Central European Journal of Economic Modelling and Econometrics

Gale Economy with Investments and Limit Technology

Emil Panek*

Submitted: 3.02.2021, Accepted: 6.11.2021

Abstract

In this article we have described a multiproduct model of economical dynamics of Gale type, in which the changes in production technology (the dynamics of Gale type production spaces) depend upon the scale of targeted investments. Under such assumptions we have proved a so-called "weak" version of a multilane turnpike theorem in the Gale type economy with varying technology which converges to a certain limit technology. It states that in the long periods of time, regardless of the initial state of the economy, the optimal growth processes almost always lie close to the family of steady growth paths with maximum growth rate called the multilane turnpike.

Keywords: Gale production model with investments, von Neumann equilibrium, limit production space, technological and economical production efficiency, multilane production turnpike

JEL Classification: C62, C67, O41, O49

*University of Zielona Góra, Institute of Economics and Finance, Zielona Góra, Poland; e-mail: e.panek@wez.uz.zgora.pl; ORCID: 0000-0002-7950-1689

1 Introduction

More than six decades ago P. A. Samuelson (1960) formulated the hypothesis about a specific convergence in long periods of time of the optimal growth paths to a certain "model" path, on which the economy achieves the maximum, even production growth. This "model" path which characterizes the economy in a specific dynamic equilibrium (known as the von Neumann equilibrium) is suggestively compared to a highway (a turnpike) in a road transport. If we are to reach a nearby town from a certain location, then we go directly to our destination using local roads. However, if our destination is far away, then we try to get in the first place a highway (a turnpike) and then move along it for as long as possible. Only the final part of our journey will be taken again on local roads. By identifying the location of our starting point and destination with the states of the economy and by denoting by $T = \{0, 1, \ldots, t_1\}$ the period of time (horizon) of the economy that we are interested in, with the contractual initial period $t = 0$ and the final period $t_1 < +\infty$, the turnpike law can be formulated as follows: Starting from the historically shaped initial state (in the period $t = 0$) a rationally functioning economy should reach its turnpike (the "model" growth path) as quickly as possible, then in the middle growth phase it should follow this turnpike and in the end phase (in the last periods of the horizon T) it can possibly move away from the turnpike to reach the final state.

The presented economic growth hypothesis aroused great interest around the world among many mathematical economists. They proved many variants of the turnpike theorems (production, capital, consumption turnpikes) in various multi-sector/multi-product models of economic dynamics, mainly of the Neumann-Gale type. As a result of the research conducted over the past half of the century, the turnpike theory has been developed. Today it is one of the pillars of mathematical economics. In all Neumann-Gale models of the economic dynamics, one of the main concepts are the so-called production spaces (in other words: technological sets). In all research on this subject it is assumed either that the production technology is stationary (time invariable) or (less frequently) that the technological changes determining the dynamics of production spaces (technological sets) do not require investment inputs and thus they are a peculiar gift of God or of nature; see e.g. Giorgi G. and Zuccotti C. (2016), Lancaster K. (1968, Part III, Chapters 10, 11), Makarov, Rubinov (1977), Nikaido (1968, Chapter 4), Panek (2000, Part 2, Chapters 5, 6), Takayama (1985, Chapters 6, 7). A comprehensive bibliography of papers on turnpike theory can be found in McKenzie (2005), Mitra and Nishimura (2009). This strand also includes the author's earlier papers on the turnpike properties of the optimal growth processes in the stationary (2016, 2017) and non-stationary (2017b, 2018, 2019a, 2019c, 2020a, 2020b) Gale economies with a multilane turnpike and papers focused on the turnpike effect in a Gale economy with a general form of the growth criterion (2019b), as well as with a minimal-time growth criterion – the so-called optimal-time growth problem (2021).

The assumption that changes in production technology in the economy do not require investment expenditures is a simplification. Hence the question arises whether in models of economic dynamics with a Gale production space the turnpike effect will also take place after including the investment mechanism? This paper gives an affirmative answer to this question. Its added value lies particularly in:

i) generalization of the classical model of Gale economic dynamics by (a) including the non-stationarity of the economy (here: changes in production technology over time) and (b) allowing in the model the existence of more than one production turnpike and replacing the standard single turnpike (single von Neumann ray) with a bundle of turnpikes (the so-called multilane turnpike),

ii) inclusion of an investment mechanism in the non-stationary Gale economy and the proof of a "weak" turnpike theorem in the case when the changes in production technology in such an economy (with the dynamics of production spaces) are determined by the investment expenditures allocated for this purpose.

Thus we obtain another confirmation of the universal nature of the turnpike theorem in the mathematical economics, which states that in long periods of time the optimal growth processes should lie near certain specific/distinguished paths of the steady growth. Those paths are called turnpikes and the economy remains on them in the so-called von Neumann (dynamical) equilibrium and this way achieves both the maximum rate of growth and the highest technological and economic production efficiency.

It is likely that the strong and very strong version of this theorem remains valid but the confirmation of such conjecture requires further work.

The structure of the paper is as follows. In Section 2 we discuss the basic properties of the Gale type production space which are need to construct models of economy dynamics of Gale type with investments and limiting technology. The way in which the investments influence the production technology (the shape of production spaces) in our model of economy is described in Section 3. In Section 4 we define the multilane production turnpike and the stationary growth rate. We also discuss specific properties of those. In Section 5 we formulate the conditions under which the dynamical von Neumann equilibrium exists in the presented Gale type economy with investments. We also clarify the link between von Neumann equilibrium and the growth of economy in the multilane production turnpike. The key result is contained in Section 6. This is where we define the optimal growth process and prove the weak version of the multilane turnpike theorem in the non-stationary Gale economy with investments.

We indicate in the summary possible further research directions which the interested readers can pursue.

2 Production space. Technological efficiency of production

We consider an economy where time is discrete, $t = 0, 1, \ldots$ In each time period there is n goods which are produced and/or used up in production. Let $x(t) = (x_1(t), \ldots,\ x_n(t)) \geqq 0$ be a vector of goods used and $y(t) = (y_1(t), \ldots,\ y_n(t)) \geqq 0$ be a vector of goods produced in period t (if $a, b \in R^n$, then $a \geqq b$ stands for $a_i \geq b_i$, $i = 1, 2, \ldots, n$; $a \geq b$ denotes $a \geqq b$ and $a \neq b$). The vector $x(t)$ is called the vector of inputs and the vector $y(t)$ is called the vector of outputs (production). If the available technology allows to obtain from the input vector $x(t)$ the output vector $y(t)$, then the pair $(x(t), y(t))$ describes a technologically feasible production process (in time period t). Let $Z(t) \subset R_+^{2n}$ denote the set of technologically admissible production processes in the period t. The condition $(x, y) \in Z(t)$ (equivalently $(x(t), y(t)) \in Z(t)$) says that in the period t one can produce y from the inputs x in the economy. The set $Z(t)$ is called the Gale production space (technological set) if the following condition are satisfied:

(G1) $\forall \left(x^1, y^1\right), \left(x^2, y^2\right) \in Z(t), \forall \lambda_1, \lambda_2 \geq 0 \ \left(\lambda_1 \left(x^1, y^1\right) + \lambda_2 \left(x^2, y^2\right) \in Z(t)\right)$ (inputs/outputs proportionality condition and the additivity of production processes).

(G2) $\forall (x, y) \in Z(t)\ (x = 0 \Rightarrow y = 0)$ ("no cornucopia" condition).

(G3) $\forall (x, y) \in Z(t)$, $\forall x' \geqq x$, $\forall\ 0 \leqq y' \leqq y\ ((x', y') \in Z(t))$ (possibility of wasting the inputs/outputs).

(G4) Production spaces $Z(t)$ are closed subsets of R_+^{2n}.

Gale production space is a convex closed cone in R_+^{2n}. If $(x, y) \in Z(t)$ and $(x, y) \neq 0$, then under **(G2)** we have $x \neq 0$. We consider only nonzero (nontrivial) production processes. Let us consider a process $(x, y) \in Z(t) \setminus \{0\}$. The number

$$\alpha(x, y) = \max \{\alpha\colon \alpha x \leqq y\}$$

is called the technological efficiency rate of the process (x, y) in the period t. The function $\alpha(\cdot)$ is defined and positively homogenous of degree 0 on $R_+^{2n} \setminus \{0\}$ and

$$\alpha(x, y) = \min_i \frac{y_i}{x_i}.$$

Theorem 1. *If the production space $Z(t)$ is of Gale type (satisfies the conditions **(G1)**–**(G4)**), then:*

$$\exists\ (\overline{x}(t), \overline{y}(t)) = (\overline{x}, \overline{y}) \in Z(t) \setminus \{0\} \left(\alpha\left(\overline{x}, \overline{y}\right) = \max_{(x,y) \in Z(t) \setminus \{0\}} \alpha(x, y) = \alpha_{M,t} \geq 0\right).$$

Proof. The function $\alpha(\cdot)$ is positive homogenous of degree 0 and nonnegative, hence the solution to maximization problem:

$$\max_{(x,y)\in Z(t)\setminus\{0\}} \alpha(x,y)$$

exists if and only if there exists a solution of the problem:

$$\max_{(x,y)\in\Omega(t)} \alpha(x,y), \tag{1}$$

where

$$\Omega(t) = \{(x,y)\in Z(t)\colon \|x\| = 1\}$$

(if $a\in R^n_+$, then $\|a\| = \sum_{i=1}^n a_i$). The set $\Omega(t)$ is compact (bounded and closed in R^{2n}). Indeed, if we consider a sequence of processes $\left(x^i,y^i\right)\in\Omega(t)\subset Z(t)$, $i=1,2,\ldots,$ and assume that $\left\|x^i,y^i\right\| \xrightarrow{i} +\infty$, hence $\left\|y^i\right\| \xrightarrow{i} +\infty$, because $\forall i\left(\left\|x^i\right\| = 1\right)$. Since $\left(x^i,y^i\right)\in Z(t)$, then under the condition **(G1)**, holds from the element where the vector η^i is defined ($\left\|y^i\right\| > 0$), we have $\left(\xi^i,\eta^i\right)\in Z(t)$, where $\xi^i = (1/\|y^i\|)x^i = x^i/\|y^i\|$, $\eta^i = (1/\|y^i\|)y^i = y^i/\|y^i\|$, $i=1,2,\ldots$ Because $\xi^i \xrightarrow{i} 0$, $\left\|\eta^i\right\| = 1$, therefore:

$$\exists\left\{\xi^{i_j},\eta^{i_j}\right\}_{j=1}^{\infty}\left(\xi^{i_j}\xrightarrow{j}0,\ \eta^{i_j}\xrightarrow{j}\ \overline{\eta}\neq 0\right)$$

and $(0,\overline{\eta})\in Z(t)$ (since the Gale production space is closed), which is in contradiction with the condition **(G2)**. We have proved that the set $\Omega(t)$ is bounded. Let $\left(x^i,y^i\right)\in\Omega(t)\subset Z(t)$, $i=1,2,\ldots,$ $\left(x^i,y^i\right)\xrightarrow{i}(\overline{x},\overline{y})$. It follows that $\|\overline{x}\| = 1$ and $(\overline{x},\overline{y})\in Z$, so $(\overline{x},\overline{y})\in\Omega$. The set $\Omega(t)$ is therefore closed, hence compact. If the function $\alpha(\cdot)$ were continuous on $\Omega(t)$, then to finish/complete the proof it would be enough to use the Weierstrass theorem about the existence of a maximum of a continuous function on the compact set. The function $\alpha(\cdot)$ is admittedly continuous on $\operatorname{int}\Omega(t)$, but unfortunately it can be discontinuous on the boundary of that set. To avoid this problem we use the property that the task (1) is equivalent with:

$$\max_{\alpha\in\alpha(\Omega(t))} \alpha, \tag{2}$$

where $\alpha\left(\Omega(t)\right) = \{\alpha\colon \exists(x,y)\in\Omega(t)\left(\alpha = \alpha(x,y)\right)\}\subset R^1_+$. The equivalence means that the process $(\overline{x},\overline{y})$ is a solution to (1) if and only if the number $\overline{\alpha} = \alpha\left(\overline{x},\overline{y}\right)$ is a solution to the task (2). We present a proof that the set $\alpha\left(\Omega(t)\right)$ is compact (bounded and closed in R^1).

(*Boundedness*) Assume that $\exists\left\{\alpha_i\right\}_{i=1}^{\infty}\left(\alpha_i\in\alpha\left(\Omega(t)\right)\ \&\ \alpha_i\xrightarrow{i}+\infty\right)$. Then:

$$\exists\left\{x^i,y^i\right\}_{i=1}^{\infty}\left(y^i = \alpha_i x^i, \left(x^i,y^i\right)\in\Omega(t)\subset Z(t), \left\|x^i\right\| = 1, i=1,2,\ldots, \left\|y^i\right\|\xrightarrow{i}+\infty\right)$$

and following the proof of compactness of the set $\Omega(t)$ we reach the conclusion that to the production space $Z(t)$ belongs a process $(0, \overline{\eta})$ with the vector $\overline{\eta} \neq 0$, which contradicts the condition **(G2)**.
(*Closed set*) Let $\alpha_i \in \alpha\left(\Omega(t)\right)$, $i = 1, 2, \ldots,$ and $\alpha_i \xrightarrow{i} \overline{\alpha}$. Then:

$$\exists \left\{x^i\right\}_{i=1}^{\infty} \left(\left\|x^i\right\| = 1 \ \& \ \left(x^i, \alpha_i x^i\right) \in \Omega(t)\right).$$

Since $\forall i \left(\left\|x^i\right\| = 1\right)$, then:

$$\exists \left\{x^{i_j}\right\}_{j=1}^{\infty} \left(\left\|x^{i_j}\right\| = 1,\ x^{i_j} \xrightarrow{j} \overline{x}, \quad \|\overline{x}\| = 1\right),$$

i.e. $\left(x^{i_j}, \alpha_{i_j} x^{i_j}\right) \xrightarrow{j} (\overline{x}, \overline{\alpha x}) \in \Omega(t)$, since the set $\Omega(t)$ is compact. That proves the property $\overline{\alpha} \in \alpha(\Omega(t))$.
The set $\alpha\left(\Omega(t)\right)$ is a compact subset of R^1_+, hence it contains the supremum of a sequence:

$$\exists \alpha_{M,t} \in \alpha\left(\Omega(t)\right), \forall \alpha \in \alpha\left(\Omega(t)\right)\left(\alpha_{M,t} \geq \alpha \geq 0\right),$$

and:

$$\exists \left(\overline{x}(t), \overline{y}(t)\right) = (\overline{x}, \overline{y}) \in \Omega(t) \left(\alpha_{M,t} = \alpha\left(\overline{x}, \overline{y}\right) = \max_{(x,y) \in \Omega(t)} \alpha(x, y) \geq 0\right).$$

□

Another version of the proof was presented in Takayama (1985, Theorem 6.A.1). The process $(\overline{x}(t), \overline{y}(t))$ is called the optimal production process in the Gale economy in period t. All such processes are defined up to multiplication by a positive constant (determine up to structure). The number $\alpha_{M,t}$ is called the optimal technological efficiency rate of the economy in the period t. We are interested in economies with positive technological efficiency.

3 Dynamics. Feasible growth processes. Limit production space

We assume the following:

i) production technology in the time period $t+1$ (represented by the production space $Z(t+1)$) depends upon the production technology in the previous period and investment inputs $i(t) = (i_1(t), \ldots, i_n(t)) \geqq 0$ (which affect the outcomes in the next year, thus, for the sake of simplification, we assume an annual investment cycle in the economy),

ii) the source of investment $i(t)$ is the production generated in the economy in the period t :

$$0 \leqq i(t) \leqq y(t). \tag{3}$$

We denote by $\sigma\left(R_+^{2n}\right)$ a family of the Gale production spaces (closed convex cones in R_+^{2n} which satisfy the conditions **(G1)**–**(G4)**). The technology dynamics is described by the following recurrence equation:

$$Z(t+1) = F_{t+1}\left(Z(t), i(t)\right), \quad t = 0, 1, \ldots, \tag{4}$$

in which the reproduction function $F_t : \sigma\left(R_+^{2n}\right) \times R_+^n \to \sigma\left(R_+^{2n}\right)$ has the following properties:

(F1) $\forall t \; \forall Z \in \sigma\left(R_+^{2n}\right) \; \left(F_t(Z, 0) = Z\right),$

(F2) $\forall t \; \forall Z \in \sigma\left(R_+^{2n}\right) \; \forall i^1 \geqq i^2 \geqq 0 \left(F_t\left(Z, i^1\right) \supseteq F_t\left(Z, i^2\right)\right),$

(F3) $\forall t \; \forall Z^1, Z^2 \in \sigma\left(R_+^{2n}\right) \; \forall i \geqq 0 \left(Z^1 \supseteq Z^2 \Rightarrow F_t\left(Z^1, i\right) \supseteq F_t\left(Z^2, i\right)\right).$

A production space $Z(0)$ is given:

$$Z(0) = Z^0 \subset R_+^{2n}. \tag{5}$$

Equation (4) informs us that the dynamics of production space (or production technology) is determined by investments. In particular, the lack of investments implies that the technology of production does not change (i.e. a vector $i(t)$ is identified with the investments netto which increase the production asset). The more investments is made in the period t, the larger are the production capabilities of the economy in the next period.
The economy is closed in the sense that the only source of inputs $x(t+1)$ (incurred in the economy in period $t+1$) may be the production $y(t)$ (generated in the previous period) reduced by the investments $i(t)$:

$$x(t+1) \leqq y(t) - i(t), \quad t = 0, 1, \ldots,$$

which under **(G3)** leads to the condition:

$$\left(y(t) - i(t), y(t+1)\right) \in Z(t+1), \quad t = 0, 1, \ldots \tag{6}$$

We fix an initial production y^0 at $t = 0$:

$$y(0) = y^0 \geq 0. \tag{7}$$

If there are production spaces $Z(t), t = 0, 1, \ldots,$ meeting the conditions **(G1)**–**(G4)**, **(F1)**–**(F3)** and subject to the growth rules (3)–(7), then we say that a Gale economic dynamic model (growth model) with investments is given. We say that a triple which consists of three sequences:

i) production vectors $\{y(t)\}_{t=0}^{\infty}$,

ii) investment vectors $\{i(t)\}_{t=0}^{\infty}$,

iii) and production spaces $\{Z(t)\}_{t=0}^{\infty}$

satisfying conditions (3)–(7) determines a (Z^0, y^0, ∞)-feasible growth process in the Gale economy with investments. The sequence $\{y(t)\}_{t=0}^{\infty}$ is a (y^0, ∞)-feasible production trajectory, the sequence $\{i(t)\}_{t=0}^{\infty}$ - a feasible investments trajectory (corresponding to the (y^0, ∞)-feasible production trajectory). The sequence $\{Z(t)\}_{t=0}^{\infty}$ forms a (Z^0, ∞)-feasible sequence of production spaces. In each (Z^0, y^0, ∞)-feasible growth process production spaces satisfy a condition: $Z(t+1) \supseteq Z(t)$. Under such conditions we have

$$\forall t \, (\alpha_{M,t+1} \geq \alpha_{M,t})$$

and to guarantee a positive optimal technological efficiency of the economy in every time period t it is enough to reach the condition:

(G5) $\alpha_{M,0} > 0$.

The rules we have imposed on the growth do not exclude the following unrealistic situation where $\lim_t \alpha_{M,t} = +\infty$. Meanwhile, if $\lim_t \alpha_{M,t} = +\infty$, then it is easy to show on the simplest model of single-good economy with production spaces $Z(t) \subseteq Z(t+1) \subset R_+^2$, $t = 0, 1, \ldots$, that in the limit the condition **(G2)** is not satisfied, i.e. there exist processes $(0, y)$, for which it is possible to produce an (arbitrarily) large production $y > 0$ under zero expenditures. To exclude such a behavior we impose the following conditions:

(F4) **(i)** There exists a convex closed set $Z \subset R_+^{2n}$ which contains all sets (cones) $Z(t)$ belonging to any (Z^0, ∞)-sequence of production spaces in any (Z^0, y^0, ∞)-feasible growth process.

(ii) Set Z is such the smallest set satisfying the condition **(i)** that if $(x, y) \in Z$ and $x = 0$, then $y = 0$.

Theorem 2. *If the conditions **(F1)**–**(F4)** are satisfied, the set Z is a space of Gale type (satisfies conditions **(G1)**–**(G4)**).*

Proof. We divide the proof into several steps.
Step 1. We denote by $\widetilde{Z}$ the set-theoretic union of production spaces in all (Z^0, y^0, ∞)-feasible growth processes, $\widetilde{Z} \subseteq Z$. If $z \in \widetilde{Z}$, then there exists a (Z^0, ∞)-feasible sequence of production spaces $\{Z(t)\}_{t=0}^{\infty}$ and a time period τ, for which $z \in Z(\tau) \subseteq \widetilde{Z}$. The space $Z(\tau)$ is a convex cone, hence $\lambda z \in Z(\tau) \subseteq \widetilde{Z}$, where λ is any nonnegative number. Thus:

$$\forall z \in \widetilde{Z} \, \forall \lambda \geq 0 \left(\lambda z \in \widetilde{Z} \right)$$

(a set with such properties is called a cone, not necessarily convex).
Step 2. Let $\operatorname{conv}\widetilde{Z}$ be the smallest convex set which contains $\widetilde{Z}$ (so-called convex hull of $\widetilde{Z}$). We prove that $\operatorname{conv}\widetilde{Z}$ is a convex cone which satisfies the conditions **(G1)–(G3)**.
(a) Let $z \in \operatorname{conv}\widetilde{Z}$. Then

$$\exists z^1, z^2 \in \widetilde{Z}\, \exists \alpha, \beta \geq 0, \alpha + \beta = 1 \left(z = \alpha z^1 + \beta z^2\right).$$

We fix any $\lambda \geq 0$. Since $z^1, z^2 \in \widetilde{Z}$, then (according to Step 1) $\lambda z^1, \lambda z^2 \in \widetilde{Z}$ and we obtain from the definition of $\operatorname{conv}\widetilde{Z}$ that:

$$\alpha\lambda z^1 + \beta\lambda z^2 = \lambda\left(\alpha z^1 + \beta z^2\right) = \lambda z \in \operatorname{conv}\widetilde{Z},$$

hence:

$$\forall z \in \operatorname{conv}\widetilde{Z}\, \forall \lambda \geq 0 \left(\lambda z \in \operatorname{conv}\widetilde{Z}\right). \tag{8}$$

Let $z^1, z^2 \in \operatorname{conv}\widetilde{Z}$ be two arbitrary vectors. Let $z = z^1 + z^2$. If $z^1 = 0$, then $z = z^2 \in \operatorname{conv}\widetilde{Z}$. Similarly, when $z^2 = 0$, then $z = z^1 \in \operatorname{conv}\widetilde{Z}$. Let us suppose that $z^1, z^2 \neq 0$ and consider:

$$\overline{z}^1 = \frac{z^1}{\|z^1\|}, \quad \overline{z}^2 = \frac{z^2}{\|z^2\|}, \quad \lambda_1 = \frac{1}{\|z^1\|}, \quad \lambda_2 = \frac{1}{\|z^2\|}.$$

It follows that $\|z^1\| = \|z^2\| = 1$, $\lambda_1, \lambda_2 > 0$, $\overline{z}^1 = \lambda_1 z^1 \in \operatorname{conv}\widetilde{Z}$ and $\overline{z}^2 = \lambda_2 z^2 \in \operatorname{conv}\widetilde{Z}$. Let $\alpha = \|z^1\| / \|z^1 + z^2\|$, $\beta = \|z^2\| / \|z^1 + z^2\|$, then $\alpha, \beta > 0$, $\alpha + \beta = 1$. Since $\operatorname{conv}\widetilde{Z}$ is a convex set, then

$$\begin{aligned}
\overline{z} = \alpha\overline{z}^1 + \beta\overline{z}^2 &= \frac{\|z^1\|\,\overline{z}^1}{\|z^1 + z^2\|} + \frac{\|z^2\|\,\overline{z}^2}{\|z^1 + z^2\|} = \\
&= \frac{\|z^1\|}{\|z^1 + z^2\|} \cdot \frac{z^1}{\|z^1\|} + \frac{\|z^2\|}{\|z^1 + z^2\|} \cdot \frac{z^2}{\|z^2\|} = \\
&= \frac{z^1}{\|z^1 + z^2\|} + \frac{z^2}{\|z^1 + z^2\|} = \\
&= \frac{z}{\|z\|} \in \operatorname{conv}\widetilde{Z}.
\end{aligned}$$

Considering $\lambda = \|z\| > 0$, according to (8) we obtain $z = \lambda\overline{z} \in \operatorname{conv}\widetilde{Z}$. Finally:

$$\forall z^1, z^2 \in \operatorname{conv}\widetilde{Z} \left(z = z^1 + z^2 \in \operatorname{conv}\widetilde{Z}\right). \tag{9}$$

The conditions (8)–(9) are equivalent to **(G1)**. The set $\operatorname{conv}\widetilde{Z}$ is a convex cone.

(b) To prove that the convex cone $\operatorname{conv}\widetilde{Z}$ satisfies condition **(G2)** let us assume, to the contrary, that the vector $z = (0, y) = (0, \ldots, 0, y_1, \ldots, y_n)$ with zero expenditures and production vector y with at least one positive coordinate belongs to this cone. Then, according to the definition of $\operatorname{conv}\widetilde{Z}$, there exist vectors $z^1 = \left(0, y^1\right) \in \widetilde{Z}$, $z^2 = \left(0, y^2\right) \in \widetilde{Z}$ and numbers $\alpha, \beta \geq 0$, such that $\alpha + \beta = 1$ and

$$z = (0, y) = \ \alpha z^1 + \beta z^2 = \left(0, \alpha y^1 + \beta y^2\right) \in \operatorname{conv}\widetilde{Z}.$$

The set $\widetilde{Z}$ is a union of all production spaces in every $\left(Z^0, y^0, \infty\right)$-feasible growth process, so that there exists $\left(Z^0, \infty\right)$-feasible sequences of spaces $\left\{Z^i(t)\right\}_{t=0}^{\infty}$, $i = 1, 2$, and time periods τ_1, τ_2, such that

$$z^1 = \left(0, y^1\right) = z^1\left(\tau_1\right) = \left(0, y^1\left(\tau_1\right)\right) \in Z^1\left(\tau_1\right), \tag{10}$$

$$z^2 = \left(0, y^2\right) = z^2\left(\tau_2\right) = \left(0, y^2\left(\tau_2\right)\right) \in Z^2\left(\tau_2\right). \tag{11}$$

Since $y = y^1 + y^2 \neq 0$ (and the vectors y^1, y^2 are non-negative), the vector $y^1 = y^1\left(\tau_1\right) \neq 0$ or $y^2 = y^2\left(\tau_2\right) \neq 0$ (or both hold). If $y^1\left(\tau_1\right) \neq 0$, then the condition (10) leads to a contradiction with **(G2)**. In a similar fashion, when $y^2\left(\tau_2\right) \neq 0$, the condition **(G2)** contradicts (11). So if $z = (0, y) \in \operatorname{conv}\widetilde{Z}$ then $y = 0$ and **(G2)** holds true.

(c) Let $z = (x, y) \in \operatorname{conv}\widetilde{Z}$. Given that $(x, y) \in \widetilde{Z} \subseteq \operatorname{conv}\widetilde{Z}$, there exists an $\left(Z^0, \infty\right)$-feasible sequence of spaces $\{Z(t)\}_{t=0}^{\infty}$ and time period τ, such that $z = (x, y) \in Z(\tau) \subseteq \widetilde{Z}$. Then according to **(G3)**

$$\forall\, x' \geqq x\, \forall\, 0 \leqq y' \leqq y\ \left(\left(x', y'\right) \in Z(\tau)\right).$$

Hence:

$$(x, y) \in \widetilde{Z} \Longrightarrow \forall x' \geqq x\, \forall 0 \leqq y' \leqq y \left(\left(x', y'\right) \in Z(\tau) \subseteq \widetilde{Z} \subseteq \operatorname{conv}\widetilde{Z}\right). \tag{12}$$

If $z = (x, y) \in \operatorname{conv}\widetilde{Z}$, $z \notin \widetilde{Z}$, then

$$\exists z^1 = \left(x^1, y^1\right) \in \widetilde{Z}\, \exists z^2 = \left(x^2, y^2\right) \in \widetilde{Z}\, \exists\, \alpha, \beta \geq 0, \alpha + \beta = 1$$
$$\left(z = (x, y) = \alpha\left(x^1, y^1\right) + \beta\left(x^2, y^2\right) \in \operatorname{conv}\widetilde{Z}\right) \tag{13}$$

Since $\left(x^1, y^1\right) \in \widetilde{Z}$, $\left(x^2, y^2\right) \in \widetilde{Z}$, the method presented above leads to:

$$\forall x'^1 \geqq x^1\ \forall\ 0 \leqq y'^1 \leqq y^1 \left(\left(x'^1, y'^1\right) \in \widetilde{Z} \subseteq \operatorname{conv}\widetilde{Z}\right),$$
$$\forall x'^2 \geqq x^2\ \forall\ 0 \leqq y'^2 \leqq y^2 \left(\left(x'^2, y'^2\right) \in \widetilde{Z} \subseteq \operatorname{conv}\widetilde{Z}\right).$$

Let $z' = (x', y') = \alpha \left(x'^1, y'^1\right) + \beta \left(x'^2, y'^2\right)$ (with α, β as in (13)). Then $z' = (x', y') \in \operatorname{conv} \widetilde{Z}$ and

$$x' = \alpha x'^1 + \beta x'^2 \geqq \ \alpha x^1 + \beta x^2 = x,$$
$$0 \leqq y' = \alpha y'^1 + \beta y'^2 \leqq \alpha y^1 + \beta y^2 = y.$$

We draw the following conclusion:

$$(x, y) \in \left(\operatorname{conv} \widetilde{Z}\right) \setminus \widetilde{Z} \Longrightarrow \forall x' \geqq x \ \forall \ 0 \leqq y' \leqq y \left((x', y') \in \operatorname{conv} \widetilde{Z}\right). \tag{14}$$

Due to (12), (14) the convex cone $\operatorname{conv} \widetilde{Z}$ satisfies the condition **(G3)**.
Step 3. Finally, let us consider the set Z again. From our assumption this is the smallest convex closed set which contains a union of all production spaces belonging to any $\left(Z^0, y^0, \infty\right)$-feasible growth process, i.e. it is the least closed subset which contains $\operatorname{conv} \widetilde{Z}$; such a set is the topological closure of $\operatorname{conv} \widetilde{Z}$, $Z = \operatorname{cl}\left(\operatorname{conv} \widetilde{Z}\right)$. We prove that the set Z satisfies conditions **(G1)**, **(G3)**. We fix two vectors $z^1 = \left(x^1, y^1\right) \in Z$, $z^2 = \left(x^2, y^2\right) \in Z$, numbers λ_1, $\lambda_2 \geq 0$ and a vector $z = (x, y) = \lambda_1 z^1 + \lambda_2 z^2 = \lambda_1 \left(x^1, y^1\right) + \lambda_2 \left(x^2, y^2\right)$. The vectors z^1, z^2 are limit points in $\operatorname{conv} \widetilde{Z}$, so there exist sequences $z^{1i} = \left(x^{1i}, y^{1i}\right), z^{2i} = \left(x^{2i}, y^{2i}\right) \in \operatorname{conv} \widetilde{Z}$, $i = 1, 2, \ldots$, convergent (respectively) to z^1, z^2. The set $\operatorname{conv} \widetilde{Z}$ is a convex cone, hence $\forall i \left(z^i = \lambda_1 z^{1i} + \lambda_2 z^2 \in \operatorname{conv} \widetilde{Z}\right)$, and then:

$$\lim_i z^i = \lim_i \left(\lambda_1 z^{1i} + \lambda_2 z^{2i}\right) = \lambda_1 z^1 + \lambda_2 z^2 = \lambda_1 \left(x^1, y^1\right) + \lambda_2 \left(x^2, y^2\right) = z \in Z.$$

The set Z satisfies the condition **(G1)** (is a convex cone). Similarly, if $z = (x, y) \in Z$, then there exists a sequence $z^i = \left(x^i, y^i\right) \in \operatorname{conv} \widetilde{Z}$, $i = 1, 2, \ldots$, convergent to z. Let:

$$x' \geqq x, \quad 0 \leqq y' \leqq y, \quad \widetilde{x}^i = x^i + x' - x,$$

$$\widetilde{y}^i = \max\left\{0, y^i + y' - y\right\} = \left(\max\left\{0, y_1^i + y_1{}' - y_1\right\}, \ldots, \max\left\{0, y_n^i + y_n{}' - y_n\right\}\right).$$

Thus $\widetilde{x}^i \geqq x^i$,$0 \leqq \widetilde{y}^i \leqq y^i$,$\widetilde{z}^i = \left(\widetilde{x}^i, \widetilde{y}^i\right) \in \operatorname{conv} \widetilde{Z} \subseteq Z$ and

$$\lim_i \widetilde{z}^i = \lim_i \left(\widetilde{x}^i, \widetilde{y}^i\right) = (x', y') = z' \in Z$$

(from its definition the set Z is closed), hence

$$\forall (x, y) \in Z \, \forall x' \geqq x \, \forall \ 0 \leqq y' \leqq y \ \ ((x', y') \in Z).$$

The set Z satisfies the condition **(G3)**.
The conditions **(G2)**, **(G4)** hold under our assumptions. □

The set Z defined above is called the limit production space. A condition $(x, y) \in Z$ means that one can produce y from the inputs x. If $(x, y) \in Z \setminus \{0\}$, a number

$$\alpha(x, y) = \max \{\alpha\colon\ \alpha x \leqq y\}$$

is called the technological efficiency rate of the limit process (x, y). Theorem 1 remains valid (upon replacing the space $Z(t)$ with its limit counterpart Z):

$$\exists\, (\overline{x}, \overline{y}) \in Z \setminus \{0\} \left(\alpha\, (\overline{x}, \overline{y}) = \max_{(x,y) \in Z \setminus \{0\}} \alpha(x, y) = \alpha_M \geq 0 \right).$$

A number α_M is called the optimal technological efficiency rate in the Gale economy with limit production space. The inclusion $Z \supseteq Z(t+1) \supseteq Z(t)$, $t = 0, 1, \ldots,$ implies under **(G5)** the following:

$$\forall t\, (\alpha_M \geq \alpha_{M,t+1} \geq \alpha_{M,t} \geq \alpha_{M,0} > 0)\, .$$

A process $(\overline{x}, \overline{y})$ is the optimal growth process in the Gale economy with limit production space.

4 Multilane production turnpike. Stationary production trajectories

We introduce the following notation:

$$Z_{\text{opt}} = \{(\overline{x}, \overline{y}) \in Z \setminus \{0\colon \alpha\, (\overline{x}, \overline{y}) = \alpha_M\}\}\, . \tag{15}$$

The set Z_{opt} is a union of all optimal growth processes in the Gale economy with limit production space Z. If the conditions **(G1)**–**(G5)**, **(F1)**–**(F4)** are satisfied, the set Z_{opt} is a convex cone in R^{2n}_{+} without 0; see Panek (2016), Theorem 1. Moreover, if $(\overline{x}, \overline{y}) \in Z_{\text{opt}}$, then under **(G1)**, **(G3)** it follows that $(\overline{x}, \alpha_M \overline{x}) \in Z_{\text{opt}}$ and $(\overline{y}, \alpha_M \overline{y}) \in Z_{\text{opt}}$. A vector $\overline{s} = \overline{y} / \|\overline{y}\|$ characterizes the production structure in the optimal process $(\overline{x}, \overline{y}) \in Z_{\text{opt}}$ in Gale economy with limit technology. We denote by S the set of vectors of the production structure in all the optimal processes in the Gale economy with limit technology:

$$S = \left\{ s\colon \exists (x, y) \in Z_{\text{opt}} \left(s = \frac{y}{\|y\|} \right) \right\}.$$

Equivalently $S = \{s\colon \exists (x, y) \in Z_{\text{opt}}\, (s = x / \|x\|)\}$. Our set S exists under the same condition which gave rise to the set Z_{opt} and exists under the same kind of conditions imposed on Z_{opt}. This set is convex and compact; Panek (2016, Theorem 2(i)).
The half line:

$$N^s = \{\lambda s\colon \lambda > 0\}\, , \quad s \in S$$

is called a von Neumanna ray (single production turnpike) in the Gale economy with limit technology. A set

$$\mathbb{N} = \bigcup_{s \in S} N^s$$

is called a multilane turnpike in Gale economy with limit technology. The multilane turnpike $\mathbb{N}$ is a convex cone without 0.

Lemma 3. *Let us assume that the non-stationary economy with investments meets conditions **(G1)**–**(G5)**, **(F1)**–**(F4)**. Then if in a certain limit production process $(x, y) \in Z \setminus \{0\}$, the structure of inputs $x/\|x\|$ or outputs $y/\|y\|$ differs from the turnpike's structure, then its technological efficiency is lower than optimal:*

$$\forall (x, y) \in Z \setminus \{0\} \left(\frac{x}{\|x\|} \notin S \vee \frac{y}{\|y\|} \notin S \Rightarrow \alpha(x, y) < \alpha_M \right).$$

Proof. (See also Panek (2018, Lemma 1)). Let $x \in \mathbb{N}$. Then the pair $(x, y) \in Z \setminus \{0\}$ with the vector $y = \alpha_M x$ represents an optimal production process in Gale economy with a limit technology, or $(x, y) \in Z_{\text{opt}}$. Therefore

$$\forall x \in \mathbb{N} \, \exists y \in R^n_+ \left((x, y) \in Z \setminus \{0\} \wedge \alpha(x, y) = \alpha_M > 0\right).$$

However, if $x \notin \mathbb{N}$, then the technological efficiency of the process $(x, y) \in Z \setminus \{0\}$ is lower than optimal. Indeed, assuming that $(x, y) \in Z \setminus \{0\}$ and $\alpha(x, y) = \alpha_M$, we get: $(x, y) \in Z_{\text{opt}}$, that is $(x, \alpha_M x) \in Z_{\text{opt}}$. Then $x \in \mathbb{N}$, contrary to our assumption.
Similarly, if $y \in \mathbb{N}$, then for the input vector $x = \alpha_M^{-1} y$ we obtain an admissible process $(x, y) \in Z \setminus \{0\}$ with technological efficiency $\alpha(x, y) = \alpha_M$, so $(x, y) \in Z_{\text{opt}}$. Therefore

$$\forall y \in \mathbb{N} \, \exists x \in R^n_+ \left((x, y) \in Z \setminus \{0\} \wedge \alpha(x, y) = \alpha_M > 0\right).$$

Let us suppose now that $(x, y) \in Z \setminus \{0\}$, $y \notin \mathbb{N}$ and $\alpha(x, y) = \alpha_M$. Then $(x, y) \in Z_{\text{opt}}$ and $(y, \alpha_M y) \in Z_{\text{opt}}$, so $y \in \mathbb{N}$ in contradiction to our assumption. To sum up:

$$\forall \, (x, y) \in Z \setminus \{0\} \left(x \notin \mathbb{N} \vee y \notin \mathbb{N} \; \Rightarrow \alpha(x, y) < \alpha_M\right). \tag{16}$$

The thesis of the theorem follows from the fact that condition $x \notin \mathbb{N}$ is equivalent to $(x/\|x\|) \notin S$ and (similarly) the condition $y \notin \mathbb{N}$ is equivalent to $(y/\|y\|) \notin S$. □

According to (16) the maximal technological efficiency is achieved by the economy only on the multilane turnpike $\mathbb{N}$. Consider a limit production space Z. Substituting in (4):

$$Z(0) = Z \quad \text{and} \quad i(t) = 0, \quad t = 0, 1, \ldots,$$

we obtain $Z(t) \equiv Z = \mathit{const}$. Let $\overline{y} \in \mathbb{N}$, then:

$$(\overline{y}, \alpha_M \overline{y}) \in Z_{\text{opt}} \subset Z, \left(\alpha_M \overline{y}, \alpha_M^2 \overline{y}\right) \in Z_{\text{opt}} \subset Z, \ldots,$$

and the sequence $\{\overline{y}(t)\}_{t=0}^{\infty}$, in which

$$\overline{y}(t) = \alpha_M^t \overline{y}, \quad t = 0, 1, \ldots, \tag{17}$$

defines a $(\overline{y}, \infty)$-feasible production trajectory in Gale economy with limit technology and initial production vector $y(0) = \overline{y} \in \mathbb{N}$. In such economy a $(Z, \overline{y}, \infty)$-feasible growth process consists of a triple $\{\overline{y}(t)\}_{t=0}^{\infty}$, $\{i(t)\}_{t=0}^{\infty}$, $\{Z(t)\}_{t=0}^{\infty}$ with production trajectory (17), investments $i(t) = 0$ and production spaces $Z(t) = Z$, $t = 0, 1, \ldots$ On the trajectory (17) the economy reaches its highest growth rate $\alpha_M > 0$ and

$$\forall t \left(\frac{\overline{y}(t)}{\|\overline{y}(t)\|} = \frac{\overline{y}}{\|\overline{y}\|} = \overline{s} \in S \right).$$

For that reason, we call such a trajectory a stationary production trajectory with maximal growth rate (shortly: optimal stationary production trajectory). Any positive multiple or sum of such two trajectories is again an optimal stationary production trajectory (always with zero investments and constant production spaces $Z(t) = Z$, $t = 0, 1, \ldots$) These trajectories all lie on the multilane turnpike $\mathbb{N}$.

5 Economic efficiency of production and von Neumann equilibrium

Let us consider any production process $(x, y) \in Z \setminus \{0\}$. Let $p = (p_1, \ldots, p_n) \geq 0$ denote a vector of commodity prices. A number:

$$\beta(x, y, p) = \frac{\sum_{i=1}^{n} p_i y_i}{\sum_{i=1}^{n} p_i x_i} = \frac{\langle p, y \rangle}{\langle p, x \rangle}$$

($\langle p, x \rangle \neq 0$) is called the rate of the economic efficiency of the process (x, y) (with prices p). A triple $\{\alpha_M, (\overline{x}, \overline{y}), \overline{p}\}$ which satisfies the following conditions:

$$\alpha_M \overline{x} \leqq \overline{y}, \tag{18}$$

$$\forall (x, y) \in Z \left(\langle \overline{p}, y \rangle \leq \alpha_M \langle \overline{p}, x \rangle \right), \tag{19}$$

$$\langle \overline{p}, \overline{y} \rangle > 0 \tag{20}$$

is called an optimal von Neumann equilibrium state in Gale economy with limit technology (shortly: von Neumann equilibrium state). The vector $\overline{p}$ is called a von Neumann (equilibrium) price vector. It follows from (18)–(20) that:

$$\alpha_M \langle \overline{p}, \overline{x} \rangle = \langle \overline{p}, \overline{y} \rangle > 0$$

and

$$\beta(\overline{x}, \overline{y}, \overline{p}) = \frac{\langle \overline{p}, \overline{y} \rangle}{\langle \overline{p}, \overline{x} \rangle} = \max_{(x,y) \in Z \setminus \{0\}} \beta(\overline{x}, \overline{y}, \overline{p}) = \alpha(\overline{x}, \overline{y}) = \alpha_M > 0.$$

The von Neumann equilibrium is such a state of the economy (represented by production and prices), in which the economic efficiency equals the technological efficiency (at its highest possible level). The equilibrium prices $\overline{p}$ and the production processes $(\overline{x}, \overline{y})$ in von Neumann equilibrium are defined up to structure (up to multiplication by a positive constant).

Theorem 4. *(i) Suppose that the conditions **(G1)–(G5)**, **(F1)–(F4)** hold, then there exist vector of prices $\overline{p}$ which satisfy* (19).

(ii) In addition, if:

(FG1) $\forall (x, y) \in Z \setminus \{0\} \left(\alpha(x, y) < \alpha_M \Rightarrow \beta(x, y, \overline{p}) = \frac{\langle \overline{p}, y \rangle}{\langle \overline{p}, x \rangle} < \alpha_M \right)$

holds (assuming that $\langle \overline{p}, x \rangle \neq 0$), then the value of production vector $\overline{y}$ expressed in terms of prices $\overline{p}$ in any optimal production process $(\overline{x}, \overline{y}) \in Z_{opt}$ is positive, i.e. condition (20) *is satisfied.*

Proof. (See Panek (2019a, Theorem 1)). *(i)* Since, in particular, the zero production process belongs to the limit production space Z, so according to **(G3)** processes $\left(e^i, 0\right) \in R_+^{2n}$ also belong to this space ($e^i = (0, \ldots, 1, \ldots, 0) \in R^n$ in an n-dimensional vector with 1 on ith place. Set

$$C = \{c \in R^n |\ c = \alpha_{M,t} x - y,\ (x, y) \in Z\}$$

is a convex cone in R^n (as a linear image of the Z cone) containing no negative vectors. Indeed, suppose that:

$$\exists\, (x', y') \in Z\, (c' = \alpha_M x' - y' < 0)\,.$$

Then:

$$\exists \varepsilon' > 0 \left(\alpha_M = \max_{(x,y) \in Z \setminus \{0\}} \alpha(x, y) \geq \alpha\, (x', y') \geq \alpha_M + \varepsilon' \right),$$

which contradicts the definition of the optimal α_M indicator. Since the processes $\left(e^i, 0\right)$, $(i = 1,\ 2,\ \ldots, n)$ belong to Z, therefore:

$$c^i = \alpha_M e^i - 0 = (0, \ldots, \alpha_M, \ldots, 0) \in C, \quad i = 1, 2, \ldots, n$$

(in the vector c^i, the number $\alpha_M > 0$ is on the ith position). From the hyperplane separation theorem we conclude that:

$$\exists\, \overline{p} \neq 0\, \forall c \in C\, (\langle \overline{p}, c \rangle \geq 0)\,, \tag{21}$$

in particular:

$$\left\langle \overline{p}, c^i \right\rangle = \alpha_M \overline{p}_i \geq 0, i = 1, 2, \ldots, n,$$

which means $\overline{p} \geq 0$. The condition (21) is equivalent to (19).

(ii) We will show that if the condition **(FG1)** holds, then the condition (20) also holds. For this purpose let us take any process $(\overline{x}, \overline{y}) \in Z_{\text{opt}}$, where the set Z_{opt} is as in (15). Then from (18) and (19) we get $\langle \overline{p}, \overline{y} \rangle = \alpha_M \langle \overline{p}, \overline{x} \rangle \geq 0$ and according to the definition of the optimal process:

$$\exists k \left(\alpha\left(\overline{x}, \overline{y}\right) = \min_i \frac{\overline{y}_i}{\overline{x}_i} = \frac{\overline{y}_k}{\overline{x}_k} = \alpha_M > 0 \right).$$

Let $\widetilde{x} = \overline{x} + e^k$, where $e^k = (0, \ldots, 1, \ldots, 0)$ is an n-dimensional vector with the kth coordinate equal to 1. In Gale economy with the limit technology, the pair $(\widetilde{x}, \overline{y})$ is an admissible process (according to **(G3)**), but not optimal, because $\alpha\left(\widetilde{x}, \overline{y}\right) < \alpha_M$. Then, regarding **(FG1)**, we get $\beta\left(\widetilde{x}, \overline{y}, \overline{p}\right) < \alpha_M$, or equivalently:

$$\langle \overline{p}, \overline{y} \rangle - \alpha_M \langle \overline{p}, \widetilde{x} \rangle < 0.$$

Suppose that $\langle \overline{p}, \overline{y} \rangle = 0$, then $\overline{p}_k = 0$, that is $\langle \overline{p}, \widetilde{x} \rangle = \langle \overline{p}, \overline{x} \rangle$. However if $\langle \overline{p}, \overline{y} \rangle = 0$, then from (18), (19) it follows that $\langle \overline{p}, \overline{x} \rangle = 0$, so also $\langle \overline{p}, \widetilde{x} \rangle = 0$. Then:

$$\langle \overline{p}, \overline{y} \rangle - \alpha_M \langle \overline{p}, \widetilde{x} \rangle = 0.$$

This contradiction closes the proof. □

Condition **(FG1)** means that in the Gale economy with limit technology does not achieve the highest economic efficiency a process $(x, y) \in Z \setminus \{0\}$ that does not have the highest technological efficiency. Let us note that the condition (18) of the definition of the optimal von Neumann equilibrium state is satisfied by every process $(\overline{x}, \overline{y}) \in Z_{\text{opt}}$. Thus, when the condition **(FG1)** is met, then the optimal equilibrium state is created by every triplet $\{\alpha_M, (\overline{x}, \overline{y}), \overline{p}\}$ with any process $(\overline{x}, \overline{y}) \in Z_{\text{opt}}$.
Let $x \geq 0$ be any commodity vector. We define a distance between x and the multilane turnpike $\mathbb{N}$:

$$d\left(x, \mathbb{N}\right) = \inf_{x' \in \mathbb{N}} \left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\|. \tag{22}$$

A key role in the proof of the turnpike theorem (Theorem 7) is played by the following lemma which is an adjusted to our purposes version of the lemma of Radner (1961).

Lemma 5. *Assume that **(G1)**–**(G5)**, **(F1)**–**(F4)** and **(FG1)**, then*

$$\forall \varepsilon > 0\ \exists \delta_\varepsilon \in (0, \alpha_M)\, \forall (x, y) \in Z \setminus \{0\}$$
$$\left(d\left(x, \mathbb{N}\right) \geq \varepsilon \Rightarrow \beta\left(x, y, \overline{p}\right) = \frac{\langle \overline{p}, y \rangle}{\langle \overline{p}, x \rangle} \leq \alpha_M - \delta_\varepsilon \right). \tag{23}$$

Proof. (See Panek (2017, Theorem 5)). Let us take any number $\varepsilon > 0$. If the process $(x, y) \in Z \setminus \{0\}$ satisfies the lemma conditions, then also any process $\lambda(x, y)$ with any number $\lambda > 0$ does (and vice versa). Therefore, for the proof it is enough to limit oneself to the admissible production processes (x, y) from the set

$$V(\varepsilon) = \{(x, y) \in Z | \, \|x\| = 1 \wedge d\,(x, \mathbb{N}) \geq \varepsilon\}\,.$$

This set is compact (limited and closed on R^{2n}). From (16) it follows that:

$$\forall (x, y) \in V(\varepsilon) \;\; (\alpha(x, y) < \alpha_M)\,,$$

so (regarding **(FG1)**)

$$\forall (x, y) \in V(\varepsilon) \;\; \left(\beta\,(x, y, \overline{p}) = \frac{\langle \overline{p}, y\rangle}{\langle \overline{p}, x\rangle} < \alpha_M\right).$$

The function $\beta\,(\cdot, \cdot, \overline{p})$ is continuous on $V(\varepsilon)$ (since everywhere on $V(\varepsilon)$ we have $\langle \overline{p}, x\rangle > 0$), so there exists a solution to the problem

$$\max_{(x,y)\in V(\varepsilon)} \beta\,(x, y, \overline{p}) = \beta_\varepsilon$$

and $\beta_\varepsilon < \alpha_M$. Then:

$$\exists \delta_\varepsilon > 0 \; \forall (x, y) \in V(\varepsilon)\, (\beta\,(x, y, \overline{p}) \leq \alpha_M - \delta_\varepsilon)$$

or equivalently:

$$\exists \delta_\varepsilon > 0 \; \forall (x, y) \in V(\varepsilon)\, (\beta\,(\langle \overline{p}, y\rangle \leq (\alpha_M - \delta_\varepsilon)\, \langle \overline{p}, x\rangle))\,.$$

□

Lemma 5 plays a key role in the proof of the "weak" multilane turnpike theorem, which will be presented later (Theorem 7).

6 Optimal growth processes. "Weak" turnpike effect

We fix a time period $t_1 < +\infty$ and denote by $T = \{0, 1, \ldots, t_1\}$ a finite horizon in the economy which is of interest to us. A finite production sequence $\{y(t)\}_{t=0}^{t_1}$, investments $\{i(t)\}_{t=0}^{t_1-1}$ and production spaces $\{Z(t)\}_{t=0}^{t_1}$ satisfying (3)–(7) define a (Z^0, y^0, t_1)-feasible growth process in the Gale economy with investments; compare with a similar definition of a (Z^0, y^0, ∞)-feasible growth process in Section 3. If a triple of sequences $\{y(t)\}_{t=0}^{\infty}$, $\{i(t)\}_{t=0}^{\infty}$, $\{Z(t)\}_{t=0}^{\infty}$ forms a (Z^0, y^0, ∞)-feasible growth process (in the unbounded time horizon), then a triple $\{y(t)\}_{t=0}^{t_1}$, $\{i(t)\}_{t=0}^{t_1-1}$, $\{Z(t)\}_{t=0}^{t_1}$ forms a (Z^0, y^0, t_1)-feasible growth process in the finite time horizon T.

A sequence $\{y(t)\}_{t=0}^{t_1}$ is called a $\left(y^0, t_1\right)$-feasible production trajectory, the sequence $\{i(t)\}_{t=0}^{t_1-1}$-feasible investments trajectory (corresponding to $\left(y^0, t_1\right)$-feasible production trajectory). We say about the sequence $\{Z(t)\}_{t=0}^{t_1}$ that it defines a (Z^0, t_1)-feasible sequence of production spaces. Under our assumptions (Z^0, y^0, t_1)-feasible processes exist for $\forall t_1 \leq +\infty$.
We denote by $u(\cdot)$ a utility function in the Gale economy defined on the production vectors in the last time period t_1 of horizon T. Function $u(\cdot)$ satisfies the following conditions:

(U1) Function $u: R_+^n \to R_+^1$ is continuous, positive homogenous of degree 1, concave and increasing.

(U2) $\exists a > 0\, \forall y \in R_+^n\, (u(y) \leq a \langle \overline{p}, y \rangle)$.

Property **(U1)** has the standard form. Property **(U2)** implies that there exists $a > 0$, such that the utility function can be approximated from above by a linear form with the coefficients vector $a\overline{p}$, where $\overline{p}$ is a vector of von Neumann prices. Conditions **(U1)**–**(U3)** are satisfied, among others, by positive homogenous of degree 1 utility functions of CES type.
We concentrate now on the following maximization problem of target growth (maximization of the production utility function in the final period of the horizon $T = \{0, 1, \ldots, t_1\}$:

$$\begin{gathered} \max u\left(y\left(t_1\right)\right) \\ \text{under conditions (3)–(7)} \\ \text{(space } Z^0 \text{ and the vector } y^0 \text{ are fixed).} \end{gathered} \tag{24}$$

A sequence of production vectors which is a solution this problem is denoted by $\{y^*(t)\}_{t=0}^{t_1}$ and called a $\left(y^0, t_1\right)$-optimal production trajectory. In addition, we have a sequence of investments $\{i^*(t)\}_{t=0}^{t_1-1}$ (optimal investments trajectory) and an optimal sequence of production spaces $\{Z^*(t)\}_{t=0}^{t_1}$. A triple $\{y^*(t)\}_{t=0}^{t_1}$, $\{i^*(t)\}_{t=0}^{t_1-1}$, $\{Z^*(t)\}_{t=0}^{t_1}$ satisfying conditions (3)–(7) is called a (Z^0, y^0, t_1)-optimal growth process.
The last condition means that the economy can reach (in at least one process) the multilane turnpike before reaching the end of horizon T:

(FG2) There exists such a $\left(Z^0, y^0, \check{t}+1\right)$-feasible growth process $\{\check{y}(t)\}_{t=0}^{\check{t}+1}$, $\{\check{i}(t)\}_{t=0}^{\check{t}}$, $\left\{\check{Z}(t)\right\}_{t=0}^{\check{t}+1}$, $\check{t} < t_1$, for which $\check{i}\left(\check{t}\right) = 0$ and: $\alpha\left(\check{y}\left(\check{t}\right), \check{y}\left(\check{t}+1\right)\right) = \alpha_M$.

If that condition is satisfied, then $\check{y}\left(\check{t}\right) \in \mathbb{N}$. Without it the turnpike $\mathbb{N}$ would be an example of a highway (in the usual traffic sense), although existing, but without connections to local roads. We will prove that such a property allows the economy not only to reach the turnpike, but to remain on the turnpike (and production increase with the pace α_M) until the end of horizon T.

Lemma 6. *If the conditions **(G1)**–**(G5)**, **(F1)**–**(F4)**, **(FG1)**, **(FG2)** are satisfied, there exists a* $\left(y^0, t_1\right)$*-feasible production trajectory* $\{\widetilde{y}(t)\}_{t=0}^{t_1}$ *of the following*

$$\widetilde{y}(t) = \begin{cases} \check{y}(t), & t = 0, 1, \ldots, \check{t}, \\ \alpha_M^{t-\check{t}} \check{y}\left(\check{t}\right), & t = \check{t}+1, \ldots, t_1, \end{cases} \tag{25}$$

where $\check{y}(t) \in \mathbb{N}$ *for* $t = \check{t}, \check{t}+1, \ldots, t_1$.

Proof. It follows from the definition of a $\left(Z^0, y^0, \check{t}+1\right)$-feasible growth process that:

$$\begin{gathered} \check{y}(0) = y^0, \check{Z}(0) = Z^0, \\ (\check{y}(t) - \check{\imath}(t), \check{y}(t+1)) \in \check{Z}(t+1), \\ \check{Z}(t+1) = F_{t+1}\left(\check{Z}(t), \check{\imath}(t)\right), \\ 0 \leqq \check{\imath}(t) \leqq \check{y}(t), \\ t = 0, 1, \ldots, \check{t}. \end{gathered}$$

Since the condition **(FG2)** is true, we have

$$\check{\imath}\left(\check{t}\right) = 0 \quad \text{and} \quad \alpha_M \check{y}\left(\check{t}\right) \leqq \check{y}\left(\check{t}+1\right).$$

Let $i(t) = 0$ for every $t = \check{t}+1, \ldots, t_1 - 1$. It implies that:

$$\forall\, t \in \left\{\check{t}+1, \ldots, t_1\right\} \left(Z(t) = \check{Z}\left(\check{t}\right) \subseteq Z\right),$$

hence:

$$\begin{gathered} \left(\check{y}\left(\check{t}\right), \alpha_M \check{y}\left(\check{t}\right)\right) \in \check{Z}\left(\check{t}\right) \subseteq Z, \\ \left(\alpha_M \check{y}\left(\check{t}\right), \alpha_M^2 \check{y}\left(\check{t}\right)\right) \in \check{Z}\left(\check{t}\right) \subseteq Z, \\ \cdots \\ \left(\alpha_M^{t_1-\check{t}-1} \check{y}\left(\check{t}\right), \alpha_M^{t_1-\check{t}} \check{y}\left(\check{t}\right)\right) \in \check{Z}\left(\check{t}\right) \subseteq Z, \end{gathered}$$

or equivalently:

$$(\overline{y}(t), \overline{y}(t+1)) \in \check{Z}\left(\check{t}\right) \subseteq Z, \quad t = \check{t}, \check{t}+1, \ldots, t_1 - 1,$$

where

$$\overline{y}(t) = \alpha_M^{t-\check{t}} \check{y}\left(\check{t}\right).$$

The production trajectory (25) with the companion investments trajectory

$$\widetilde{\imath}(t) = \begin{cases} \check{\imath}(t), & t = 0, 1, \ldots, \check{t} - 1, \\ 0, & t = \check{t}, \check{t} + 1, \ldots, t_1 - 1, \end{cases}$$

and corresponding production spaces

$$\widetilde{Z}(t) = \begin{cases} \check{Z}(t), & t = 0, 1, \ldots, \check{t}, \\ \check{Z}\left(\check{t}\right), & t = \check{t} + 1, \ldots, t_1, \end{cases}$$

determines a $\left(Z^0, y^0, t_1\right)$-feasible growth process, in which the economy from the period $\check{t}$ until the end of horizon T stays on the turnpike. $\square$

There are at least three known terms that correspond to different types of turnpike theorems: "weak", "strong" and "very strong" turnpike theorem. Weak turnpike theorems claim that the optimal growth processes in almost all periods of the fixed time horizon T (i.e. in all periods except their finite number, independent of the horizon length) lie in any arbitrarily close neighborhood of the turnpike. Strong turnpike theorems specify the time at which the optimal process can be precipitated from the turnpike neighborhood. They proof that all such possible events can only take place in the initial and/or final periods of the horizon. Also here the number of time periods in which an optimal growth process may be precipitated from the turnpike neighborhood is also limited and independent of the length of the entire horizon. Finally, very strong turnpike theorems refer to processes/trajectories that almost always (for almost all periods in horizon T) lie on the turnpike. Particularly, this group of theorems includes theorems that state that the entry of the optimal growth process onto the turnpike is irreversible.
The meaning of the theorem which we formulate below is that every optimal production trajectory – a solution of the task (24) – independently of the length of horizon T, almost always remains in an arbitrarily close neighborhood of the multilane turnpike $\mathbb{N}$ (in the sense of measure (22)) or lies on that turnpike. In simple words, no optimal economy growth can hold away from the multilane turnpike.

Theorem 7. *Let $\{y^*(t)\}_{t=0}^{t_1}$ be a $\left(y^0, t_1\right)$-optimal production trajectory (solution of the problem* (24)*). If the conditions **(G1)–(G5), (F1)–(F4), (FG1), (FG2)** hold, then $\forall\ \varepsilon > 0$ there exists a natural number k_ε, such that the number of time periods for which*

$$d\left(y^*(t), \mathbb{N}\right) \geq \varepsilon \tag{26}$$

does not excess k_ε. The number k_ε does not depend on the length of the time horizon T. The metric function $d(\cdot)$ is defined in (22).

Proof. The definition of a $\left(y^0, t_1\right)$-optimal production trajectory $\{y^*(t)\}_{t=0}^{t_1}$, properties (6), (19) and **(F4) (i)** imply that:

$$\langle \overline{p}, y^*(t+1)\rangle \leq \alpha_M \langle \overline{p}, y^*(t) - i^*(t)\rangle, \quad t = 0, 1, \ldots, t_1 - 1,$$

hence:

$$\begin{aligned} \langle \overline{p}, y^*(t_1)\rangle &\leq \alpha_M \langle \overline{p}, y^*(t_1 - 1) - i^*(t_1 - 1)\rangle \\ &\leq \alpha_M^2 \langle \overline{p}, y^*(t_1 - 2)\rangle - \alpha_M \langle \overline{p}, i^*(t_1 - 1)\rangle - \alpha_M^2 \langle \overline{p}, i^*(t_1 - 2)\rangle \\ &\leq \ldots \leq \alpha_M^{t_1} \left\langle \overline{p}, y^0\right\rangle - \sum_{k=1}^{t_1} \alpha_M^k \langle \overline{p}, i^*(t_1 - k)\rangle. \end{aligned} \tag{27}$$

We denote by $\mathcal{A} = \{\tau_1, \ldots, \tau_k\}$ the set of time periods, for which the condition (26) holds, $0 \leq \tau_1 < \tau_2 < \ldots < \tau_k < t_1$. According to Lemma 5:

$$\langle \overline{p}, y^*(t+1)\rangle \leq (\alpha_M - \delta_\varepsilon) \langle \overline{p}, y^*(t) - i^*(t)\rangle, \quad t \in \mathcal{A}. \tag{28}$$

It follows from (27)–(28) that:

$$\begin{aligned} &\langle \overline{p}, y^*(t_1)\rangle \\ &\leq \alpha_M^{t_1 - k} (\alpha_M - \delta_\varepsilon)^k \left\langle \overline{p}, y^0\right\rangle - \sum_{\substack{\tau=0 \\ \tau \notin \mathcal{A}}}^{t_1 - 1} \alpha_M^{t_1 - \tau} \langle \overline{p}, i^*(\tau)\rangle - \sum_{\tau \in \mathcal{A}} (\alpha_M - \delta_\varepsilon)^{t_1 - \tau} \langle \overline{p}, i^*(\tau)\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \alpha_M^{t_1 - k} (\alpha_M - \delta_\varepsilon)^k \left\langle \overline{p}, y^0\right\rangle, \end{aligned}$$

and under **(U2)** we reach the conclusion:

$$u(y^*(t_1)) \leq a\alpha_M^{t_1 - k} (\alpha_M - \delta_\varepsilon)^k \left\langle \overline{p}, y^0\right\rangle. \tag{29}$$

On the other hand, Lemma 6 shows that there exists a (y^0, t_1)-feasible production trajectory $\{\widetilde{y}(t)\}_{t=0}^{t_1}$ of the form (25). The positive homogeneity of degree 1 of the utility function (condition **(U1)**) implies:

$$u(y^*(t_1)) \geq u(\widetilde{y}(t_1)) = u\left(\alpha_M^{t_1 - \check{t}} \check{y}(\check{t})\right) = \sigma \alpha_M^{t_1 - \check{t}} u(\check{s}) > 0, \tag{30}$$

$\sigma = \|\check{y}(\check{t})\| > 0$, $\check{s} = \dfrac{\check{y}(\check{t})}{\|\check{y}(\check{t})\|} \in S$. Conditions (29)–(30) justify the inequality:

$$0 < \sigma \alpha_M^{t_1 - \check{t}} u(\check{s}) \leq a\alpha_M^{t_1 - k} (\alpha_M - \delta_\varepsilon)^k \left\langle \overline{p}, y^0\right\rangle,$$

which allows us to bound k :

$$k \leq \frac{\ln A}{\ln \alpha_M - \ln(\alpha_M - \delta_\varepsilon)} = B,$$

where $A = \max_{s \in S}\{(a\alpha_M^{\check{t}} \langle \overline{p}, y^0 \rangle)/(\sigma u(s))\} > 0$. If $B \leq 0$, then $k_\varepsilon = 0$ and the (y^0, t_1)-optimal production trajectory $\{y^*(t)\}_{t=0}^{t_1}$ stays on the turnpike in all time periods $t \in T$. This is true as well for the initial time period $t = 0$, which means that a (y^0, t_1)-feasible production trajectory $\{\widetilde{y}(t)\}_{t=0}^{t_1}$ of the form (25) is also a (y^0, t_1)-optimal one, hence $\forall\ t\ (\widetilde{y}(t) = y^*(t))$.
If $B > 0$, it is enough to consider the number k_ε time periods for which the condition (26) holds to be the least integer not smaller than $\{0, B\}$. $\square$

7 Final remarks

Conditions **(F1)–(F3)** which determine the properties of the map $F_t(\cdot)$ have a very general form. These properties condition the production technology (dynamics of production spaces) upon investments. This was a conscious decision, which allows us to extend/generalize the presented model of Gale type with investments in many different directions. We can build a whole generation of such models which opens an interesting research direction.
What remains to do, is to investigate the "strong" and "very strong" turnpike effect in the Gale economy with investments which were mentioned in the introduction. We would like to study this effect from the point of view of the target growth of type (24), as well as, from the point of view of maximization of production utility, generated in all time periods of the horizon $T = \{0, 1, \ldots, t_1\}$. In the classic variant of the non-stationary model of Gale type economy, this last problem was described, among others, in the paper Panek (2019b).
An interesting research problem is to investigate turnpike properties of optimal growth processes in Gale type economy with investments without assuming the existence of limit technology space. Some results of research in the classic variant of Gale nonstationary economy are contained in Panek (2019c, 2020a, 2020b).
The weakness of the model presented in this paper is the implicit assumption that the only consequence of the suspension of production investments in the economy in time t is the stabilization of its production technology in the next period (i.e. no depreciation of fixed production assets). The first results of the research currently conducted by the author in this field lead to the conclusion that also taking into account the depreciation of fixed production assets does not deprive the economy of its turnpike properties.

References

[1] Giorgi G., Zuccotti C., (2016), Equilibrium and Optimality in Gale-von Neumann Models, DEM Working Papers 118, University of Pavia, Department of Economics and Management.

[2] Lancaster K., (1968), *Mathematical Economics*, The Macmillan Company, New York-London.

[3] Makarov V. L., Rubinov A. M., (1977), *Mathematical Theory of Economic Dynamics and Equilibria*, Springer-Verlag, New York, Heidelberg, Berlin.

[4] McKenzie L. W., (2005), Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics, [in:] *Handbook of Mathematical Economics*, [eds.:] K. J. Arrow, M. D. Intriligator, 1281–1355.

[5] Mitra T., Nishimura K., (2009), *Equilibrium, Trade and Growth. Selected papers of L.W. McKenzie*, The MIT Press, Cambridge, Massaschusetts, London, England.

[6] Nikaido H., (1968), *Convex Structures and Economic Theory*, Academic Press, New York.

[7] Panek E., (2000), *Ekonomia matematyczna*, AEP, Poznań.

[8] Panek E., (2016), Gospodarka Gale'a z wieloma magistralami. "Słaby" efekt magistrali, *Przegląd Statystyczny* 63(4), 355–374.

[9] Panek E., (2017), "Słaby" efekt magistrali w niestacjonarnej gospodarce Gale'a z graniczną technologią i wielopasmową magistralą produkcyjną, [in:] *Matematyka i informatyka na usługach ekonomii*, [ed.:] D. Appenzeller, UEP, 94–110.

[10] Panek E., (2018), Niestacjonarna gospodarka Gale'a z graniczną technologią i wielopasmową magistralą produkcyjną. "Słaby", "silny" i "bardzo silny" efekt magistrali, *Przegląd Statystyczny* 65(4), 373–392.

[11] Panek E., (2019a), O pewnej wersji twierdzenia o wielopasmowej magistrali w niestacjonarnej gospodarce Gale'a, *Przegląd Statystyczny* 66(2), 143–157.

[12] Panek E., (2019b), Non-Stationary Gale Economy with Limit Technology, Multilane Turnpike and General Form of Optimality Criterion, *Argumenta Oeconomica Cracoviensia* 1(20), 9–22.

[13] Panek E., (2019c), Optimal growth proceses in non-stationary Gale economy with multilane production turnpike, *Economic and Business Review* 5(2), 3–23.

[14] Panek E., (2020a), A multilane turnpike in a non-stationary input-output economy with a temporary von Neumann equilibrium, *Statistical Review* 67(1), 77–92.

[15] Panek E., (2020b), Almost "very strong" multilane turnpike effect in the non-stationary Gale economy with a temporary von Neumann equilibrium and price constraints, *Economic and Business Review* 6(2), 66–80.

[16] Panek E., (2021), The minimal-time growth problem and turnpike effect in the stationary Gale economy, *Economic and Business Review* 7(1), 7–25.

[17] Samuelson P. A., (1960), Efficient Path of Capital Accumulation in Terms of the Calculus of Variations, [in:] *Mathematical Methods in the Social Sciences*, [eds:] K. J. Arrow, S. Karlin, P. Suppes, Stanford University Press, Stanford.

[18] Radner R., (1961), Path of Economic Growth that are Optimal with Regard to Final States: A Turnpike Theorem, *The Review of Economic Studies* 28(2), 98–104.

[19] Takayama A., (1985), *Mathematical Economics*, Cambrige University Press, Cambridge.