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CONTROL AND INFORMATICS

New stability tests for positive descriptor linear systems

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Abstract. The asymptotic stability of positive descriptor continuous-time and discrete-time linear systems is considered. New sufficient conditions for stability of positive descriptor linear systems are established. The efficiency of the new stability conditions are demonstrated on numerical examples of continuous-time and discrete-time linear systems.

Key words: descriptor; positive; continuous-time; discrete-time; system; sufficient condition; stability.

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1–4]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems with different fractional orders have been addressed in [3–5]. Descriptor positive systems have been analyzed in [6–8] and descriptor fractional linear systems in [8–11]. Linear positive electrical circuits with state feedbacks have been addressed in [2,4,6]. The stability of fractional linear discrete-time systems has been investigated in [12–16].

In this paper the asymptotic stability of positive descriptor continuous-time and discrete-time linear systems will be addressed.

The paper is organized as follows. In Section 2 the new stability tests for positive descriptor linear continuous-time systems are presented. The corresponding stability tests for positive descriptor discrete-time linear systems are given in Section 3. Concluding remarks are given in Section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

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2. POSITIVE DESCRIPTOR CONTINUOUS-TIME LINEAR SYSTEMS

Consider the autonomous continuous-time linear system

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t)\in\Re^n$ is the state vector and $E\in\Re^{n\times n}$, $A\in\Re^{n\times n}$.

The system (1) satisfies the conditions $\det E = 0$ and

$$\det[Es - A] \neq 0. \tag{2}$$

The following elementary operations on real matrices will be used [2, 3]:

1. Multiplication of any i-th row (column) by the number a. This operation will be denoted by $L[i\times a]$ for row operation and by $R[i\times a]$ for column operation.
2. Addition to any i-th row (column) of the j-th row (column) multiplied by any number b. This operation will be denoted by $L[i + j\times b]$ for row operation and by $R[i + j\times b]$ for column operation.
3. The interchange of rows i and j will be denoted by $L[i, j]$ and for columns by $R[i, j]$.

The elementary operations do not change the rank of the matrices [2, 3].

Performing elementary row operations on the array

$$E, A \tag{3}$$

or equivalently on (1) by elimination the linearly dependent rows in the matrix E we obtain

$$\begin{matrix} E_1 & A_1 \\ 0 & A_2 \end{matrix} \tag{4}$$

and

$$\begin{bmatrix} E_1 \\ 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{5}$$

where

$$
\begin{aligned}
&x_1(t) \in \Re^{n_1}, && x_2(t) \in \Re^{n_2}, && A_1 = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}, \\
&A_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}, && A_{11} \in \Re^{n_1 \times n_1}, && A_{12} \in \Re^{n_1 \times n_2}, \\
&A_{21} \in \Re^{n_2 \times n_1}, && A_{22} \in \Re^{n_2 \times n_2}, && n_1 + n_2 = n.
\end{aligned}
\tag{6}
$$

The matrix $A \in \Re_+^{n_2 \times n_2}$ is called monomial if each its column (row) contains only one positive entry and its remaining entries are zero.

Lemma 1. The system (1) is (internally) positive if the matrix $E_1 \in \Re_+^{n_1 \times n}$ contains n_1 monomial columns and

$$
\begin{aligned}
&A_{11} \in M_{n_1}, \quad A_{12} \in \Re_+^{n_1 \times n_2}, \\
&A_{21} \in \Re_+^{n_2 \times n_1}, \qquad A_{22} \in M_{n_2}, \quad n_1 + n_2 = n.
\end{aligned}
\tag{7}
$$

Proof. If the matrix E_1 contains n_1 monomial columns than by elementary column operations the remaining its columns may be eliminated and we obtain $E_1 = \begin{bmatrix} E_{11} & 0 \end{bmatrix}$, where $E_{11} \in M_{n_1}$. In this case from (5) we obtain $\dot{x}_1(t) = E_{11}^{-1} A_{11} x_1(t) + E_{11}^{-1} A_{12} x_2(t)$, where $E_{11}^{-1} A_{11} \in M_{n_1}$, $E_{11}^{-1} A_{12} \in \Re_+^{n_1 \times n_2}$. This completes the proof. □

A vector is called strictly positive if all its components are positive.

Theorem 1. The positive descriptor system (1) is asymptotically stable if there exists strictly positive vectors $\lambda_1 \in \Re_+^{n_1}$, $\lambda_2 \in \Re_+^{n_2}$ such that

$$
A_{11}\lambda_1 + A_{12}\lambda_2 < 0 \quad \text{and} \quad A_{21}\lambda_1 + A_{22}\lambda_2 = 0. \tag{8}
$$

Proof. Integrating (5) we obtain

$$
E_1 \int_0^\infty \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \mathrm{d}t = A_{11} \int_0^\infty x_1(t)\,\mathrm{d}t + A_{12} \int_0^\infty x_2(t)\,\mathrm{d}t \tag{9}
$$

$$
0 = A_{21} \int_0^\infty x_1(t)\,\mathrm{d}t + A_{22} \int_0^\infty x_2(t)\,\mathrm{d}t. \tag{10}
$$

Taking into account that

$$
E_1 \int_0^\infty \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \mathrm{d}t = E_1 \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} - E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{11}
$$

and for asymptotically stable system $x_1(\infty) = 0$, $x_2(\infty) = 0$ from (9) and (10) we obtain

$$
-E_1 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} < 0 \tag{12}
$$

and

$$
\begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 0, \tag{13}
$$

where $\lambda_1 = \int_0^\infty x_1(t)\,\mathrm{d}t$, $\lambda_2 = \int_0^\infty x_2(t)\,\mathrm{d}t$ which is equivalent to (8).

If $A_{22} \in M_{n_2}$ is asymptotically stable then $-A_{22}^{-1} \in \Re_+^{n_2 \times n_2}$ and from (10) we obtain

$$
\lambda_2 = -A_{22}^{-1} A_{21} \lambda_1 \in \Re_+^{n_2}. \tag{14}
$$

Substituting (14) into $A_{11}\lambda_1 + A_{12}\lambda_2 < 0$ we obtain

$$
\bar{A} = A_{11} - A_{12} A_{22}^{-1} A_{21} \in M_{n_1} \tag{15}
$$

which is asymptotically stable. This completes the proof. □

In particular case when the system is standard and $\det E \neq 0$ then from (1) we have $\dot{x} = \bar{A}x$, $\bar{A} = E^{-1}A$ and the relations (8) take the well-known form [3, 4]

$$
\bar{A}\lambda < 0, \tag{16}
$$

where $\lambda \in \Re_+^n$ is strictly positive vector.

Example Consider the autonomous descriptor system (1) with the matrices

$$
E = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \qquad
A = \begin{bmatrix} -3 & 1 & 0 & 1 \\ 0 & 4 & -1 & -2 \\ -2 & 1 & -1 & 2 \\ 1 & -2 & 1 & 0 \end{bmatrix}. \tag{17}
$$

The system satisfies the assumption (2) since

$$
\begin{aligned}
\det[Es - A] &= \begin{bmatrix} 3 & s-1 & 0 & 2s-1 \\ -s & -4 & 1 & -s+2 \\ 2 & s-1 & 1 & 2s-2 \\ s-1 & 2 & -1 & s \end{bmatrix} \\
&= 7s^2 - 28s + 5.
\end{aligned}
\tag{18}
$$

Performing on the array

$$
E, A = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 2 & -3 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 4 & -1 & -2 \\ 0 & 1 & 0 & 2 & -2 & 1 & -1 & 2 \\ 1 & 0 & 0 & 1 & 1 & -4 & 1 & 0 \end{array}\right], \tag{19}
$$

the following elementary operations $L[2,4]$, $L[4+2]$ and $L[3+1(-1)]$ we obtain

$$
\begin{matrix} E_1 & A_1 \\ 0 & A_2 \end{matrix} = \left[\begin{array}{cc:cc|cc:cc} 0 & 1 & 0 & 2 & -3 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & -4 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{array}\right]. \tag{20}
$$

Note that the first two columns of E_1 are the columns of monomial matrix and we may eliminate the fourth column of E_1. In this case we have

$$
A_{11}\lambda_1 + A_{12}\lambda_2 = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}\lambda_1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\lambda_2 < \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}
$$

and

$$
A_{21}\lambda_1 + A_{22}\lambda_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\lambda_1 + \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}\lambda_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{22}
$$

From (22) and (21) we obtain

$$
\begin{aligned} \lambda_2 &= -A_{22}^{-1}A_{21}\lambda_1 \\ &= \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\lambda_1 = \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}\lambda_1 \end{aligned} \tag{23}
$$

and

$$
\begin{aligned} \bar{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ &= \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2.5 & 1 \\ 2.5 & -4 \end{bmatrix}. \end{aligned} \tag{24}
$$

The descriptor system with (17) satisfies the condition (8) for $\lambda_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, $\lambda_2 = \begin{bmatrix} 1.5 & 0.5 \end{bmatrix}^T$ and by Theorem 1 it is asymptotically stable.

3. POSITIVE DISCRETE-TIME DESCRIPTOR LINEAR SYSTEMS

Consider the autonomous descriptor discrete-time linear system

$$
\bar{E}x_{i+1} = \bar{A}x_i\,, \tag{25}
$$

where $x_i \in \Re^n$ is the state vector and $\bar{E} \in \Re^{n\times n}$, $\bar{A} \in \Re^{n\times n}$.

It is assumed that the system (25) satisfies the conditions $\det\bar{E} = 0$ and

$$
\det\left[\bar{E}z - \bar{A}\right] \neq 0. \tag{26}
$$

Performing elementary row operations on the array

$$
\bar{E},\ \bar{A} \tag{27}
$$

or equivalently on the system (25) by elimination the linearly dependent rows in the matrix $\bar{E}$ we obtain

$$
\begin{matrix} \bar{E}_1 & \bar{A}_1 \\ 0 & \bar{A}_2 \end{matrix} \tag{28}
$$

and

$$
\begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix}\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}\begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \tag{29}
$$

where

$$
\begin{aligned} &x_{1,i} \in \Re^{n_1}, && x_{2,i} \in \Re^{n_2}, \\ &\bar{A}_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \end{bmatrix}, && \bar{A}_2 = \begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \\ &\bar{A}_{11} \in \Re^{n_1\times n_1}, && \bar{A}_{12} \in \Re^{n_1\times n_2}, \\ &\bar{A}_{21} \in \Re^{n_2\times n_1}, && \bar{A}_{22} \in \Re^{n_2\times n_2}, \\ &\bar{E}_1 = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \end{bmatrix}, && \bar{E}_{11} \in \Re_+^{n_1\times n_1}, \\ &\bar{E}_{12} \in \Re_+^{n_1\times n_2}, && n_1 + n_2 = n. \end{aligned} \tag{30}
$$

The discrete-time system (25) is positive if

$$
\begin{aligned} &\bar{A}_{11} - \bar{E}_{11} \in \Re_+^{n_1\times n_1}, \quad \bar{A}_{12} - \bar{E}_{12} \in \Re_+^{n_1\times n_2} \\ &\text{and } \ \bar{A}_{22} \in M_{n_2} \ \text{ is asympotically stable.} \end{aligned} \tag{31}
$$

Note that $-\bar{A}_{22} \in \Re^{n_2\times n_2}$ if and only if $\bar{A}_{22}$ is asymptotically stable Metzler matrix [3].

Theorem 2. The positive descriptor system (25) is asymptotically stable if there exists strictly positive vectors $\lambda_1 \in \Re_+^{n_1}$, $\lambda_2 \in \Re_+^{n_2}$, such that

$$
\begin{aligned} \left(\bar{A}_{11} - \bar{E}_{11}\right)\lambda_1 + \left(\bar{A}_{12} - \bar{E}_{12}\right)\lambda_2 &< 0, \\ \bar{A}_{21}\lambda_1 + \bar{A}_{22}\lambda_2 &= 0. \end{aligned} \tag{32}
$$

Proof. Taking into account (29) we obtain

$$
\bar{E}_{11}\sum_{i=0}^{\infty}x_{1,i+1} + \bar{E}_{12}\sum_{i=0}^{\infty}x_{2,i+1} = \bar{A}_{11}\sum_{i=0}^{\infty}x_{1,i} + \bar{A}_{12}\sum_{i=0}^{\infty}x_{2,i}\,, \tag{33}
$$

$$
0 = \bar{A}_{21}\sum_{i=0}^{\infty}x_{1,i} + \bar{A}_{22}\sum_{i=0}^{\infty}x_{2,i}\,. \tag{34}
$$

Note that

$$
\begin{aligned} \bar{E}_{11}\sum_{i=0}^{\infty}x_{1,i+1} &= \bar{E}_{11}\left(\sum_{i=0}^{\infty}x_{1,i} - x_{1,0}\right), \\ \bar{E}_{12}\sum_{i=0}^{\infty}x_{2,i+1} &= \bar{E}_{12}\left(\sum_{i=0}^{\infty}x_{2,i} - x_{2,0}\right) \end{aligned} \tag{35}
$$

and using (33) we obtain

$$
\begin{aligned} -\bar{E}_{11}x_{1,0} - \bar{E}_{12}x_{2,0} = \left[\bar{A}_{11} - \bar{E}_{11}\right]\lambda_1 + \left[\bar{A}_{12} - \bar{E}_{12}\right]\lambda_2 \\ < 0, \end{aligned} \tag{36}
$$

$$
\bar{A}_{21}\lambda_1 + \bar{A}_{22}\lambda_2 = 0, \tag{37}
$$

where

$$\lambda_1 = \sum_{i=0}^{\infty} x_{1,i}, \qquad \lambda_2 = \sum_{i=0}^{\infty} x_{2,i}. \tag{38}$$

This completes the proof. □

Remark 1. The condition (36) is not satisfied if at least one row of the matrix $\bar{A}_{11} + \bar{A}_{12} - \bar{E}_{11} - \bar{E}_{12}$ is not negative.

Remark 2. Assuming $\lambda_1 \in \Re_+^{n_1}$ is strictly positive vector we may find the desired vector $\lambda_2 \in \Re_+^{n_2}$ from the equation

$$\lambda_2 = -\bar{A}_{22}^{-1} \bar{A}_{21} \lambda_1. \tag{39}$$

Example 3.1. Consider the positive descriptor system (25) with the matrices

$$E = \begin{bmatrix} 0.8 & 0.1 & \vdots & 0.6 & 0.2 \\ 0.1 & 0.7 & \vdots & 0.3 & 0.8 \\ \dots & \dots & \vdots & \dots & \dots \\ 0.9 & 0.8 & \vdots & 0.9 & 1 \\ -0.1 & -0.7 & \vdots & -0.3 & -0.8 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3 & 0.3 & \vdots & 0.8 & 0.3 \\ 0.2 & 0.4 & \vdots & 0.4 & 0.5 \\ \dots & \dots & \vdots & \dots & \dots \\ 1.3 & 1 & \vdots & -0.2 & 1.1 \\ -0.1 & -0.2 & \vdots & 0 & -1.2 \end{bmatrix}. \tag{40}$$

The system satisfied the assumption (26) since

$$\det[Ez - A] = -0.0414z^3 + 1.0981z^2 - 1.1839z + 2.174. \tag{41}$$

Performing on the array

$$E, A = \left[E \mid A \right]. \tag{42}$$

the following elementary operations $L[3+1(-1)]$, $L[3+2(-1)]$, $L[4+2(+1)]$ we obtain

$$\left[\begin{array}{cc|cc} \bar{E}_{11} & \bar{E}_{12} & \bar{A}_{11} & \bar{A}_{12} \\ 0 & 0 & \bar{A}_{21} & \bar{A}_{22} \end{array}\right] = \left[\begin{array}{cccc|cccc} 0.8 & 0.1 & \vdots & 0.6 & 0.2 & 0.3 & 0.3 & \vdots & 0.2 & 0.3 \\ 0.1 & 0.7 & \vdots & 0.3 & 0.8 & 0.2 & 0.4 & \vdots & 0.4 & 0.5 \\ \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \vdots & \dots & \dots \\ 0 & 0 & \vdots & 0 & 0 & 0.8 & 0.3 & \vdots & -1.4 & 0.3 \\ 0 & 0 & \vdots & 0 & 0 & 0.1 & 0.2 & \vdots & 0.4 & -0.7 \end{array}\right]. \tag{43}$$

In this case from (39) for $\lambda_1 = [1 \ \ 1]^T$ we have

$$\lambda_2 = -\bar{A}_{22}^{-1} \bar{A}_{21} \lambda_1 = \begin{bmatrix} -1.4 & 0.3 \\ 0.4 & -0.7 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 & 0.3 \\ 0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{44}$$

and

$$\left(\bar{A}_{11} - \bar{E}_{11}\right) \lambda_1 + \left(\bar{A}_{12} - \bar{E}_{12}\right) \lambda_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.4 & 0.1 \\ 0.1 & -0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{45}$$

Therefore, the positive descriptor system with (40) is asymptotically stable.

4. CONCLUDING REMARKS

The asymptotic stability of positive descriptor continuous-time-time and discrete-time linear systems has been investigated. New sufficient conditions for stability of positive descriptor continuous-time linear systems have been given in Section 2 and for discrete-time systems in Section 3. The efficiency of the new stability conditions are demonstrated on numerical examples of continuous-time and discrete-time linear systems. The considerations can be extended to positive descriptor fractional orders continuous-time and discrete-time systems.

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