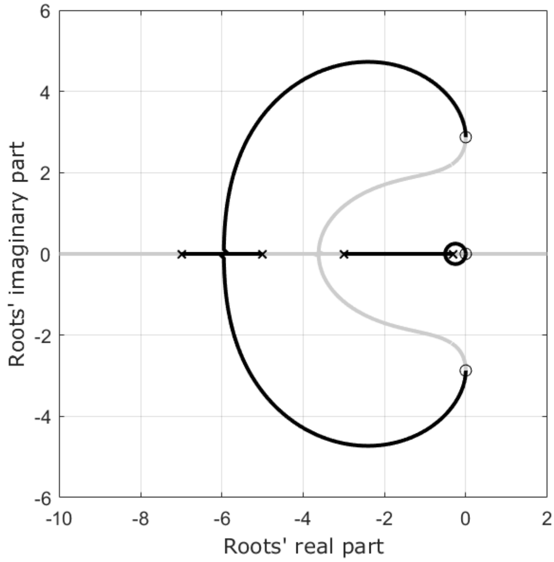

Figure 1: Complete root locus for (3) with $q_{i,i}(s) = q_{4,4}(s) = s^4 + 75.5s^2 + 31.5$ and $q_{i,i-1}(s) = q_{4,3}(s) = 15.3s^3 + 126.3s$. Black line: direct locus ($k > 0$); grey line: inverse locus ($k < 0$)

2. The Lepschy test

The basic two-term step-down recursion of the Lepschy test (see, e.g., [39]) determines the polynomial $p_{i-1}(s)$ from the preceding polynomial $p_i(s)$ according to

$$(s+1)\, p_{i-1}(s) = q_{i,i}(s) + \rho_i\, q_{i,i-1}(s), \tag{6}$$

where $q_{i,i}$ and $q_{i,i-1}$ are defined as in (1) and

$$\rho_i = -\frac{q_{i,i}(-1)}{q_{i,i-1}(-1)} \tag{7}$$

which is the ratio between the sum of the even-order coefficients of $p_i(s)$ and the sum of its odd-order coefficients if i is even, and vice versa if i is odd.

The right-hand side of the recurrence relation (6) can be embedded into the locus described by the roots of

$$q_{i,i}(s) + h\, q_{i,i-1}(s) = 0 \tag{8}$$

as parameter h varies over the reals ($-\infty < h < +\infty$). This locus passes, for $h = 1$, through the roots of polynomial (1) and, for $h = -1$, through those of

$$p_i(-s) = (-1)^i\, q_{i,i}(s) + (-1)^{i-1}\, q_{i,i-1}(s) \tag{9}$$

whose roots are symmetric to the roots of $p_i(s)$ with respect to the imaginary axis. It follows that, according to the invariance property pointed out in [20], the complete root locus for (8) is the same, except for graduation, as the complete locus described by the roots of

$$p_i(s) + g\, p_i(-s) = 0, \quad -\infty < g < +\infty, \tag{10}$$

and this locus is clearly symmetric also with respect to the imaginary axis, as shown, e.g., in Fig. 2. Precisely, if the conjugate pair of points $P_1 \equiv (x, y)$ and $P_2 \equiv (x, -y)$ belongs to this locus for $g = g_P$ also the conjugate pair of points $P_3 \equiv (-x, y)$ and $P_4 \equiv (-x, -y)$ belongs to it for $g = 1/g_P$. Indeed, recursion (6) can be replaced by the alternative equivalent recursion

$$\left[1 + (-1)^i \sigma_i\right] (s+1) p_{i-1}(s) = p_i(s) + \sigma_i\, p_i(-s), \tag{11}$$

where the constant $[1 + (-1)^i \sigma_i]$ at the left-hand side is simply a "normalisation factor" whose task is to make polynomials $p_i(s)$, $i = n-1, n-2, \ldots, 0$, monic if $p_n(s)$ is so, and

$$\sigma_i = -\frac{p_i(-1)}{p_i(1)}\,. \tag{12}$$

Remark 1 *Quantities ρ_i and σ_i are not well defined if $q_{i,i-1}(-1) = 0$ (see equation (7)) or $p_i(1) = 0$ (see Eq. (12)). However, these occurrences happen with zero probability zero and may be overcome by means of perturbation methods.*

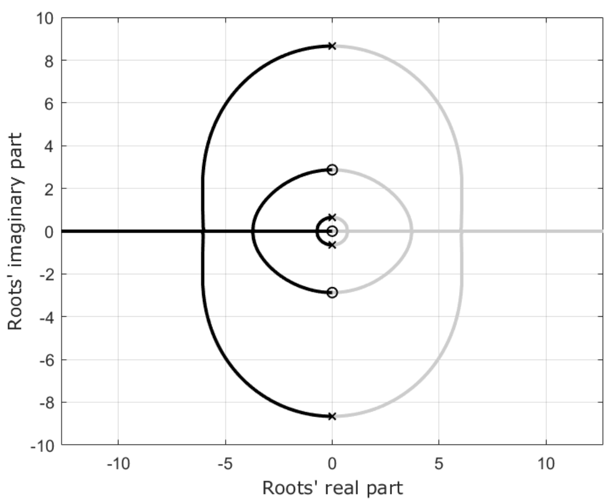

Figure 2: Complete root locus for (8) with $q_{i,i}(s) = q_{4,4}(s) = s^4 + 75.5s^2 + 31.5$ and $q_{i,i-1}(s) = q_{4,3}(s) = 15.3s^3 + 126.3s$. Black line: direct locus ($h > 0$); grey line: inverse locus ($h < 0$)

As observed in [23], the Lepschy test can be considered as the continuous-time counterpart of the Jury–Blanchard test for discrete-time systems [16] (see, also, the eminent precursors [8, 33, 37]) because the latter can be embedded into a root locus that is symmetric with respect to the unit circumference of the z-plane [27].

The rationale of the Lepschy test can be detailed as follows:

(i) Since $p_i(s)$, and thus $q_{i,i}(s)$ and $q_{i,i-1}(s)$, are real polynomials, the entire real axis belongs to the complete root locus associated with equation (8). As a consequence, a real value ρ_i of h for which $s + 1$ is a factor of the right-hand side of (6) certainly exists.

(ii) By assuming $q_{i,i}(s)$ and $q_{i,i-1}(s)$ *coprime*, branches of the root locus of (8) may cross the imaginary axis only at: 1) the imaginary roots of $q_{i,i}(s)$, if any, for $h = 0$, and 2) at the imaginary roots of $q_{i,i-1}(s)$, if any, and the point at infinity for $h = \pm\infty$. Consequently, the root distribution of (8) with respect to the imaginary axis may change only if the sign of h changes.

(iii) From (ii) it follows that the root distribution of the right-hand side of (6), i.e., $q_{i,i}(s) + \rho_i\, q_{i,i-1}(s)$ (and, thus, also of its left-hand side) coincides with that of $p_i(s) = q_{i,i}(s) + q_{i,i-1}(s)$ if ρ_i is positive, because in this case the roots of both combinations of $q_{i,i}(s)$ and $q_{i,i-1}(s)$ belong to the same part of the locus for (8) (either the positive locus or the negative locus depending on the sign of the leading coefficient of $q_{i,i-1}(s)$), whereas, if ρ_i is negative. the root distribution of the right-hand side of (6) coincides with that of $p_i(-s)$ whose roots are opposite to those of $p_i(s)$.

(iv) From (iii) it follows that, for $\rho_i > 0$, the number of RHP roots of $p_{i-1}(s)$ is the same as the number of RHP roots of $p_i(s)$, whereas the number of LHP roots of $p_{i-1}(s)$ is equal to the number of LHP roots of $p_i(s)$ minus 1, since the left-hand side of (6) is forced to exhibit the factor $s + 1$ corresponding to the root -1 which lies in the LHP. The situation is the opposite for $\rho_i < 0$.

By recursively applying the previous considerations to all pairs of consecutive polynomials successively generated according to recursion (6) from an original polynomial $p_n(s)$, the root distribution of $p_n(s)$ can be determined only from the signs of the n parameters ρ_i, $i = n, n-1, \dots, 1$. In particular, the following theorem can be proved [23].

Theorem 1 (Root distribution) *Assuming that the sequence of all n polynomials generated from $p_n(s)$ can be completed, the number of strictly LHP roots of $p_n(s)$ is equal to the number of positive entries in the sequence*

$$\left\{ \lambda_i = \prod_{j=i}^{n} \operatorname{sgn}(\rho_j), \quad i = 1, 2, \dots, n \right\}. \tag{13}$$

For the treatment of the critical cases, occurring when either $q_{i,i-1}(-1) = 0$ in (7) or $p_i(1) = 0$ in (12), the reader is referred to [23] and for a table-form implementation of the test to [40]. In the following examples, however, to better understand how the method operates, we do not use this table form; instead, we write down explicitly all of the $n+1$ polynomials in the sequence.

An immediate consequence of Theorem 1 is the following corollary.

Corollary 1 (Hurwitz property) *Polynomial $p_n(s)$ is Hurwitz if and only if $\rho_i > 0$, $i = 1, 2, \ldots, n$.*

Example 1 *Consider the polynomial*

$$p_4(s) = s^4 + 15.3s^3 + 75.5s^2 + 126.3s + 31.5 \tag{14}$$

whose even and odd parts are, respectively

$$q_{4,4}(s) = s^4 + 75.5s^2 + 31.5,$$
$$q_{4,3}(s) = 15.3s^3 + 126.3s.$$

The root locus of the equation $q_{4,4}(s) + h_4 q_{4,3}(s) = 0$ is depicted in Fig. 2. Polynomials $p_3(s)$, $p_2(s)$, $p_1(s)$, $p_0(s)$ generated from (14) according to (6) are listed below together with the values of parameters ρ_4, ρ_3, ρ_2, ρ_1 computed according to (7):

$$\begin{aligned}
\rho_4 &= 0.76\,, \qquad & p_3(s) &= s^3 + 10.67s^2 + 64.83s + 31.5\,,\\
\rho_3 &= 1.56\,, & p_2(s) &= s^2 + 15.66s + 49.17\,,\\
\rho_2 &= 3.20\,, & p_1(s) &= s + 49.17\,,\\
\rho_1 &= 0.02\,, & p_0(s) &= 1\,.
\end{aligned}$$

Since all the values of ρ_i, $i = 1, 2, 3, 4$, are positive, according to Corollary 1 polynomial (14) is Hurwitz. Indeed, its roots are $-0.3, -3, -5, -7$.

Like the Routh test, the Lepschy test can by adapted to the determination of the root distribution with respect to different straight lines. The case of lines parallel to the imaginary axis can be accommodated easily by the change of variable

$$z = s - \delta \tag{15}$$

which corresponds to a horizontal translation of the imaginary axis to the right if $\delta > 0$ and to the left if $\delta < 0$. The test can then by applied with no further modifications to the *real* polynomial

$$\overleftrightarrow{p}_n(z) \triangleq p_n(z + \delta). \tag{16}$$

Instead, a rotation of the imaginary axis of an angle θ around the origin implies the variable transformation

$$y = s\, e^{-\jmath\theta} \quad (\text{whence } \; s = y\, e^{\jmath\theta}) \tag{17}$$

which corresponds to a counterclockwise rotation if $\theta > 0$ and to a clockwise rotation if $\theta < 0$. The resulting polynomial in the new variable is

$$\check{p}_n(y) \triangleq p_n(ye^{\jmath\theta}) = p_n(y\cos\theta + \; y\,\jmath\sin\theta) \tag{18}$$

whose coefficients are no longer real. Therefore, the standard version of the test cannot directly be applied to (18). However, by recalling that, according to de Moivre's theorem, $(\cos\theta + \jmath\sin\theta)^n = \cos n\theta + \jmath\sin n\theta$, it is immediately seen that the coefficients of the polynomial

$$\check{p}_n^\star(y) \triangleq p_n(ye^{-\jmath\theta}) \tag{19}$$

are conjugate to those of (18) and, thus, also the roots of $\check{p}_n^\star(y)$ are conjugate to those of $\check{p}_n(y)$ (see, e.g., [33,42]). It follows that the standard test can be applied to the *real* polynomial

$$P_{2n}(y) \triangleq \check{p}_n(y)\; \check{p}_n^\star(y) \tag{20}$$

which has twice as many LHP and RHP roots as $\check{p}_n(y)$.

The next section motivates the recent renewed interest for the determination of the root distribution of a polynomial with respect to suitable contours of the complex plane (cf., e.g., [6,31]), and then adapts the Lepschy test to the problem of finding the aforementioned distribution with respect to the boundary of a minor circular sector using standard root-locus concepts and elementary notions of polynomial theory.

3. Root distribution with respect to a sector

Polynomial root clustering techniques have long history [3]. Today, the interest in this kind of studies is justified by the need for efficient numerical algorithms of root search [15] and for practical tools of stability and robustness analyses of integer- and fractional-order systems [2,6,31,38]. Concerning the latter, resort has sometimes been made to suitable variable transformations and mappings [4,22] which, however, are not well suited to the case of circular sectors which is of particular relevance in the stability analysis of fractional-order systems. On the other hand, the methods based on frequency sweeping and Nyquist diagrams are useful but, often, not easily applicable to fractional-order systems with many fractional powers [1,6].

Before presenting the aforementioned adaptation of the Lepschy test, we briefly recall the essentials of fractional-order system stability analysis.

The transfer function of a continuous-time LTI strictly-proper system in *commensurate-order form* (i.e. when all fractional powers of the independent variable w multiples of the same fraction $1/q$) can be written as

$$\widehat{G}(w) = \frac{b_m w^{\frac{m}{q}} + b_{m-1} w^{\frac{m-1}{q}} + \ldots + b_1 w^{\frac{1}{q}} + b_0}{w^{\frac{n}{q}} + a_{n-1} w^{\frac{n-1}{q}} + \ldots + a_1 w^{\frac{1}{q}} + a_0}, \tag{21}$$

where q, m, n are positive integers, $m < n$, $q \geqslant 1$. The numerator and denominator coefficients are assumed to be real. By the change of variable

$$s = w^{\frac{1}{q}}, \tag{22}$$

function (21) is transformed into the following strictly-proper *rational* function of s:

$$G(s) = \frac{B(s)}{A(s)}, \tag{23}$$

where

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0, \tag{24}$$

$$A(s) = s^s + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0. \tag{25}$$

As shown, e.g., in [34], the denominator of (21), interpreted as a function from $\mathbb{C}$ to $\mathbb{C}$ is, indeed, a multivalued function of w. For any integer $k \in \{1, \ldots, q-1\}$, in fact, the point $\widetilde{w} \triangleq w e^{\jmath 2\pi k}$ coincides, in $\mathbb{C}$, with w; nevertheless, the denominator of (21) maps it to

$$w^{\frac{n}{q}} e^{2\pi \frac{nk}{q}} + a_{n-1} w^{\frac{n-1}{q}} e^{2\pi \frac{(n-1)k}{q}} + \ldots + a_1 w^{\frac{1}{q}} e^{2\pi \frac{k}{q}} + a_0$$

which, in general, is different from

$$w^{\frac{n}{q}} + a_{n-1} w^{\frac{n-1}{q}} + \ldots + a_1 w^{\frac{1}{q}} + a_0 .$$

However, the denominator of (21) becomes a single-valued function on a Riemann surface consisting of q sheets with branch cuts along the negative real semi-axis. The first, or principal, sheet contains the so-called physical poles of (21) [35]. The stability of the rational-order system depends on their location with respect to the imaginary axis. Now, the right half of the first sheet, corresponding to the "unstable region", maps into the minor sector of the s plane defined by

$$\mathcal{S} \triangleq \left\{ s = \mu e^{\jmath \phi} : \ \mu \in \mathbb{R}_+, \ \phi \in \left[-\frac{\pi}{2q}, \frac{\pi}{2q} \right] \right\}. \tag{26}$$

Therefore it is important to check whether any root of (25) lies in this sector which is symmetric with respect to the real axis. In [6] a condition based on the argument principle has been provided for the absence of roots in such sector. Essentially, checking this condition requires plotting the (generalised) Nyquist diagram of a function with complex coefficients. An algebraic path based on the analysis of equations (8) and (10) is followed here.

First, note that the current real parameter g on the root locus for (10) is related to the current real parameter h on the root locus for (8) via

$$g = (-1)^i \, \frac{1-h}{1+h} \tag{27}$$

so that: a) for $h = 0$, $g = +1$ when i is even and $g = -1$ when i is odd, and b) for $h = \pm\infty$, $g = -1$ when i is even and $g = +1$ when i is odd. In both cases, $|g| = 1$. As already observed, the *shape* of the underlying complete root locus does not change by changing the end points of the locus in the considered way [20]. In the sequel, reference will be made to the alternative version (11) of the Lepschy algorithm because the interval of interest of the current parameter g along the root locus turn out to be centred at zero.

In Section 2 it has been shown that the Hurwitz nature of a polynomial can be checked by evaluating the sign of the value ρ_i taken by the current parameter h of (8) at $s = -1$ at any step of the recursive procedure. More precisely, in conformity with Corollary 1, if all n values of ρ_i, $i = 1, 2, \ldots, n$, are positive, then $p_n(s)$ is Hurwitz. Based on the previous considerations, this conditions can equivalently be expressed in terms of the value σ_i taken by the current parameters g of (10) at $s = -1$.

Corollary 2 *Polynomial $p_n(s)$ is Hurwitz if and only if $|\sigma_i| < 1$, $i = 1, 2, \ldots, n$.*

Note that if $\jmath\overline{\omega}$ with $\overline{\omega}$ real is a solution of (10) for some (real) value $\overline{g}$ of g, then also $-\jmath\overline{\omega}$ is a solution associated with the same value $\overline{g}$ since the coefficients of $p_i(s)$ are real. Hence $p_i(-\jmath\overline{\omega}) + \overline{g}p_i(\jmath\overline{\omega}) = 0$, i.e.,

$$p_i(\jmath\overline{\omega}) + \frac{1}{\overline{g}}\, p_i(-\jmath\overline{\omega}) = 0\,,$$

so that $\overline{g} = 1/\overline{g}$, which means that either $\overline{g} = 1$ or $\overline{g} = -1$.

When $p_n(s)$ is not Hurwitz, the count of its LHP roots could be done in terms of the values of σ_i's, too, but would result in an expression that is more cumbersome than the one given in Theorem 1 in terms of the ρ_i's and is therefore omitted.

To determine the root distribution of $p_n(s)$ with respect to the boundary of a sector (26), it is necessary to find the values of $|g|$ on the radii delimiting the sector. Clearly, these values are no longer equal to 1 nor are they the same on all locus branches. However, due to the locus symmetry, two conjugate branches of the positive or negative root locus enter or leave the considered sector "simultaneously", i.e., they cross the aforementioned radii for the *same* value of g, as was the case for the intersections of two conjugate branches with the imaginary axis. Now, the intersections with the sector radii, if any, occur on the half-lines defined by

$$s = \mu\, e^{\pm \jmath\psi} = \mu(\cos\psi \pm \jmath \sin\psi), \qquad \mu \in \mathbb{R}_+, \tag{28}$$

with $\psi = \pi/2q$. By substituting $\mu e^{+\jmath\frac{\pi}{2q}}$ for s in the root locus equation (10), the following equation in the new variable μ is obtained

$$c_{i,u}(\mu) \triangleq p_i\left(\mu e^{+\jmath\frac{\pi}{2q}}\right) + g\, p_i\left(-\mu e^{+\jmath\frac{\pi}{2q}}\right) = 0 \tag{29}$$

whose left-hand side is a polynomial in μ with *complex* coefficients. To the present purpose, it is necessary to find the real values of g, if any, for which (29) admits at least a positive finite real solution, say μ_i. Clearly, due to the locus symmetry with respect to the real axis, for the same values of g also the equation

$$c_{i,\ell}(\mu) \triangleq p_i\left(\mu e^{-\jmath\frac{\pi}{2q}}\right) + g\, p_i\left(-\mu e^{-\jmath\frac{\pi}{2q}}\right) = 0 \tag{30}$$

admits μ_i as a solution. If no such g exits, the sector boundary is traversed only at the origin of the s-plane where $|g| = 1$.

The aforementioned problem can be solved fairly easily by observing that, if (30) admits the positive real solution μ_i, then the polynomial

$$c_{i,\ell}(-\mu) = \frac{1}{g}\, c^{\star}_{i,u}(\mu), \tag{31}$$

where the $\star$ at exponent denotes again complex conjugate, admits $-\mu_i$ as a solution for the same value of g and, therefore, the polynomial with *real* coefficients

$$C_{2i}(\mu) \triangleq c_{i,u}(\mu)\, c^{\star}_{i,u}(\mu) \tag{32}$$

admits, for that value of g, the pair of solutions $\pm\mu_i$ that are symmetric with respect to the origin. In this way, the problem of finding the values of g at the intersections of the root locus with the sector boundary becomes that of detecting the presence of roots of (32) that are symmetric with respect to the origin. As is well known, this situation corresponds to the presence of an all-zero row in the Routh table for (32). An alternative way to find the intersection of a complex root locus with the imaginary axis has been suggested in [11, 12] where use is made of either the complex Hurwitz test or a system of two equations obtained by separating the locus equation into the real and the imaginary parts and finding values of the locus parameter for which real solutions exist for μ.

The rules for counting the roots of $p_n(s)$ outside and inside (26) can be formulated with reference to the values σ_i in (11) as well as to the values t_i taken by g at the intersections with the sector boundary in all the n steps of the recursive procedure leading from $p_n(s)$ to $p_0(s)$. As may be expected, however, these rules are more intricate than those based on the sequence (13) for finding the root distribution with respect to the imaginary axis. Therefore, we prefer to illustrate how the root distribution with respect to a sector can be determined by means of the examples in Section 4 instead of stating the aforementioned rules in formal terms.

The general criterion is that of proceeding backwards from the last polynomial $p_0(s) = 1$ in the sequence up to the starting one $p_n(s)$. At any step, given the root distribution of $p_{i-1}(s)$, the distribution of the preceding polynomial $p_i(s)$ is determined from the knowledge of σ_i and the value t_i of the current parameter of the root locus at the transition, if any, of a pair of its branches from the outside to the inside of the sector, or vice versa, through its delimiting radii. Precisely, the value σ_i taken by the current parameter g_i on the root locus at $s = -1$ determines the root distribution of $p_{i-1}(s)$ with respect to the imaginary axis (see point (iii) in Section 2), whereas the comparison of σ_i and t_i indicates whether the RHP sector radii have been crossed on passing from the roots of $p_i(s)$ to those of $p_{i-1}(s)$ and point -1.

Remark 2 *If there are no intersections of pairs of root-locus branches with the upper and lower radii of the sector, locus branches may enter or leave the sector only through the origin and the point at infinity along the real axis, where $g = \pm 1$ (as for all points on the imaginary axis – see note after Corollary 2).*

4. Examples

Example 2 *Consider the monic polynomial*

$$p_4(s) = s^4 + 3s^3 - 2.75s^2 - 5.75s + 7.5$$

and its root distribution with respect to the boundary of

$$\mathcal{S} \triangleq \left\{ s = \mu e^{j\phi} :\ \mu \in \mathbb{R}_+,\ \phi \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \right\}. \tag{33}$$

Note that polynomial (2) is certainly non-Hurwitz because not all of its coefficients are positive like the leading coefficient. Polynomials $p_{i-1}(s)$, $i = 4, 3, 2, 1$, generated from $p_4(s)$ by recursion (11) are listed below aligned with the corresponding σ_i-values:

$$\sigma_4 = -2.83\,, \qquad p_3(s) = s^3 - 7.27s^2 + 4.52s + 7.5\,,$$

$$\sigma_3 = 0.92\,, \qquad p_2(s) = s^2 - 177.94s + 182.47\,,$$
$$\sigma_2 = -65.43\,, \qquad p_1(s) = s + 182.47\,,$$
$$\sigma_1 = -0.99\,, \qquad p_0(s) = 1\,.$$

The values t_i at the crossings of the sector boundary (through either the sector radii or through the real axis) turn out to be (by inspection of the associated root locus): $t_4 = -1$, $t_3 = -0.4$ and -1, $t_2 = -1$ and 0.81, $t_1 = -1$. Proceeding backwards, we may argue, with some understandable redundancy, as follows:

(i) polynomial $p_0(s) = 1$ is obtained from polynomial $p_1(s) + \sigma_1 p_1(-s)$ by deleting an LHP root (the root at -1); since $|\sigma_1| = 0.99 < 1$, the root distribution of $p_1(s) + \sigma_1 p_1(-s)$ is the same as that of $p_1(s)$ itself; it follows that **the only root of $p_1(s)$ lies in the LHP***;*

(ii) polynomial $p_1(s)$ with 1 LHP root is obtained from polynomial $p_2(s) + \sigma_2 p_2(-s)$ by deleting an LHP root (the root at -1); it follows that $p_2(s) + \sigma_2 p_2(-s)$ has 2 LHP roots (the one of $p_1(s)$ and the one at -1); since $\sigma_2 = -65.43$, so that $|\sigma_2| > 1$, the root distribution of $p_2(s)$ is opposite to the root distribution of $p_2(s) + \sigma_2 p_2(-s)$ and, thus, $p_2(s)$ has 2 RHP roots. To understand whether the three roots are inside or outside the sector (33) we need to check if, when varying from σ_2 to zero, g_2 assumes the value t_2, i.e. if $t_2 \in I_2 \triangleq (-65.43, 0)$. It is immediate to check that the value $t_2 = 0.81$, which might be associated with a pair of complex conjugate roots, does not belong to I_2 while the value $t_2 = -1$ belongs to I_2. Hence at least one root enters the sector through the origin and remains inside it; the same must happen for the other root. In conclusion, $p_2(s)$ **has 2 RHP roots inside (33)***;*

(iii) polynomial $p_2(s)$ with 2 RHP roots is obtained from polynomial $p_3(s) + \sigma_3 p_3(-s)$ by deleting the LHP root at -1; it follows that $p_3(s) + \sigma_3 p_3(-s)$ has 2 RHP roots (the 2 RHP roots of $p_2(s)$) and 1 LHP root (the one at -1); since $|\sigma_3| < 1$, the root distribution of $p_3(s)$ is the same as the root distribution of $p_3(s) + \sigma_3 p_3(-s)$ and, thus, also $p_3(s)$ has 2 RHP roots and 1 LHP root; furthermore, neither of the two values of t_3 belongs to the interval $I_3 \triangleq (0, 0.92)$ hence no root crosses the boundary of the sector (33); in conclusion, $p_3(s)$ **has 2 RHP roots inside (33) and 1 LHP root***;*

(iv) polynomial $p_3(s)$ with 2 RHP roots and 1 LHP root is obtained from polynomial $p_4(s) + \sigma_4 p_4(-s)$ by deleting the LHP root at -1; it follows that $p_4(s) + \sigma_4 p_4(-s)$ has 2 RHP roots and 2 LHP roots (the 2 RHP and 1 LHP root of $p_2(s)$ and the additional LHP root at -1); since $|\sigma_4| > 1$, the root distribution of $p_4(s)$ is opposite to the root distribution of $p_4(s) + \sigma_4 p_4(-s)$ and, thus, also $p_4(s)$ has 2 RHP roots and 2 LHP root; the sector boundary is crossed only for $g_4 = t_4 = -1 \in I_4 \triangleq (-2.83, 0)$ at the origin and the point

at infinity but the two corresponding branches never intersect the upper and lower sector radii; in conclusion, $p_4(s)$ **has 2 RHP roots inside (33) and 2 LHP roots**.

Example 3 *Consider the monic polynomial*

$$p_4(s) = s^4 + 3s^3 + s^2 + 13s + 30$$

and its root distribution with respect to the boundary of (33). Polynomials $p_{i-1}(s)$, $i = 4, 3, 2, 1$, *generated from* $p_4(s)$ *by recursion (11) are listed below aligned with the corresponding* σ_i*-values:*

$$\begin{aligned}
\sigma_4 &= -0.33\,, & p_3(s) &= s^3 + 5s^2 - 4s + 30\,,\\
\sigma_3 &= -1.19\,, & p_2(s) &= s^2 - 1.43s - 2.57\,,\\
\sigma_2 &= -0.05\,, & p_1(s) &= s - 2.57\,,\\
\sigma_1 &= -2.27\,, & p_0(s) &= 1\,.
\end{aligned}$$

The values t_i *at the crossings of the sector boundary (through either the sector radii or through the real axis where* $g_i = \pm 1$*) turn out to be:* $t_4 = -1$ *and* 1.253, $t_3 = -0.67$ *and* -1, $t_2 = -1$, $t_1 = -1$. *Proceeding backwards, we may argue as follows:*

- *(i) polynomial* $p_0(s)$ *is obtained from polynomial* $p_1(s) + \sigma_1 p_1(-s)$ *by deleting an LHP root (the root at* -1*); since* $|\sigma_1| = 2.27 > 1$, *the root distribution of* $p_1(s) + \sigma_1 p_1(-s)$ *is opposite to that of* $p_1(s)$*; it follows that* **the only root of $p_1(s)$ lies in the RHP (necessarily inside (33))***;*

- *(ii) polynomial* $p_1(s)$ *with 1 RHP root is obtained from polynomial* $p_2(s) + \sigma_2 p_2(-s)$ *by deleting an LHP root (the root at* -1*); it follows that* $p_2(s) + \sigma_2 p_2(-s)$ *has 1 RHP root (the one of* $p_1(s)$ *) and 1 LHP root (the deleted root at* -1*); since* $|\sigma_2| = 0.05 < 1$, *the root distribution of* $p_2(s)$ *is the same as that of* $p_2(s) + \sigma_2 p_2(-s)$ *and, thus,* $p_2(s)$ **has 1 real RHP root (necessarily inside (33)) and 1 LHP root***;*

- *(iii) polynomial* $p_2(s)$ *with 1 RHP root and 1 LHP root is obtained from polynomial* $p_3(s) + \sigma_3 p_3(-s)$ *by deleting the LHP root at* -1*; it follows that* $p_3(s) + \sigma_3 p_3(-s)$ *has 1 RHP root (the RHP root of* $p_2(s)$*) and 2 LHP roots (the one at* -1 *and the LHP root of* $p_2(s)$*); since* $|\sigma_3| = 1.19 > 1$, *the root distribution of* $p_3(s)$ *is opposite to the root distribution of* $p_3(s) + \sigma_3 p_3(-s)$ *and, thus,* $p_3(s)$ *has 2 RHP roots and 1 LHP root. Since both* $t_3 = -0.67 \in I_3 \triangleq (-1.19, 0)$ *and* $t_3 = -1 \in I_3$, *two branches enter the sector through the upper and lower radii for* $t_3 = -0.67$ *and one branch leaves the sector through the origin for* $t_3 = -1$*: it follows that the two RHP roots of* $p_3(s)$ *are outside the sector; in conclusion,* $p_3(s)$ **has 1 LHP root and 2 RHP roots outside (33)***;*

(iv) *polynomial $p_3(s)$ with 2 RHP roots and 1 LHP root is obtained from polynomial $p_4(s) + \sigma_4 p_4(-s)$ by deleting the LHP root at -1; it follows that $p_4(s)+\sigma_4 p_4(-s)$ has 2 RHP roots (the 2 RHP of $p_3(s)$) and 2 LHP roots (the LHP root of $p_2(s)$) and one LHP root at -1); since $|\sigma_4| = 0.33 < 1$ the root distribution of $p_4(s)$ is the same as the root distribution of $p_4(s)+\sigma_4 p_4(-s)$ and, thus, also $p_4(s)$ has 2 RHP roots and 2 LHP roots; the sector boundary is never crossed in the interval $I_4 \triangleq (-0.33, 0)$, therefore, $p_4(s)$* **has 2 RHP roots outside (33) and 2 LHP roots**.

5. Conclusions

The Lepschy test seems to be a valid alternative to the Routh test from the operative and conceptual points of view. The interpretation of its basic recursion by means of a root locus allows us to better understand how the the roots of the polynomials generated by the algorithm deploy in the complex plane. The original Lepschy test was conceived for determining the root distribution of a polynomial with respect to the imaginary axis, but it can easily be extended to find the root distribution with respect to any straight line and, at the expense of an increased computational effort, even to sector boundaries, making it a suitable tool for checking the stability of fractional-order systems. Note that the alternative Routh-type test suggested in [31] for the same purpose is not computationally simpler.

In these authors' opinion, the adopted geometric approach exhibits remarkable didactic advantages over alternative more formal approaches for testing the stability of linear systems and can be considered as an interesting application of basic notions taught in introductory control courses, such as root loci and stability criteria.

Possible directions of future research comprise: (i) the derivation of more systematic rules for the evaluation of the root distribution with respect to sector boundaries, and (ii) the use of the polynomials of descending degree recursively generated by the Lepschy algorithm in the approximation of complex fractional- and integer-order systems [7].

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