GEODEZJA I KARTOGRAFIA
t. LIII, z. 1, s. 7–19 (2004)

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General accuracy characteristics of resection

Determination of the position by resection requires accuracy analysis and establishment of general accuracy characteristics.

The obtained accuracy characteristics, among others, depend on indicator of position determinability introduced for positioning.

INTRODUCTION

Determination of a point by resection is convenient because the field work is reduced to the measurement of two angles at one point only. However, it should be noted, that if three known points and the point which is to be determined lie on the same circle, called dangerous circle, the problem becomes indeterminate.

The statement that position cannot be determined involves consideration of accuracy. Therefore, general accuracy characteristics of the resection have been developed in this paper.

The given examples illustrate the practical application of the established functions.

1. *Inverse matrix from normal equations*

Let us consider the resection of which left, central and right points, of known coordinates, are denoted by 1, 0, 2, respectively. Let the point we determine be P, the observed angles α_1, α_2, and the known angle 1-0-2 be β, Fig. 1. The known distances 0-1, 0-2 we denote by a_1, a_2, and unknown distances P-1, P-0, P-2 by s_1, s_0, s_2, respectively.

For establishing inverse matrix from normal equations we introduce XY coordinate system of which the X axis is parallel to the line between the points P and 0.

In this coordinate system for the angles α_1 and α_2 we obtain the differentials calculated for approximate values of the coordinates

$$\mathrm{d}\alpha_1 = \frac{\sin\alpha_1}{s_1}\,\mathrm{d}X_P + \left(\frac{\cos\alpha_1}{s_1} - \frac{1}{s_0}\right)\mathrm{d}Y_P \tag{1}$$

$$\mathrm{d}\alpha_2 = \frac{\sin\alpha_2}{s_2}\,\mathrm{d}X_P - \left(\frac{\cos\alpha_1}{s_2} - \frac{1}{s_0}\right)\mathrm{d}Y_P \tag{2}$$

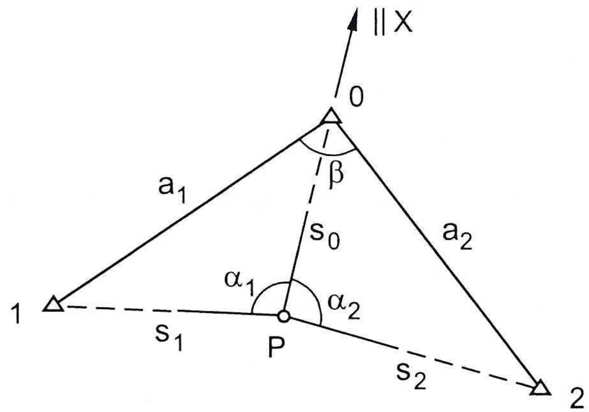

Fig. 1. Resection

Introducing angles δ_1, δ_2, Fig. 2, we state that

$$\frac{\sin\alpha_1}{s_1} = \frac{s_0\sin\alpha_1}{s_0 s_1} = \frac{a_1\sin\delta_1}{s_0 s_1} \tag{3}$$

$$\frac{\cos\alpha_1}{s_1} - \frac{1}{s_0} = -\frac{s_1 - s_0\cos\alpha_1}{s_0 s_1} = -\frac{\alpha_1\cos\delta_1}{s_0 s_1} \tag{4}$$

$$\frac{\sin\alpha_2}{s_2} = \frac{s_0\sin\alpha_2}{s_0 s_2} = \frac{a_2\sin\delta_2}{s_0 s_2} \tag{5}$$

$$\frac{\cos\alpha_2}{s_2} - \frac{1}{s_0} = -\frac{s_2 - s_0\cos\alpha_2}{s_0 s_2} = -\frac{a_2\cos\delta_2}{s_0 s_2} \tag{6}$$

where:

$$\delta_1 = 180° - (\alpha_1 + \beta_1) \tag{7}$$

$$\delta_2 = 180° - (\alpha_2 + \beta_2) \tag{8}$$

Design matrix of the coefficients, after substituting (3), (4) into (1) and (5), (6) into (2), can be written in the form

$$\mathbf{a} = \frac{1}{s_0}\begin{bmatrix} \frac{a_1}{s_1}\sin\delta_1 & -\frac{a_1}{s_1}\cos\delta_1 \\ \frac{a_2}{s_2}\sin\delta_2 & \frac{a_2}{s_2}\cos\delta_2 \end{bmatrix} \tag{9}$$

hence

$$\mathbf{a}^T\mathbf{a} = \frac{1}{s_0^2}\begin{bmatrix} \frac{a_1^2}{s_1^2}\sin^2\delta_1 + \frac{a_2^2}{s_2^2}\sin^2\delta_2 & -\frac{1}{2}\left(\frac{a_1^2}{s_1^2}\sin 2\delta_1 - \frac{a_2^2}{s_2^2}\sin 2\delta_2\right) \\ -\frac{1}{2}\left(\frac{a_1^2}{s_1^2}\sin 2\delta_1 - \frac{a_2^2}{s_2^2}\sin 2\delta_2\right) & \frac{a_1^2}{s_1^2}\cos^2\delta_1 + \frac{a_2^2}{s_2^2}\cos^2\delta_2 \end{bmatrix} \tag{10}$$

which is a square and symmetrical matrix.

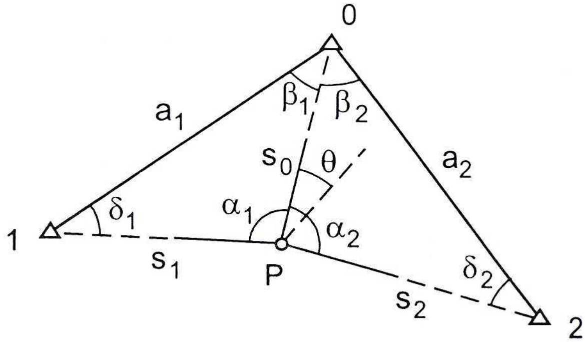

Fig. 2. Geometric elements for determination of accuracy

The matrix $\mathbf{a}^T\mathbf{a}$ is invertible if its determinant is not equal to zero. Finding the determinant we write

$$\det(\mathbf{a}^T\mathbf{a}) = \frac{1}{s_0^4}\left[\left(\frac{a_1^2}{s_1^2}\sin^2\delta_1 + \frac{a_2^2}{s_2^2}\sin^2\delta_2\right)\left(\frac{a_1^2}{s_1^2}\cos^2\delta_1 + \frac{a_2^2}{s_2^2}\cos^2\delta_2\right) - \right.$$
$$\left. -\frac{1}{4}\left(\frac{a_1^2}{s_1^2}\sin 2\delta_1 - \frac{a_2^2}{s_2^2}\sin 2\delta_2\right)^2\right] \tag{11}$$

and after some manipulation we state that

$$\det(\mathbf{a}^T\mathbf{a}) = \frac{a_1^2 a_2^2}{s_0^4 s_1^2 s_2^2}\sin^2(\delta_1 + \delta_2) \tag{12}$$

so the determinant is not equal to zero.

Taking into account that

$$\delta_1 + \delta_2 = 360° - (\alpha_1 + \beta + \alpha_2) \tag{13}$$

where β is the angle between the points 1-0-2, Fig. 1, and introducing indicator of position determinability, [9], we write

$$\omega = \alpha_1 + \beta + \alpha_2 \qquad \omega \neq 180° \tag{14}$$

and after substituting (14) into (13), we have

$$\delta_1 + \delta_2 = 360° - \omega \tag{15}$$

If now we substitute (15) into (12) we finally obtain

$$\det(\mathbf{a}^T\mathbf{a}) = \frac{a_1^2 a_2^2}{s_0^4 s_1^2 s_2^2}\sin^2\omega \tag{16}$$

Establishing the inverse of the matrix (10), after taking into account (16) and some manipulation, we state that

$$(\mathbf{a}^T\mathbf{a})^{-1} = \frac{s_0^2}{\sin^2\omega}\begin{bmatrix} \frac{s_2^2}{a_2^2}\cos^2\delta_1 + \frac{s_1^2}{a_1^2}\cos^2\delta_2 & \frac{1}{2}\left(\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2\right) & \frac{s_2^2}{a_2^2}\sin^2\delta_1 + \frac{s_1^2}{a_1^2}\sin^2\delta_2 \end{bmatrix} \tag{17}$$

which is the inverse matrix from the normal equations.

2. *Mean position error*

Finding the sum of the main diagonal of the matrix given by (17), called the trace of the matrix, we have

$$\operatorname{tr}(\mathbf{a}^T\mathbf{a})^{-1} = \frac{s_0^2}{\sin^2\omega}\left(\frac{s_1^2}{a_1^2} + \frac{s_2^2}{a_2^2}\right) \tag{18}$$

Taking into account the standard deviations of measured angles α_1 and α_2, σ_α, we write

$$\sigma_P^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\left(\frac{s_1^2}{a_1^2} + \frac{s_2^2}{a_2^2}\right) \tag{19}$$

and after introducing the angles β_1 and β_2, Fig.2, we state that

$$\sigma_P^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1} + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\right) \tag{20}$$

where $\omega = \alpha_1 + \beta + \alpha_2 = \alpha_1 + \beta_1 + \beta_2 + \alpha_2$.

The obtained functions, (19), (20), establish the mean position error of the point P.

3. *Accuracy of central distance and of central azimuth*

Taking into account (17) we can determine the accuracy of the central distance, s_0, between the points P and 0. Because the azimuth of central distance is equal to zero then basing on (17) and the results established in [8], we write

$$\sigma_{s_0}^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2 \omega}\left(\frac{s_1^2}{a_1^2}\cos^2\delta_2 + \frac{s_2^2}{a_2^2}\sin^2\delta_1\right) \tag{21}$$

and after introducing the angles α_1, β_1, α_2, β_2 we have

$$\sigma_{s_0}^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2 \omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1\right) \tag{22}$$

where δ_1, δ_2 are given by (7), (8).

Accuracy of the central azimuth between the points P and 0 on the base of (17) and error propagation we write in the form

$$\sigma_{(s_0)}^2 = \frac{\sigma_\alpha^2}{\sin^2 \omega}\left(\frac{s_1^2}{a_1^2}\sin^2\delta_2 + \frac{s_2^2}{a_2^2}\sin^2\delta_1\right) \tag{23}$$

or

$$\sigma_{(s_0)}^2 = \frac{\sigma_\alpha^2}{\sin^2 \omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1\right) \tag{24}$$

where δ_1, δ_2 we have from (7), (8).

The obtained functions (21), (22) and (23), (24) establish accuracy of the central distance and the accuracy of central azimuth, respectively.

4. *Positional error of a point in any direction*

Let us find the standard deviation in any direction, θ, which we denote by σ_{θ_P}. Rotating the XY coordinate system through the angle θ, shown in Fig. 2 , we write

$$X' = X\cos\theta + Y\sin\theta \tag{25}$$

On the basis of the error propagation and (17) we state that

$$\sigma_{\theta_P}^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}^T \begin{bmatrix} \frac{s_2^2}{a_2^2}\cos^2\delta_1 + \frac{s_1^2}{a_1^2}\cos^2\delta_2 & \frac{1}{2}\left(\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2\right) & \frac{s_2^2}{a_2^2}\sin^2\delta_1 + \frac{s_1^2}{a_1^2}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \tag{26}$$

or introducing the angles α_1, β_1, α_2, β_2

$$\sigma_{\theta_P}^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}^T \begin{bmatrix} \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \tag{27}$$

The above functions, (26), (27), determine the positional error of a point in any direction.

5. *Standard error ellipse*

The most general description of the positional accuracy of a point is given by a standard error ellipse. Usually, we want to know the maximum and minimum standard deviations for the point and their directions.

Denoting the semimajor and semiminor axes of the standard ellipse by $A = \sigma_{\max}$, $B = \sigma_{\min}$, we first determine the direction of $\sigma_{\max}$, ϕ, measured from the line between the points P and 0. Taking into account the expression (26) in which we write ϕ instead of θ, and finding $\partial\sigma_{\phi_P}^2 / \partial\phi = 0$, we have the equation

$$2\left(\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2\right)\cos 2\phi - \left(\frac{s_2^2}{a_2^2}\cos 2\delta_1 + \frac{s_1^2}{a_1^2}\cos 2\delta_2\right)\sin 2\phi = 0 \tag{28}$$

so that

$$\mathrm{tg}2\phi = \frac{\frac{s_2^2}{a_2^2}\sin 2\delta_1 - \frac{s_1^2}{a_1^2}\sin 2\delta_2}{\frac{s_2^2}{a_2^2}\cos 2\delta_1 + \frac{s_1^2}{a_1^2}\cos 2\delta_2} \tag{29}$$

and after introducing the angles α_1, β_1, α_2, β_2

$$\mathrm{tg}2\phi = \frac{\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2}{\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\cos 2\delta_1 + \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\cos 2\delta_2} \tag{30}$$

and also

$$\mathrm{tg}2\phi = \frac{\sin^2\alpha_1 \sin^2\beta_2 \sin 2\delta_1 - \sin^2\alpha_2 \sin^2\beta_1 \sin 2\delta_2}{\sin^2\alpha_1 \sin^2\beta_2 \cos 2\delta_1 + \sin^2\alpha_2 \sin^2\beta_1 \cos 2\delta_2} \tag{31}$$

which establish the direction of the semimajor axis of the error ellipse.

Taking into account the angle ϕ and the expressions (26), (27), we write

$$A^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}^T \begin{bmatrix} \dfrac{s_2^2}{a_2^2}\cos^2\delta_1 + \dfrac{s_1^2}{a_1^2}\cos^2\delta_2 & \dfrac{1}{2}\left(\dfrac{s_2^2}{a_2^2}\sin 2\delta_1 - \dfrac{s_1^2}{a_1^2}\sin 2\delta_2\right) \\ \dfrac{1}{2}\left(\dfrac{s_2^2}{a_2^2}\sin 2\delta_1 - \dfrac{s_1^2}{a_1^2}\sin 2\delta_2\right) & \dfrac{s_2^2}{a_2^2}\sin^2\delta_1 + \dfrac{s_1^2}{a_1^2}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} \tag{32}$$

or

$$\mathrm{b}A^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}^T \begin{bmatrix} \dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \dfrac{1}{2}\left(\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \dfrac{1}{2}\left(\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix} \tag{33}$$

The direction of $\sigma_{\min}$ is perpendicular to $\sigma_{\max}$ and it is equal to $\phi + 90°$. Then, on the base of (26), (27), we have

$$B^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix}^T \begin{bmatrix} \dfrac{s_2^2}{a_2^2}\cos^2\delta_1 + \dfrac{s_1^2}{a_1^2}\cos^2\delta_2 & \dfrac{1}{2}\left(\dfrac{s_2^2}{a_2^2}\sin 2\delta_1 - \dfrac{s_1^2}{a_1^2}\sin 2\delta_2\right) \\ \dfrac{1}{2}\left(\dfrac{s_2^2}{a_2^2}\sin 2\delta_1 - \dfrac{s_1^2}{a_1^2}\sin 2\delta_2\right) & \dfrac{s_2^2}{a_2^2}\sin^2\delta_1 + \dfrac{s_1^2}{a_1^2}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix} \tag{34}$$

and also

$$B^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}^T \begin{bmatrix} \dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \dfrac{1}{2}\left(\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \dfrac{1}{2}\left(\dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \dfrac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \dfrac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix} \tag{35}$$

The obtained functions allow for direct determination of the direction and the principal axes of the error ellipse.

6. *Application of the obtained functions*

Resection requires, before its realisation, analysis of the accuracy of the construction. The simple examples, presented below, illustrate the practical application of the obtained functions.

Example 1

Let us consider a design of the resection with the angles $\alpha_1 = 60°$, $\alpha_2 = 30°$, $\beta_1 = 60°$, and $\beta_2 = 90°$. Find the accuracy of the point P for $s_0 = 900$ m and $\sigma_\alpha = 5''$.

Because the indicator of position determinability is

$$\omega = \alpha_1 + \beta_1 + \beta_2 + \alpha_2 = 240° \neq 180°$$

then the position is determinable, and from (20) , we have

$$\sigma_P^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1} + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\right) = \frac{4\sigma_\alpha^2 s_0^2}{3}(1 + 4) = \frac{20}{3}s_0^2\sigma_\alpha^2$$

that is

$$\sigma_P = \frac{2\sqrt{5}}{\sqrt{3}}s_0\sigma_\alpha$$

For $s_0 = 900$ m, $\sigma_\alpha = 5''$, and $\rho'' = 2\times10^5$

$$\sigma_P = \frac{2\sqrt{5}\times900\times5}{\sqrt{3}\times2\times10^5} = 0.058 \text{ m}$$

which establishes the accuracy of the point P.

Example 2

Taking into account the data from Example 1, $\alpha_1 = 60°$, $\alpha_2 = 30°$, $\beta_1 = 60°$, $\beta_2 = 90°$, determine the accuracy of central distance and central azimuth for $s_0 = 1000$ m, $\sigma_\alpha = 5''$.

Finding

$$\delta_1 = 180° - (\alpha_1 + \beta_1) = 60°, \qquad \delta_2 = 180° - (\alpha_2 + \beta_2) = 60°$$

from (22) we get

$$\sigma_{s_0}^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1\right) = \frac{4\sigma_\alpha^2 s_0^2}{3}\left(\frac{1}{4}+1\right) = \frac{5}{3}s_0^2\sigma_\alpha^2$$

$$\sigma_{s_0} = \frac{\sqrt{5}}{\sqrt{3}}s_0\sigma_\alpha$$

and from (24) we obtain

$$\sigma_{(s_0)}^2 = \frac{\sigma_\alpha^2}{\sin^2\omega}\left(\frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 + \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1\right) = \frac{4\sigma_\alpha^2}{3}\left(\frac{3}{4}+3\right) = 5\sigma_\alpha^2$$

$$\sigma_{(s_0)} = \sqrt{5}\sigma_\alpha$$

For $s_0 = 1000$ m, $\sigma_\alpha = 5''$, and $\rho'' = 2\times10^5$

$$\sigma_{s_0} = \frac{\sqrt{5}\times1000\times5}{\sqrt{3}\times2\times10^5} = 0.032 \text{ m}$$

and

$$\sigma_{(s_0)} = \sqrt{5}\times5'' = 11''$$

which establish the accuracy of the central distance and the central azimuth.

Example 3

For resection with angles $\alpha_1 = \alpha_2 = 90°$, $\beta_1 = 60°$, $\beta_2 = 30°$, determine the positional error in the direction of $\theta = 150°$ for $s_0 = 1600$ m and $\sigma_\alpha = 5''$.

Establishing indicator of position determinability

$$\omega = \alpha_1 + \beta_1 + \beta_2 + \alpha_2 = 270° \neq 180°$$

after calculations

$$\delta_1 = 180° - (\alpha_1 + \beta_1) = 30°, \qquad \delta_2 = 180° - (\alpha_2 + \beta_2) = 60°$$

from (27) we have

$$\sigma^2_{\theta_P} = \frac{\sigma^2_\alpha s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}^T \begin{bmatrix} \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

$$= \frac{\sigma^2_\alpha s_0^2}{1}\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{bmatrix}^T \begin{bmatrix} \frac{3}{16}+\frac{3}{16} & \frac{1}{2}\left(\frac{\sqrt{3}}{8}-\frac{3\sqrt{3}}{8}\right) \\ \frac{1}{2}\left(\frac{\sqrt{3}}{8}-\frac{3\sqrt{3}}{8}\right) & \frac{1}{16}+\frac{9}{16}\end{bmatrix}\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{bmatrix}$$

$$= \frac{\sigma^2_\alpha s_0^2}{32}\begin{bmatrix} -\sqrt{3} \\ 1\end{bmatrix}^T \begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 5\end{bmatrix}\begin{bmatrix} -\sqrt{3} \\ 1\end{bmatrix} = \frac{5}{8}s_0^2\sigma^2_\alpha$$

then

$$\sigma_{\theta_P} = \frac{\sqrt{5}}{2\sqrt{2}} s_0 \sigma_\alpha$$

Substituting into above expression s_0 = 1600 m, $\sigma_\alpha = 5''$, and $\rho'' = 2\times 10^5$

$$\sigma_{\theta_P} = \frac{\sqrt{5}\times 1600\times 5}{2\sqrt{2}\times 2\times 10^5} = 0.032 \text{ m}$$

which determines positional error in direction $\theta = 150°$.

Example 4

For the resection, considered in Example 3, with $\alpha_1 = \alpha_2 = 90°$, $\beta_1 = 60°$, $\beta_2 = 30°$, establish the orientation angle and the semimajor and the se miminor axes of the standard ellipse knowing that s_0 = 1600 m, $\sigma_\alpha = 5''$.

As stated in Example 3, $\omega = 270°$, $\delta_1 = 30°$, $\delta_2 = 60°$, taking into account (31), we have

$$\mathrm{tg}2\phi = \frac{\sin^2\alpha_1 \sin^2\beta_2 \sin 2\delta_1 - \sin^2\alpha_2 \sin^2\beta_1 \sin 2\delta_2}{\sin^2\alpha_1 \sin^2\beta_2 \cos 2\delta_1 + \sin^2\alpha_2 \sin^2\beta_1 \cos 2\delta_2} = \frac{\frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}}{\frac{1}{8} - \frac{3}{8}} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

then

$$2\phi = 240°, \qquad \phi = 120°$$

From the expressions (33), (35) we obtain

$$A^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}^T \begin{bmatrix} \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}\cos\phi\\ \sin\phi\end{bmatrix}$$

$$= \frac{\sigma_\alpha^2 s_0^2}{32}\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}^T\begin{bmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 5\end{bmatrix}\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix} = \frac{3}{4}s_0^2\sigma_\alpha^2$$

$$A = \frac{\sqrt{3}}{2}s_0\sigma_\alpha$$

and

$$B^2 = \frac{\sigma_\alpha^2 s_0^2}{\sin^2\omega}\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix}^T \begin{bmatrix} \frac{\sin^2\beta_2}{\sin^2\alpha_2}\cos^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\cos^2\delta_2 & \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) \\ \frac{1}{2}\left(\frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin 2\delta_1 - \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin 2\delta_2\right) & \frac{\sin^2\beta_2}{\sin^2\alpha_2}\sin^2\delta_1 + \frac{\sin^2\beta_1}{\sin^2\alpha_1}\sin^2\delta_2 \end{bmatrix}\begin{bmatrix}-\sin\phi\\ \cos\phi\end{bmatrix}$$

$$= \frac{\sigma_\alpha^2 s_0^2}{32}\begin{bmatrix}-\sqrt{3}\\ -1\end{bmatrix}^T\begin{bmatrix}3 & -\sqrt{3}\\ -\sqrt{3} & 5\end{bmatrix}\begin{bmatrix}-\sqrt{3}\\ -1\end{bmatrix} = \frac{1}{4}s_0^2\sigma_\alpha^2$$

$$B = \frac{1}{2} s_0 \sigma_\alpha$$

So for $s_0 = 1600$ m, $\sigma_\alpha = 5''$, $\rho'' = 2\times10^5$

$$A = \frac{\sqrt{3}\times1600\times5}{2\times2\times10^5} = 0.035 \text{ m}$$

$$B = \frac{1\times1600\times5}{2\times2\times10^5} = 0.020 \text{ m}$$

which establish the semimajor and the semiminor axes of the standard error ellipse.

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Received February 3, 2004
Accepted March 1, 2004

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Ogólne charakterystyki dokładnościowe wcięcia wstecz

S t r e s z c z e n i e

Wcięcie wstecz stanowi szczególnie użyteczną konstrukcję geodezyjną ze względu na zredukowanie obserwacji terenowych do pomiaru dwóch kątów na punkcie wyznaczanym. Należy jednak zauważyć, że w przypadku kiedy trzy punkty o znanych współrzędnych i punkt wyznaczany leżą na tym samym okręgu koła pozycja jest niewyznaczalna.

Niewyznaczalność pozycji powoduje konieczność przeprowadzenia analiz dokładnościowych przed założeniem tych konstrukcji. W przedstawionej pracy ustalono ogólne charakterystyki dokładnościowe dla bezpośredniego dokonywania tych analiz.

Podano ogólne funkcje pozwalające na bezpośrednie wyznaczanie średniego błędu pozycji, dokładności odległości centralnej i dokładności azymutu centralnego. Ustalono także funkcje określające błąd wyznaczenia pozycji w dowolnym kierunku i podano orientację oraz wielkości półosi standardowych elips błędów.

Załączone przykłady numeryczne ilustrują praktyczne zastosowania otrzymanych funkcji.

Януш Мартусевич

Общие характеристики точности обратной засечки

Резюме

Обратная засечка является весьма выгодной геодезической конструкцией ввиду приведения полевых наблюдений к измерениям двух глов на определяемом пункте. Следует однако заметить, что в случае, когда три пункта с известными координатами и определяемый пункт расноложены на одной и той же окружности круга, позиция является неопределяемой.

Неопределяемость позиции вызывает необходимость проведения анализов точности перед закладкой этой конструкции. В представленной работе установлены общие характеристики точности для непосредственного проведения этих анализов.

Даны общие функции разрешающие непосредственно определять среднюю ошибку позиции, точности центрального расстояния и точности центрального азимута. Установлена также функция, определяющая ошибку установления позиции в любом направлении, и дана ориентировка, а также величины полуосей стандартных элипсов ошибок.

Приведённые цифровые примеры иллюстрируют практическое применение полученных функций.