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Distributed event-triggered generalized Nash equilibrium seeking for aggregative games on unbalanced digraphs

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This paper addresses the problem of seeking generalized Nash equilibrium for constrained aggregative games with double-integrator agents who communicate with each other on an unbalanced directed graph. An auxiliary variable is introduced to balance the consensus terms in the designed algorithm by estimating the left eigenvector of the Laplacian matrix associated with the zero eigenvalue in a distributed manner. Moreover, an event-triggered broadcasting scheme is proposed to reduce communication loads in the network. It is shown that the proposed communication scheme is free of the Zeno behavior and the asymptotic convergence of the designed algorithm is obtained. Simulation results are demonstrated to validate the proposed methods.

Key words: aggregative games, coupled constraints, generalized Nash equilibrium, multi-agent systems

1. Introduction

Aggregative games, as a class of noncooperative games, have been applied in many engineering scenarios, such as communication networks [1], smart grids [2] and charging scheduling of electric vehicles [3]. By virtue of the coordination control of multi-agent systems, many distributed algorithms have been proposed for a group of agents to seek generalized Nash equilibrium (GNE) of games with coupled constraints.

In the last decades, the GNE seeking algorithms have been extended from discrete-time to continuous-time setups, due to the various applications of engi-

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neering [4–8]. Moreover, recent work was not restricted to the consideration of single-integrator agents [9]. The results for agents with double-integrator dynamics and nonlinear dynamics [10,11] have also been reported. In addition to coupled equality constraints considered in [7,8], global inequality constraints [12–15] have been also considered in GNE seeking problems. It is known that communication networks among agents play an important role in the implementation of GNE seeking algorithms in a distributed manner. In the most of the above-mentioned literature, fixed undirected and connected graphs were assumed for the communication among agents (see [4, 9, 14, 15] and the references therein). For more flexible applications, different structures of communication graphs were considered in the GNE seeking problem, such as weight-balanced directed graphs (digraphs) and time-varying networks [16,17]. Although various distributed GNE seeking algorithms have been designed on undirected graphs or weight-balanced digraphs, there is no study to solve the GNE seeking problem on unbalanced digraphs, which are more generally seen in practice, such as mobile sensor networks composed of agents with nonuniform communication ranges [18].

In addition, we find that the existing continuous-time GNE seeking algorithms are designed under the assumption that the communications among agents are implemented in a continuous-time manner. However, this assumption limits the application of distributed seeking algorithms to practical scenarios, due to limited communication resources. Recently, based on the event-triggered communication schemes proposed in networked control systems, a few work has been reported on the NE seeking algorithms with discrete-time interactions. A self-triggered communication scheme was designed in continuous-time best-response dynamics for static noncooperative games [19]. Cai *et al.* designed event-triggered communication schemes for static noncooperative games with double-integrator agents [20, 21]. Whereas, the design of a discrete-time interaction scheme for the constrained aggregative games for double-integrator agents on unbalanced digraphs has not been reportes.

From the above observations, a distributed GNE seeking problem of aggregative games on unbalanced digraphs is studied. Compared with the existing work, the main contributions of this paper are summarized as follows.

1. A new distributed algorithm to solve a GNE seeking problem for double-integrator agents on unbalanced digraphs is designed. Unlike the existing methods that were restricted to undirect graphs or balanced digraphs [4, 7, 9, 14, 15, 17], the proposed algorithm allows the interactions among agents to be unbalanced at event-triggered communication instants.

2. A distributed estimator for the left eigenvector of the Laplacian matrix associated with the zero eigenvalue is proposed to balance the consensus terms in the designed seeking algorithm. Combined with the estimator, the designed seeking algorithm can converge to GNE of the aggregative game

on unbalanced digraphs. The convergence of the designed algorithm is analyzed by the stability of a perturbed system and the small-gain approach, which is different from the Lyapunov stability theory used in the existing work [11, 12, 16, 17, 22].

3. Different from event-triggered schemes proposed on undirected graphs or balanced digraphs [19, 20], an event-triggered broadcasting scheme is designed by agents' own information for the discrete-time interactions in the coordination of Lagrange multipliers, the estimation of the aggregator and the regulation of the auxiliary variable. Due to the introduction of the auxiliary variable to deal with unbalanced digraphs, the design and analysis of the proposed algorithm become more challenging.

The organization of this paper is given as follows. In Section 2, the considered problem is formulated. In Section 3, a distributed event-triggered GNE seeking algorithm on unbalanced digraphs is designed and analyzed. In Section 4, a simulation example is presented. The conclusions are stated in Section 5.

Notations: $\mathbb{R}$ and $\mathbb{R}^n$ denote the real numbers set and the n-dimensional Euclidean space, respectively. $\mathbb{Z}_{\geqslant 0}$ denotes the set of non-negative integers. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ is the Euclidean norm. $\otimes$ denotes the Kronecker product. A^T and $\|A\|$ are the transpose and the spectral norm of matrix A, respectively. Denote $\mathrm{col}(x_1, \ldots, x_n) = [x_1^T, \ldots, x_n^T]^T$. Given matrices $A_1, \ldots, A_n$, $\mathrm{blk}\{A_1, \cdots, A_n\}$ denotes the block diagonal matrix with A_i on the diagonal. I_n is the $n \times n$ identity matrix. $\mathbf{0}$ denotes a zero matrix with appropriate dimensions. 1_n and 0_n are n-dimensional column vectors consisting of all 1s and 0s, respectively.

2. Problem formulation

An aggregative game is considered here and denoted by $G = (\mathcal{I}, \Omega, J)$, where $\mathcal{I} = \{1, \ldots, N\}$ is the set of agents, $\Omega = \Omega_1 \times \cdots \times \Omega_N$ is the strategy space and $J = (J_1, \ldots, J_N)$ is the cost profile. For agent $i \in \mathcal{I}$, $\Omega_i \subset \mathbb{R}^n$ is its strategy set, and $J_i \in \mathbb{R}$ is its cost which can be described by function $J_i(x_i, \sigma(x)) : \Omega_i \times \mathbb{R}^r \to \mathbb{R}$. $\sigma(x) : \Omega \to \mathbb{R}^r$ is an aggregative function describing the influence of population strategy on the individual cost and is defined by $\sigma(x) = \sum_{i=1}^{N} \phi_i(x_i)$, where $\phi_i : \mathbb{R}^n \to \mathbb{R}^r$ expresses the local contribution to the aggregation. The strategy profile of the game is $x = \mathrm{col}(x_1, \ldots, x_N) \in \Omega$. Define $x_{-i} = \mathrm{col}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ as agents' strategies except agent i. The strategies of all agents are subject to a group of equality constraints, which can be denoted by $\sum_{i=1}^{N} A_i x_i = \sum_{i=1}^{N} d_i$ with $A_i \in \mathbb{R}^{l \times n}$ and $d_i \in \mathbb{R}^l$. Assume that A_i

is row full rank, for all $i \in \mathcal{I}$. For agent i, given the other agents' strategies x_{-i}, it always wants to choose its strategy x_i to minimize its own cost, that is,

$$\begin{aligned} &\min_{x_i \in \Omega_i} J_i(x_i, \sigma(x)) \\ &s.t. \quad \sum_{i=1}^{N} A_i x_i = \sum_{i=1}^{N} d_i \,. \end{aligned} \tag{1}$$

In addition, each agent in game G has an inherent dynamics which is modeled by the following double-integrator

$$\begin{aligned} \dot{x}_i &= v_i \,, \\ \dot{v}_i &= u_i \,. \end{aligned} \tag{2}$$

In (2), $x_i \in \mathbb{R}^n$, $v_i \in \mathbb{R}^n$, and $u_i \in \mathbb{R}^n$ are agent i's strategy, auxiliary state and control input, respectively. To realize a distributed setting, we assume that agents can interact with each other to obtain some information from neighbors. The interaction among agents can be described by a digraph $\mathcal{G} = (\mathcal{I}, \mathcal{E}, \mathcal{A})$, where $\mathcal{I} = \{1, \ldots, N\}$ is the set of nodes, $\mathcal{E} \in \mathcal{I} \times \mathcal{I}$ is the set of edges and $\mathcal{A} \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix. For digraph $\mathcal{G}$, $(i, j) \in \mathcal{E}$ implies that agent j can receive information from agent i. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ of graph $\mathcal{G}$ is defined as $l_{ii} = \sum\limits_{j=1, j \neq i}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$, where a_{ij} is the weight on edge $(i, j) \in \mathcal{E}$. $a_{ij} > 0$ indicates edge $(i, j) \in \mathcal{E}$; $a_{ij} = 0$, otherwise. A digraph is strongly connected if there exists a directed path between any pair of distinct nodes [23]. For a strongly connected graph, some properties of L are given below.

Lemma 1 *[24] For a strongly connected digraph $\mathcal{G}$ with Laplacian matrix L, there is a positive left eigenvector $\xi = [\xi_1, \ldots, \xi_N]^T$ associated with the zero eigenvalue of L such that $\xi^T L = 0$ and $\sum_{i=1}^{N} \xi_i = 1$. Moreover, $\hat{L} = (\Xi L + L^T \Xi)/2$ is positive semidefinite and it's eigenvalues denoted by $0 = \varrho_1 < \varrho_2 \leqslant \varrho_3 \leqslant \ldots \leqslant \varrho_N$, where $\Xi = diag(\xi_1, \ldots, \xi_N)$.*

Define the map $F(x) = \mathrm{col}(\nabla_{x_1} J_1, \ldots, \nabla_{x_N} J_N)$, where $\nabla_{x_i} J_i = \dfrac{\partial J_i(x_i, \sigma(x))}{\partial x_i}$, $\forall i \in \mathcal{I}$. Some necessary assumptions are given below to ensure a well defined problem.

Assumption 1 *The communication topology $\mathcal{G}$ is a strongly connected digraph.*

Assumption 2 *For all $i \in \mathcal{I}$, $\Omega_i = \mathbb{R}^n$, the cost function $J_i(x_i, \sigma(x))$ is continuously differentiable in x and convex in x_i for every fixed x_{-i}. The map $\phi_i(x_i)$ is differentiable with respect to x_i and κ_3-Lipschitz continuous on Ω_i for some constant $\kappa_3 > 0$.*

Assumption 3 *The map $F(x)$ is μ-strongly monotone on Ω for some positive constant μ.*

Definition 1 *For an aggregative game G, the GNE is a strategy profile $x^* = col(x_1^*, \ldots, x_N^*)$ satisfying*

$$J_i(x_i^*, \sigma(x^*)) \leqslant J_i(x_i, \sigma(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_N^*)) \tag{3}$$

and $\sum_{i=1}^{N} A_i x_i^* = \sum_{i=1}^{N} d_i$ for any $x_i \in \Omega_i$ and for all $i \in \mathcal{I}$.

Similar to the Nash equilibrium, the GNE is the point at which no agent has the willingness to change its strategy unilaterally for less cost under the condition that all agents' strategies need to satisfy the coupled constraints. Under Assumption 2, problem (1) can be considered as a constrained convex optimization problem. Via the primal-duality theory, the Lagrange multiplier $\lambda_i \in \mathbb{R}^l$ is introduced in the problem (1) to construct a Lagrangian function. The solution of (1) and the optimal Lagrange multiplier $\bar{\lambda}$ exist and satisfy the KKT conditions, which is stated in the following lemma.

Lemma 2 [25, Theorem 4.8] *Under Assumptions 2 and 3, aggregative game G admits a unique GNE x^* if*

$$\nabla_{x_i} J_i(x_i^*, \sigma(x^*)) + A_i^T \bar{\lambda} = 0, \quad \forall i \in \mathcal{I}, \tag{4}$$

$$\sum_{i=1}^{N} A_i x_i^* = \sum_{i=1}^{N} d_i, \tag{5}$$

where $\bar{\lambda} \in \mathbb{R}^l$ is a common Lagrange multiplier.

The objective of this paper mainly includes three points: 1) the design of distributed GNE seeking algorithm for double-integrator agents on an unbalanced digraph; 2) the design of event-triggered communication scheme for communication networks with limited resources; 3) the convergence analysis of the proposed method to the GNE of aggregative game G.

3. Distributed event-triggered GNE seeking algorithm on an unbalanced digraph

In this section, a distributed event-triggered GNE seeking algorithm is designed on an unbalanced digraph. There are four parts in the following designed algorithm, that is, the strategy update, the coordination of Lagrange multipliers, and the estimations of aggregator $\sigma(x)$ and the left eigenvector ξ of Laplacian matrix L. Let $\eta_i \in \mathbb{R}^r$ be the agent i's estimation of aggregator $\sigma(x)$. Let z_i be an auxiliary variable. r_i is defined to estimate ξ. And let k_i, α, and β be positive constants to be designed. In addition, define $t_1^i, t_2^i, t_3^i, \ldots$ to be a sequence of time instants when agent i communicates with its neighbors, for all $i \in \mathcal{I}$. At t_k^i, agent i broadcasts its information $\lambda_i(t_k^i)$, $\eta_i(t_k^i)$ and $r_i(t_k^i)$ to its neighbors. Let $\hat{\lambda}_i = \lambda_i(t_k^i)$, $\hat{\eta}_i = \eta_i(t_k^i)$ and $\hat{r}_i = r_i(t_k^i)$ for $t \in [t_k^i, t_{k+1}^i)$. Then, for agent i, a distributed GNE seeking algorithm is designed as follows.

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= -k_i v_i - \nabla_{x_i} J_i(x_i, \eta_i) - A_i^T \lambda_i, \end{aligned} \tag{6}$$

where $\nabla_{x_i} J_i(x_i, \eta_i)$ is the gradient of cost function $J_i(x_i, \eta_i)$ with estimated aggregation η_i. The coordination of Lagrange multiplier λ_i associated with the coupled constraints is given by

$$\begin{aligned} \dot{\lambda}_i &= -\alpha \sum_{j=1}^{N} a_{ij}(\hat{\lambda}_i - \hat{\lambda}_j) - z_i + (r_i^i)^{-1}(A_i x_i - d_i), \\ \dot{z}_i &= \alpha \sum_{j=1}^{N} a_{ij}(\hat{\lambda}_i - \hat{\lambda}_j), \end{aligned} \tag{7}$$

where λ_i and z_i start from $\lambda_i(0) = 0_l$ and $z_i(0) = 0_l$, respectively. The estimation of aggregator $\sigma(x)$ is

$$\dot{\eta}_i = \beta \sum_{i=1}^{N} a_{ij}(\hat{\eta}_i - \hat{\eta}_j) + (r_i^i)^{-1} \frac{d}{dt} \phi_i(x_i), \tag{8}$$

where $\eta_i(0) = \phi_i(x_i(0))$. In (7)–(8), r_i^i is the ith element of vector $r_i = [r_i^1, \ldots, r_i^N]^T$ with initial condition $r_i^i = 0$, $r_i^j(0) = 1$, $i \neq j$, $i, j \in \mathcal{I}$ to handle the unbalance incurred by graph $\mathcal{G}$. The vector r_i is regulated by

$$\dot{r}_i = -\sum_{j=1}^{N} a_{ij}\left(\hat{r}_i - \hat{r}_j\right). \tag{9}$$

Let $x = \mathrm{col}(x_1, \ldots, x_N)$, $v = \mathrm{col}(v_1, \ldots, v_N)$, $\lambda = \mathrm{col}(\lambda_1, \ldots, \lambda_N)$, $\hat{\lambda} = \mathrm{col}(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$, $z = \mathrm{col}(z_1, \ldots, z_N)$, $\eta = \mathrm{col}(\eta_1, \ldots, \eta_N)$, $\hat{\eta} = \mathrm{col}(\hat{\eta}_1, \ldots, \hat{\eta}_N)$, $r = \mathrm{col}(r_1, \ldots, r_N)$, $\hat{r} = \mathrm{col}(\hat{r}_1, \ldots, \hat{r}_N)$, $R = \mathrm{diag}\{r_1^1, \ldots, r_N^N\}$, $k = \mathrm{diag}\{k_1, \ldots, k_N\}$, $A = \mathrm{blk}\{A_1, \ldots, A_N\}$, $d = \mathrm{col}(d_1, \ldots, d_N)$, $\phi(x) = \mathrm{col}(\phi_1(x_1), \ldots, \phi_N(x_N))$ and $G(x, \eta) = \mathrm{col}(\nabla_{x_1} J_1(x_1, \eta_1), \ldots, \nabla_{x_N} J_N(x_N, \eta_N))$. The compact form of system (6)–(9) is

$$
\begin{aligned}
\dot{x} &= v, \\
\dot{v} &= -kv - G(x, \eta) - A^T \lambda, \\
\dot{\lambda} &= -\alpha(L \otimes I_l)\hat{\lambda} - z + (R^{-1} \otimes I_l)(Ax - d), \\
\dot{z} &= \alpha(L \otimes I_l)\hat{\lambda}, \\
\dot{\eta} &= -\beta(L \otimes I_r)\hat{\eta} + (R^{-1} \otimes I_r)\frac{d}{dt}\phi(x), \\
\dot{r} &= -(L \otimes I_N)\hat{r}.
\end{aligned}
\tag{10}
$$

Assumption 4 *The map $G(x, \eta)$ is κ_1-Lipschitz continuous with respect to $x \in \Omega$ and κ_2-Lipschitz continuous with respect to η for some constants $\kappa_1, \kappa_2 > 0$.*

The relationship between the equilibrium point of system (10) and the GNE of aggregative game $G = (\mathcal{I}, \Omega, J)$ is given in the following lemma.

Lemma 3 *Under Assumptions 1-3, x^* is a GNE of aggregative game $G = (\mathcal{I}, \Omega, J)$ if and only if there exist $v^* = 0$, $\bar{\lambda} \in \mathbb{R}^l$, $z^* \in \mathbb{R}^l$ and $\bar{\eta} \in \mathbb{R}^r$ such that $(x^*, v^*, 1_N \otimes \bar{\lambda}, z^*, 1_N \otimes \bar{\eta})$ is an equilibrium of system (10).*

Proof. Let $(x^*, v^*, \lambda^*, z^*, \eta^*)$ with $\lambda^* = 1_N \otimes \bar{\lambda}$ and $\eta^* = 1_N \otimes \bar{\eta}$ be an equilibrium of the system (10) satisfying the following equations.

$$0_{Nn} = v^*, \tag{11a}$$
$$0_{Nn} = -kv^* - G(x^*, \eta^*) - A^T \lambda^*, \tag{11b}$$
$$0_{Nl} = -\alpha(L \otimes I_l)\lambda^* - z^* + ((R^*)^{-1} \otimes I_l)(Ax^* - d), \tag{11c}$$
$$0_{Nl} = \alpha(L \otimes I_l)\lambda^*, \tag{11d}$$
$$0_{Nr} = -\beta(L \otimes I_r)\eta^* + ((R^*)^{-1} \otimes I_r)\phi(x^*), \tag{11e}$$
$$0_{N^2} = -(L \otimes I_N)r^*. \tag{11f}$$

From (11f) and under Assumption 1, since the digraph $\mathcal{G}$ is strongly connected, it is easy to see that $r_1^* = \cdots = r_N^* = \bar{r}$ for some $\bar{r} \in \mathbb{R}^N$. Moreover, $\lim_{t\to\infty} e^{-Lt} = 1_N \xi^T$ and $\lim_{t\to\infty} e^{-(L\otimes I_N)t} r(0) = (1_N \xi^T \otimes I_N) r(0) = 1_N \otimes \xi$. Thus, we have that $\bar{r} = \xi$ and $R^* = \mathrm{diag}\{\xi_1, \ldots, \xi_N\}$, where $\xi = [\xi_1, \ldots, \xi_N]^T$.

Left-multiplying both sides of (11e) and the fifth equation in (10) by $(1_N^T R^* \otimes I_r)$, it follows that $-(1_N^T R^* L \otimes I_r)\eta^* + 1_N^T \frac{d}{dt}\phi(x^*) = 0$, $\sum_{i=1}^{N} \xi_i \dot{\eta}_i = \frac{d}{dt}\sigma(x)$ and $\sum_{i=1}^{N} \xi_i \eta_i(0) = \sigma(0)$ with $\sum_{i=1}^{N} \xi_i = 1$. As a result, $\sum_{i=1}^{N} \xi_i \eta_i = \sigma(x)$, which implies that $\bar{\eta} = \sigma(x^*)$. From (11d), it is easy to see that $\lambda_1^* = \cdots = \lambda_N^* = \bar{\lambda}$ for some $\bar{\lambda} \in \mathbb{R}^l$ and $\sum_{i=1}^{N} \xi_i z_i^* = 0$. Left-multiplying both sides of the fourth equation in (10) by $(1_N^T R^* \otimes I_l)$, it yields that $\sum_{i=1}^{N} \xi_i \dot{z}_i = 0_l$, which implies $\sum_{i=1}^{N} \xi_i z_i(t) = \sum_{i=1}^{N} \xi_i z_i(0) = 0_l$, $\forall t > 0$. Then, it follows from (11c) that $0 = -(1_N^T \alpha R^* L \otimes I_l)\lambda^* - 1_N^T R^* z^* + 1_N^T (Ax^* - d)$ which further implies that $\sum_{i=1}^{N} A_i x_i^* = \sum_{i=1}^{N} d_i$. It indicates that (5) holds. It follows from (11a)–(11b) that $\nabla_{x_i} J_i(x_i^*, \sigma(x^*)) + A_i^T \bar{\lambda} = 0$, which satisfies (4). Thus, according to Lemma 2, the equilibrium of system (10) is the GNE of aggregative game G. And, vice versa.$\square$

Let $e_{i1} = \hat{\lambda}_i - \lambda_i$, $e_{i2} = \hat{\eta}_i - \eta_i$, $e_{i3} = \hat{r}_i - r_i$, and $e = \mathrm{col}(e_{i1}, e_{i2}, e_{i3})$. The triggering instants $\{t_k^i\}_{k=1}^{\infty}$ for agent i to broadcast λ_i, η_i, and r_i are determined by the following rule,

$$t_{k+1}^i = \inf\left\{t > t_k^i,\ \|e_i\| - \zeta_1 e^{-\zeta_2 t} \geqslant 0\right\}, \tag{12}$$

where $\zeta_{i1}, \zeta_{i2} > 0$. Define $e_1 = \mathrm{col}(e_{11}, \ldots, e_{N1})$, $e_2 = \mathrm{col}(e_{12}, \ldots, e_{N2})$, and $e_3 = \mathrm{col}(e_{13}, \ldots, e_{N3})$. The compact form of (10) is rewritten by

$$\begin{aligned}
\dot{x} &= v,\\
\dot{v} &= -kv - G(x, \eta) - A^T \lambda,\\
\dot{\lambda} &= -\alpha(L \otimes I_l)\lambda - z + (R^{-1} \otimes I_l)(Ax - d) - \alpha(L \otimes I_l)e_1,\\
\dot{z} &= \alpha(L \otimes I_l)\lambda + \alpha(L \otimes I_l)e_1,\\
\dot{\eta} &= -\beta(L \otimes I_r)\eta + (R^{-1} \otimes I_r)\frac{d}{dt}\phi(x) - \beta(L \otimes I_r)e_2,\\
\dot{r} &= -(L \otimes I_N)r - (L \otimes I_N)e_3.
\end{aligned} \tag{13}$$

Theorem 1 *Suppose that Assumptions 1-4 hold. The strategies of all agents who follow the designed seeking algorithm (6)–(9) with event-triggered communication scheme (12), can arrive at the unique GNE of aggregative game $G = (\mathcal{I}, \Omega, J)$, if the following conditions are satisfied.*

$$\underline{k} > \max\left\{\frac{a_1}{2\mu}, 2\kappa_1^2\right\}, \quad \alpha > \frac{2\xi_{\max}^2}{\varrho_2 \xi_{\min}}, \quad 0 < \beta < \frac{\sqrt{2\rho_1\rho_2\varepsilon_1\kappa_3}}{(\delta_1 - 3\sqrt{10}\|A\|\rho_1\rho_2\varepsilon_1)\varrho_2},$$

where $\underline{k} = \min\{k_1, \ldots, k_N\}$, $\xi_{\min} = \min\{\xi_1, \ldots, \xi_N\}$, $\xi_{\max} = \max\{\xi_1, \ldots, \xi_N\}$, $a_1 > 1 + \frac{\sqrt{2}}{\kappa_1}$, $\delta_1 = \min\left\{\mu\underline{k} - \frac{1}{2}, a_1\underline{k} - \frac{(a_1+1)^2\kappa_1^2}{2}\right\}$, $\rho_1 = \min\{\underline{k}, a_1 + 1\}$, $\rho_2 = \max\{\kappa_2, \|A\|\}$, *and* $\varepsilon_1 = \max\left\{a_1 + \frac{3}{2}, \frac{3}{2}\underline{k}^2\right\}$.

Proof. Denote $\chi = [x, v, \lambda, z, \eta]^T$,

$$f(\chi) = \begin{bmatrix} v \\ -kv - G(x,\eta) - A^T\lambda \\ -\alpha(L\otimes I_l)\lambda - z + ((R^*)^{-1}\otimes I_l)(Ax-d) - \alpha(L\otimes I_l)e_1 \\ \alpha(L\otimes I_l)\lambda + \alpha(L\otimes I_l)e_1 \\ -\beta(L\otimes I_r)\eta + ((R^*)^{-1}\otimes I_r)\frac{d}{dt}\phi(x) - \beta(L\otimes I_r)e_2 \end{bmatrix}.$$

and

$$w = \begin{bmatrix} 0 \\ 0 \\ ((R^{-1} - (R^*)^{-1})\otimes I_l)(Ax-d) \\ 0 \\ ((R^{-1} - (R^*)^{-1})\otimes I_r)\frac{d}{dt}\phi(x) \end{bmatrix}.$$

System (13) can be rewritten by

$$\dot{\chi} = f(\chi) + w. \tag{14}$$

First, the stability of system

$$\dot{\chi} = f(\chi) \tag{15}$$

is analyzed by the small-gain theorem. Then, the input-to-state stability of system (14) is analyzed. For (x, v)-subsystem in (15), consider a Lyapunov candidate function $V_1 = \frac{1}{2}(a_1\|v - v^*\|^2 + \|k(x - x^*) + v - v^*\|^2)$, where $a_1 > 0$. Let $\underline{k} = \min\{k_1, \ldots, k_N\}$. The derivative of V_1 with respect to time t is

$$\begin{aligned}\dot{V}_1 &= a_1(v - v^*)^T(\dot{v} - \dot{v}^*) + (k(x - x^*) + v - v^*)^T(k(\dot{x} - \dot{x}^*) + \dot{v} - \dot{v}^*)\\ &\leqslant -a_1\underline{k}\|v - v^*\|^2 - (a_1 + 1)(v - v^*)^T(G(x,\eta) - G(x^*,\eta^*)\\ &\quad - A^T(\lambda - \lambda^*)) - k(x - x^*)^T(G(x,\eta) - G(x^*,\eta^*) + A^T(\lambda - \lambda^*)).\end{aligned}$$

Under Assumption 3, we have that $-k(x - x^*)^T(G(x,\eta^*) - G(x^*,\eta^*)) \leqslant -\mu\underline{k}\|x - x^*\|^2$. By Assumption 4, it yields that $-(a_1 + 1)(v - v^*)^T(G(x,\eta^*) - G(x^*,\eta^*)) \leqslant \kappa_1(a_1 + 1)\|x - x^*\|\|v - v^*\|$. Thus we have that

$$\begin{aligned}\dot{V}_1 &\leqslant -a_1\underline{k}\|v - v^*\|^2 - \mu\underline{k}\|x - x^*\|^2 + (a_1 + 1)\kappa_1\|x - x^*\|\|v - v^*\|\\ &\quad - k(x - x^*)^T(G(x,\eta) - G(x,\eta^*) + A^T(\lambda - \lambda^*))\\ &\quad - (a_1 + 1)(v - v^*)^T(G(x,\eta) - G(x,\eta^*) + A^T(\lambda - \lambda^*)).\end{aligned}$$

By Young's Inequality, $(a_1 + 1)\kappa_1\|x - x^*\|\|v - v^*\| \leqslant \frac{1}{2}\|x - x^*\|^2 + \frac{(a_1 + 1)^2\kappa_1^2}{2}\|v - v^*\|^2$. Thus,

$$\begin{aligned}\dot{V}_1 &\leqslant -\delta_1\|\mathrm{col}(x - x^*, v - v^*)\|^2 + \sqrt{2}\rho_1\|\mathrm{col}(x - x^*, v - v^*)\|\\ &\quad \times \|G(x,\eta) - G(x,\eta^*) + A^T(\lambda - \lambda^*)\|\\ &\leqslant -\delta_1\|\mathrm{col}(x - x^*, v - v^*)\|^2 + \sqrt{2}\rho_1\rho_2\|\mathrm{col}(x - x^*, v - v^*)\|(\|\eta - \eta^*)\|\\ &\quad + \|\lambda - \lambda^*\|),\end{aligned}$$

where $\delta_1 = \min\left\{\mu\underline{k} - \frac{1}{2}, a_1\underline{k} - \frac{(a_1 + 1)^2\kappa_1^2}{2}\right\}$ with $\underline{k} > \max\left\{\frac{1}{2\mu}, 2\kappa_1^2\right\}$ and $a_1 > 1 + \frac{\sqrt{2}}{\kappa_1}$, $\rho_1 = \min\{\underline{k}, a_1 + 1\}$, and $\rho_2 = \max\{\kappa_2, \|A\|\}$. Recall the definition of Lyapunov function V_1, we have that $\|\mathrm{col}(x - x^*, v - v^*)\|^2 \leqslant V_1 \leqslant \varepsilon_1\|\mathrm{col}(x - x^*, v - v^*)\|^2$ with $\varepsilon_1 = \max\left\{a_1 + \frac{3}{2}, \frac{3}{2}\underline{k}^2\right\}$. Thus, we have that

$$\begin{aligned}\|\mathrm{col}(x - x^*, v - v^*)\| &\leqslant \sqrt{\varepsilon_1}\|\mathrm{col}(x - x^*, v - v^*)\|e^{-\frac{\delta_1}{2\varepsilon_1}t}\\ &\quad + \frac{\sqrt{2}\rho_1\rho_2\varepsilon_1}{\delta_1}\sup_{0\leqslant\tau\leqslant t}(\|\eta(\tau) - \eta^*\| + \|\lambda(\tau) - \lambda^*\|).\end{aligned} \tag{16}$$

For the (λ, z)-subsystem in (15), consider a Lyapunov candidate function $V_2 = \frac{1}{2}(\lambda-\lambda^*)^T(R^*\otimes I_l)(\lambda-\lambda^*)+\frac{1}{2}(\lambda-\lambda^*+z-z^*)^T(R^*\otimes I_l)(\lambda-\lambda^*+z-z^*)$. The derivative of V_2 with respect to time t is

$$\begin{aligned}\dot{V}_2 = &-\alpha(\lambda-\lambda^*)^T((R^*L+L^TR^*)\otimes I_l)(\lambda-\lambda^*)-2(\lambda-\lambda^*)^T(R^*\otimes I_l)(z-z^*)\\&+2(\lambda-\lambda^*)^TA(x-x^*)+(z-z^*)^TA(x-x^*)\\&-(z-z^*)^T(R^*\otimes I_l)(z-z^*)+\alpha(\lambda-\lambda^*)^T(R^*L\otimes I_l)e_1.\end{aligned}$$

Since digraph $\mathcal{G}$ is strongly connected, it yields that $-\frac{1}{2}\alpha(\lambda-\lambda^*)^T((R^*L+L^TR^*)\otimes I_l)(\lambda-\lambda^*)\leqslant -\alpha\varrho_2\|\lambda-\lambda^*\|^2$, where ϱ_2 is the second least eigenvalue of matrix $R^*L+L^TR^*$. Additionally, by Young's Inequality, $-2(\lambda-\lambda^*)^T(R^*\otimes I_l)(z-z^*)\leqslant \frac{2\xi_{\max}}{\xi_{\min}}\|\lambda-\lambda^*\|^2+\frac{1}{2}\xi_{\min}\|z-z^*\|^2$, where $\xi_{\max}=\max\{\xi_1,\ldots,\xi_N\}$ and $\xi_{\min}=\min\{\xi_1,\ldots,\xi_N\}$. Recall the definition of Lyapunov function V_2, we have that $\frac{1}{2}\|\mathrm{col}(\lambda-\lambda^*,z-z^*)\|\leqslant V_2\leqslant \frac{3}{2}\|\mathrm{col}(\lambda-\lambda^*,z-z^*)\|$. By the triggering rule (12), we have that

$$\begin{aligned}&\|\mathrm{col}(\lambda(t)-\lambda^*,z(t)-z^*)\|\\&\quad\leqslant\sqrt{3}\|\mathrm{col}(\lambda(0)-\lambda^*,z(0)-z^*)\|e^{-\frac{\delta_2}{3}t}\\&\qquad+2\sqrt{2}\|A\|\int_0^t e^{-(t-\tau)\frac{\delta_2}{3}}\|x(\tau)-x^*\|\mathrm{d}\tau+\frac{\alpha\rho_N}{2}\int_0^t e^{-(t-\tau)\frac{\delta_2}{3}}\|e_1\|\mathrm{d}\tau\\&\quad\leqslant\sqrt{3}\|\mathrm{col}(\lambda(0)-\lambda^*,z(0)-z^*)\|e^{-\frac{\delta_2}{3}t}+12\sqrt{2}\|A\|\sup_{0\leqslant\tau\leqslant t}\|x(\tau)-x^*\|+\varphi_1(t),\end{aligned}\tag{17}$$

where $\zeta_1=\min\{\zeta_{11},\ldots,\zeta_{N1}\}$, $\zeta_2=\min\{\zeta_{12},\ldots,\zeta_{N2}\}$, $\delta_2=\min\left\{\alpha\varrho_2-\frac{2\xi_{\max}^2}{\xi_{\min}},\frac{1}{2}\xi_{\min}\right\}$, and $\alpha>\frac{2\xi_{\max}^2}{\varrho_2\xi_{\min}}$. $\varphi_1(t)=\frac{3\zeta_1\alpha\varrho_N}{2(\delta_2-3\zeta_2)}(e^{-\zeta_2 t}-e^{-\delta_2/3t})$ if $\zeta_2\neq\delta_2/3$, and $\varphi_1(t)=\frac{\alpha\varrho_N\zeta_1}{2}te^{-\delta_2/3t}$ otherwise.

For the η-subsystem in (15), consider a Lyapunov candidate function $V_3=\frac{1}{2}(\eta-\eta^*)^T(R^*\otimes I_m)(\eta-\eta^*)$, where $\eta^*=1_N\otimes\sigma(x)$. The derivative of V_3 with respect to time t is

$$\begin{aligned}\dot{V}_2 = &-\beta(\eta-\eta^*)^T((R^*L+L^TR^*)\otimes I_r)(\eta-\eta^*)\\&+(\eta-\eta)^T\frac{d}{dt}(\phi(x)-I_N\otimes\sigma(x))-\beta(\eta-\eta^*)^T(R^*L\otimes I_r)e_2.\end{aligned}$$

By Assumption 2, we have that

$$\left\|\frac{d}{dt}(\phi(x)-(1_N\otimes\sigma(x)))\right\| \leqslant \|\nabla\phi(x)-\nabla(1_N\otimes\sigma(x)\|\|v-v^*\| \\ \leqslant \kappa_3\|v-v^*\|.$$

By the triggering rule (12), we have that

$$\|\eta(t)-\eta^*\| \leqslant \|\eta(0)-\eta^*\|e^{-\beta\varrho_2 t}+\frac{\kappa_3}{\beta\varrho_2}\sup_{0\leqslant\tau\leqslant t}\|v(\tau)-v^*\|+\varphi_2(t), \tag{18}$$

where $\varphi_2(t)=\dfrac{\beta\varrho_N\zeta_1}{\beta\varrho_2-\zeta_2}\left(e^{-\zeta_2 t}-e^{-\beta\varrho_2 t}\right)$ if $\zeta_2\neq\beta\varrho_2$ and $\varphi_2(t)=\beta\varrho_N\zeta_1 te^{-\beta\varrho_2 t}$ otherwise.

Thus, it follows from the results in (16), (17) and (18) that

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant t}\|\mathrm{col}(x(\tau)-x^*,v(\tau)-v^*)\| \\ &\leqslant\frac{1}{1-\gamma_1\gamma_2}\Big(\sqrt{\varepsilon_1}\big\|\mathrm{col}(x(0)-x^*,v(0)-v^*)\big\|e^{-\frac{\delta_1}{2\varepsilon_1}}+\gamma_1\sup_{0\leqslant\tau\leqslant t}\big(\|\eta(0)-\eta^*\|e^{-\beta\varrho_2 t} \\ &\quad+\sqrt{3}\|\mathrm{col}\left(\lambda(0)-\lambda^*,z(0)-z^*\right)\|e^{-\frac{\delta_2}{3}t}\big)\Big),\end{aligned} \tag{19}$$

where $\gamma_1=\dfrac{\sqrt{2}\rho_1\rho_2\varepsilon_1}{\delta_1}$, $\gamma_2=\dfrac{\kappa_3}{\beta\varrho_2}+12\sqrt{2}\|A\|$. According to Small-Gain Theorem in [26], if $\gamma_1\gamma_2<1$, system (15) can converge to its equilibrium point.

It follows from r-subsystem in (15) that R converges to R^* as $t\to\infty$. Thus, we know that w in (14) is bounded, which suggests that (14) is input-to-state stable. According to Lemma 4.7 in [27], system (14) asymptotically converges to its equilibrium point $(x^*,v^*,\lambda^*,z^*,\eta^*,r^*)$. Furthermore, by Lemma 3, the designed GNE seeking algorithm can converge to the GNE of aggregative game $G=(\mathcal{I},\Omega,J)$. □

Remark 1 *Different from the existing distributed continuous-time GNE seeking algorithm designed for multi-agent systems on undirected graphs and weight-balanced digraphs [6, 10, 12, 15–17, 20], the designed algorithm can be applied in more general case of unbalanced digraphs.*

Remark 2 *This work is an extension of [21] from connected and undirected graphs to weight-unbalanced directed graphs. Under the setup of weight-unbalanced directed communication networks, the design and the analysis of the GNE seeking algorithm become more difficult and challenging. Compared with [21], the designed algorithm (6)–(9) can be flexibly applied in different scenarios.*

Theorem 2 *The designed event-triggered communication scheme* (12) *is free of the Zeno behavior.*

Proof. The analysis of the Zeno behaviors is given as follows. For agent i, the dynamics of measurement error e_i is

$$\dot{e}_{i1} = -\dot{\lambda}_i\,, \quad \dot{e}_{i2} = -\dot{\mu}_i\,, \quad \dot{e}_{i3} = -\dot{r}_i\,. \tag{20}$$

$$\begin{aligned}
\frac{d\|e_i\|}{dt} \leqslant \left\|\frac{de_i}{dt}\right\| &\leqslant \left\|\frac{de_{i1}}{dt}\right\| + \left\|\frac{de_{i2}}{dt}\right\| + \left\|\frac{de_{i3}}{dt}\right\| \\
&\leqslant \alpha\|(L\otimes I_l)\|\|\lambda-\lambda^*\| + \|z-z^*\| + \|A\|\|x-x^*\| + \alpha\|(L\otimes I_l)\|\|e_1\| \\
&\quad + \beta\|(L\otimes I_r)\|\|\eta-\eta^*\| + \left\|\frac{d}{dt}(\phi(x) - I_N\otimes\sigma(x))\right\| + \|(L\otimes I_N)\|\|r\| \\
&\quad + \beta\|(L\otimes I_r)\|\|e_2\| + \|(L\otimes I_N)\|\|e_3\|.
\end{aligned}$$

According to the above analysis, there is a positive constant D such that

$$\frac{d\|e_i\|}{dt} \leqslant D. \tag{21}$$

Assume that t_k^i is the latest triggering instant. The next triggering instant is the time when the inequality $\|e_i\| \leqslant \zeta_{i1}e^{-\zeta_{i2}t}$ is violated. Thus, from (21), we have that

$$\zeta_{i1}e^{-\zeta_{i2}t_{k+1}^i} \leqslant \|e_i\| \leqslant D(t_{k+1}^i - t_k^i), \tag{22}$$

which implies that $(t_{k+1}^i - t_k^i) > 0$ for any finite horizon. If this is not true, by denoting $t_\infty^i = \lim_{k\to\infty} t_k^i$, then $t_\infty^i < \infty$ and moreover $\lim_{k\to\infty}(t_{k+1}^i - t_k^i) = 0$. Thus, it is known from (22) that, for $k \to \infty$, $0 < \zeta_{i1}e^{-\zeta_{i2}t_\infty^i} \leqslant 0$, which is a contradiction. As a result, no Zeno behavior is exhibited. □

Remark 3 *Compared with [22], the significant differences are coupled constraints considered in this paper and the event-triggered communication scheme for reducing communication loads. In addition, double-integrator agents are considered in this paper, which is different from single-integrator agents considered in [22]. It is known that agents with complex dynamics bring the difficulties in the design and analysis of the designed algorithm.*

Remark 4 *In contrast to event-triggered communication schemes presented in [6, 20, 28], our designed event-triggering rule* (12) *only depends on agents own information rather than neighbors' broadcasting information. As a result, under the event-triggering rule* (12)*, agents do not monitor neighbors' information in real-time.*

4. Simulations

In this section, the demand management of distributed energy resources considered as an example is used to test the proposed distributed GNE seeking algorithm. Regard the distributed energy resources as agents in the network to compose a multi-agent system. The dynamics of distributed energy resources can be reduced by (2). The communication topology is depicted in Fig. 1. For agent i, the cost function is given by

$$J_i(x_i, \sigma(x)) = (x_i - b_i)^2 + (c_0 + cN\sigma(x))x_i\,,$$

where $[b_1, \ldots, b_6]^T = [50, 55, 60, 65, 70]^T$, $c_0 = 5$, and $c = 0.04$. The coupled constraint is described by $\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} b_i - 40$. The parameters in the designed distributed event-triggered GNE seeking algorithm (6)–(9) are set by $k_i = 5$, $\alpha = 2$, and $\beta = 15$ for all $i \in \{1, \ldots, 5\}$. In addition, $\zeta_1 = [0.4, 0.8, 0.4, 0.51, 1]^T$ and $\zeta_2 = [0.2, 0.15, 0.2, 0.15, 0.5]^T$ are chosen in the event-triggered condition (12). Figure 2 depicts the evolution of all agents' strategies by following the designed distributed algorithm (6)–(9) with event-triggered communication scheme (12). It indicates that the strategies of all agents can arrive at the GNE of aggregative game $G = (\mathcal{I}, \Omega, J)$ with discrete-time communication. For the designed event-triggered communication scheme, the event intervals of the five agents are analyzed in Table 1. Figure 3 shows the triggering instants of the five agents and it indicates that the Zeno behavior is excluded.

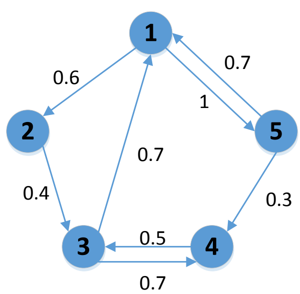

Figure 1: Communication graph among the five agents

Table 1: Event intervals of the five agents

Agent	1	2	3	4	5
Event times	151	122	114	147	168
Min interval	0.01	0.01	0.01	0.01	0.01
Mean interval	0.121	0.118	0.122	0.113	0.112
Max interval	0.65	0.74	0.56	0.63	0.47

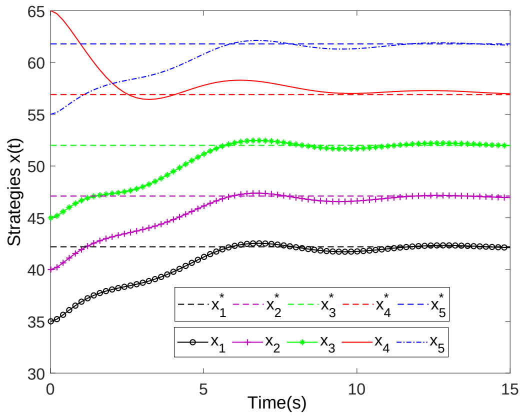

Figure 2: Evolutions of strategies updated by event-triggered GNE seeking algorithm (6)–(9)

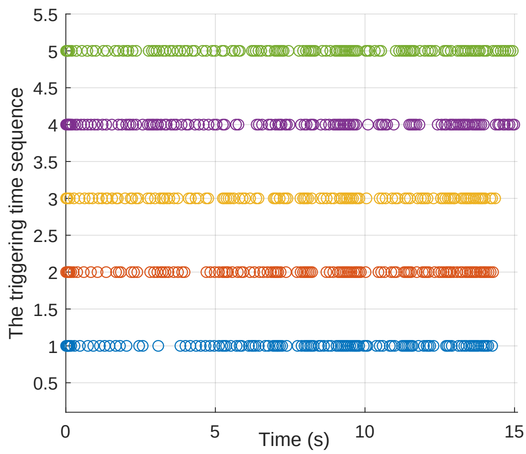

Figure 3: Triggering instants of the five agents

5. Conclusion

In this paper, a distributed event-triggered GNE seeking algorithm was firstly proposed for aggregative games with coupled constraints on an unbalanced graph. To deal with the unbalance incurred by communication graphs, an auxiliary state for estimation of the left eigenvector of the Laplacian matrix associated with zero eigenvalue is introduced in the designed algorithm. Moreover, an event-triggered broadcasting scheme was designed to realize the discrete-time interactions among agents. The convergence of the designed seeking algorithm with discrete-time communications was analyzed by the stability of the perturbed system and the small-gain approach. It was also shown that the proposed event-triggered communication scheme did not exhibit the Zeno behavior. A numerical example was presented to demonstrate the effectiveness of the proposed method.

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