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# Pricing Marriage Insurance with Mortality Dependence

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**Abstract**

Accurate determination of the probability structure of the multistate model is significant from the valuation and profitability assessment of insurance contracts standpoint. This article aims to analyse the effect of spouses' future lifetime dependence on premiums and prospective reserves for marriage insurance contracts. As a result, under the assumptions that the evolution of the insured risk is described by a nonhomogeneous Markov chain and the dependence between spouses' future lifetime is modelled by the copula, we derive formulas for the elements of the transition matrices. Based on actual data, we conduct a comparative analysis of actuarial values for three scenarios related to future lifetimes of husband and wife. We test the robustness of premium value to the changing degree of dependency between spouses' future lifetimes.

**Keywords:** dependent lifetimes, modified multistate model, copula, joint-life status, last surviving status

**JEL Classification**: G22, J22

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# 1 Introduction

Insurers offer contracts to provide a contingent payment on the death of an individual (death benefit) in return for a single or a series of periodical payments (premiums). These life insurance contracts are designed to protect against the serious financial impact that results from a person's death. An important variation of individual life insurance is the so-called joint life insurance, where multiple lives are involved.

In this paper, we concentrate on marriage insurance, which is a special case of multilife insurance contract. We distinguish between two types of such contracts: *joint-life status insurance* issued to a married couple where the death benefit is paid if husband or wife dies, and *last surviving status insurance* by which the lump sum is paid after the death of each of the spouses. The analysis of marriage insurance contracts is usually based on the assumption of the independence between the insured. In contrast to this classical approach, the dependence of future lifetimes between the spouses is assumed. It is a more realistic assumption than independence because the husband and the wife are exposed to the same risks, which cause the dependence of their future lifetimes. We may observe the influence of the death of a spouse on the future lifetime of the second one. The 'broken heart syndrome' occurs in such a situation. Moreover, a common external event called 'shock' e. g., the car or aeroplane crash, which causes the death of both spouses may occur. We applied copulas to model the dependence structure of the length of the spouses' lives. The modelling of spouses' lifelengths taking into account the dependence by the use of copula was considered, for example in Carriere (2001); Gourieroux and Lu (2015); Luciano et al. (2008); Luciano et al. (2016), and Spreeuw (2006). Moreover, in the article Gourieroux and Lu (2015), Luciano et al. (2016) and Spreeuw (2006), the authors have applied these copulas to determine the actuarial value of the different kinds of life annuities. However, according to our discernment, no application of these copulas to the valuation of the marriage life insurance contract consisting of different types of benefits (lump sum and annuity benefits) has been found in the literature. We hope that this article will bridge this gap in the research of marriage insurance.

We focus on a discrete-time model, where insurance payments are made at the ends of time units. We assume that the evolution of the insured risk is described by the time-nonhomogeneous Markov chain. Moreover, actuarial values are considered under the assumption of stochastic interest rates.

The purpose of the article is twofold.

In the first part, we applied and developed a matrix representation to the analysis of all types of future cash flows arising from the marriage insurance contract. For this purpose, we derive transition probabilities under the assumption that future lifetimes of spouses are dependent and such dependence is modelled by a copula. The appropriate accommodation of the modified multiple state model enables us to obtain matrix formulas for actuarial values of multilife insurance contract under assumptions on the dependence between the insured. Matrix notation allows for efficient analysis of the stochastic structure of the model and cash flows resulting from the realisation of

contracts. Moreover, it provides a compact form for both the joint-life status and the last surviving status. In particular, this tool will facilitate the analysis of the impact of the probabilistic structure of the model (with dependence and independence) of marriage insurances for actuarial values, such as premiums and reserves.
In the second part, we focus on empirical comparative analysis. In particular, we looked at how the way the relationship between the future lifetime of spouses is modelled affects:

i) the probabilistic structure of the modified multistate model, which is important in determining actuarial values,

ii) the pricing of the insurance contract,

iii) insurance reserves to provide liquidity to the insurer, as well as a guarantee of stability.

The paper is organised as follows. In Section 2, after a brief description of multiple state model for marriage insurance according to two statuses, we present the main result in Theorem 1, where formulas for elements of the transition matrix under the assumption that future lifetimes of spouses are dependent and modelled by copulas are derived. The matrix approach to pricing marriage insurances is developed in Section 3. Some numerical examples based on actual data for the Czech Republic are given in Section 4. Suggestions for further possible applications of obtained results are presented in Section 5.

# 2 Marriage insurance

## 2.1 Multiple state model

Let $x$ and $y$ be the *age at a policy issue* of the husband and wife, respectively. Moreover, let $T_x^M$ and $T_y^W$ be the *remaining lifetimes* of $x$-year-old man (husband) and $y$-year-old woman (wife). These lifetimes take values in $[0, w_x^M]$ and $[0, w_y^W]$, where $w_x^M$ (resp. $w_y^W$) denotes the difference between the border age $\omega$ (for example, 100, 105 or 110 years old) of the man (resp. woman) and $x$ (resp. $y$).
In this paper, we consider an insurance contract issued at time 0 (defined as the time of issue of the insurance contract) and terminating according to the plan at a later time $n$, which is called the *term of policy* or the *insurance period*. The insurance period depends on the status:

i) $n = \max\left\{w_x^M, w_y^W\right\}$ for the last surviving status,

ii) $n = \min\left\{w_x^M, w_y^W\right\}$ for the joint-life status.

Let us note that, irrespective of the status, we are indeed dealing with whole-life insurance contracts, because in this article we have chosen to equate term of policy with upper bound for insurance period for technical reasons. Firstly, the life of the insured as determined by the life tables is limited by the border age, and secondly, we use a matrix notation for calculating actuarial quantities, and the matrix needs the determination of its dimensions. This means that at the time of taking out the insurance contract, the theoretical length of the insurance period is determined on the basis of the possible maximum age to which the husband and wife can live (included in the life tables). For example, if the husband is 65, the wife 60 years old and $\omega = 100$, then

i) $n = \max\left\{w_x^M, w_y^W\right\} = \max\{100 - 65; 100 - 60\} = \max\{35; 40\} = 40$ for the last surviving status,

ii) $n = \min\left\{w_x^M, w_y^W\right\} = \min\{100 - 65; 100 - 60\} = \min\{35; 40\} = 35$ for the joint-life status.

Following Haberman and Pitacco (1999), with a given insurance contract, we assign a *multiple state model*. That is, at any time, the insured risk is in one of a finite number of states labelled by $1, 2, \ldots, N$. Let $S = \{1, 2, \ldots, N\}$ be the *state space*. Each state corresponds to an event that determines the cash flows (premiums and benefits). Additionally, by $T = \{(i, j) : i, j \in S\}$ we denote a *set of direct transitions* between states of the state space. The pair $(S, T)$ is called a *multiple state model* and describes all possible insured risk events up to the end of insurance.

The graphic representation of the multiple state model $(S, T) = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\})$ for marriage insurance is shown in Figure 1, where circles represent the states and arcs correspond to direct transitions between them. Next to the arcs, benefits related to the transition between states are marked, where $c^M$ $(c^W)$ is a given lump sum paid when the husband (wife) dies and $c$ is the lump sum paid if both spouses die.

The length of the insurance contract with JLS is shorter, and the decisive influence on the duration of the insurance contract is the future lifetime of the older spouse, while for LSS the decisive influence is the future lifetime of the younger partner (therefore, the length of the insurance contract is usually greater for this status). The time horizon over which benefits are realized (the potential length of the insurance contract) have a decisive impact on actuarial quantities such as premiums and reserves. For both statuses, the insurance benefit is an increasing function of age. This indicates that (in terms of present value at the start of the insurance contract) the unit benefit payable immediately on the death of the first spouse is greater than the same payable after the death of a widow/widower. This is because the variation in the expected time until the death in JLS is much smaller than LSS, and as such, insurance payments on lives with shorter time until death pay more than those with more time until death. Note that in LSS the sum of nominal payments is always equal to $c^M + c^W$, while for JLS it might be equal to $c^M$ or $c^W$ (death of one spouse) or $c^M + c^W$ (simultaneous death

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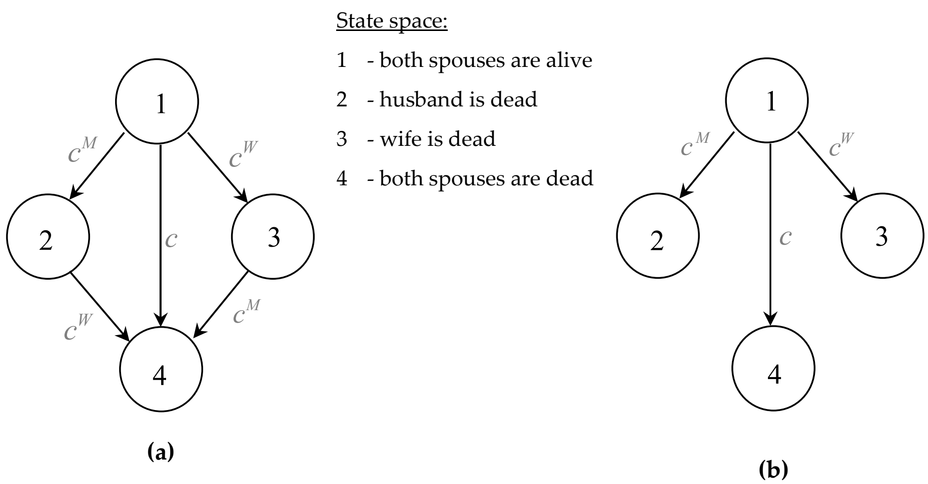
Figure 1: A multiple state model for marriage insurance contract for: (a) Last Surviving Status – LSS; (b) Joint-Life Satus - JLS




of both spouses). It follows directly from that the multiple state model for marriage insurance with JLS (diagram (b) in Figure 1) is included in the multiple state model with LSS (diagram (a) in Figure 1). This means that insurance premiums for LSS insurance are higher than those for JLS. In this paper, we focus on the impact of the assumptions made regarding the future lifetimes of the spouses on the actuarial values, and not on comparing these values depending on the statuses. Therefore, in further considerations, we focus on the more popular in practice insurance contact, namely marital insurance with the status of the last survivor.

## 2.2 Transition matrix for mortality dependence

We focus on a discrete-time model, where $X(k)$ denotes the state of an individual (the contract) at time $k$ ($k \in \{0, 1, 2, \ldots, n\}$). Hence the evolution of the insured risk is described by a discrete-time stochastic process $\{X(k); k = 0, 1, 2, \ldots\}$ (in this article we will use the short form $\{X(k)\}$). The basic quantities describing the evolution of process $\{X(k)\}$ are finite dimensional distributions, which allows for the determination of the conditional distributions (necessary in the valuation of each contract). Finite dimensional distributions depend directly on the type of contract. For example, in marriage insurance, the distribution is influenced by husband's further lifetime as well as wife's. In practice, determining these probabilities is often very difficult due to the limited access to data. Therefore, the process $\{X(k)\}$ is assumed to be a nonhomogeneous Markov chain (see, e.g. Denuit et al. 2001; Dębicka 2013; Hoem 1969; Hoem 1988; Wolthuis 1994). Although this is quite a considerable simplification, because, for example, in the case of marital insurance, both the future lifetime of the husband and the wife shortens after the death of the spouse (broken heart syndrome), but as research show, after 2-3 years it returns to the level specified in life tables e.g. Ji et al. (2011). From the methodological point of view, under this assumption we have described the probabilistic structure of the model based on a chain of transition matrices $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$, where $\mathbf{Q}(k) = (q_{ij}(k))_{i,j=1}^{N}$ and $q_{ij}(k) = P(X(k+1) = j|X(k) = i)$.

In the case of marriage insurance with the last surviving status, the transition matrix has the following form

$$\mathbf{Q}(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) \\ 0 & q_{22}(k) & 0 & q_{24}(k) \\ 0 & 0 & q_{33}(k) & q_{34}(k) \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

For the joint-life status, we have $q_{22}(k) = q_{33}(k) = 1$ and $q_{24}(k) = q_{34}(k) = 0$.

Under the assumption that $T_x^M$ and $T_y^W$ are independent, transition probabilities are determined using only life tables, where $l_x^M$ ($l_y^W$) is the number of living x-years-old men (y-years-old women) out of initial $l_0^M$= 10 000 men (women).

If we want to derive the transition probabilities $q_{ij}(k)$, we must know the joint distribution of lifetimes ($T_x^M$, $T_y^W$). We assume that we know the cumulative distribution function $F(w,z)$ of pair ($T_{x_0}^M$,$T_{y_0}^W$), where $x_0$ and $y_0$ are base ages for men and women, such that $x = x_0 + t$, $y = y_0 + s$, $t \geq 0$ and $s \geq 0$. We also know the copula $C(u,v)$, a link between the joint and marginal cumulative distribution functions (Genest and McKay 1986; Nelsen 1999), connected with this joint distribution, i.e. $F(w,z) = C(F_M(w), F_W(z))$, where $F_M(w) = P(T_{x_0}^M \leq w)$ and $F_W(z) = P(T_{y_0}^W \leq z)$ are the marginal distributions of the lifetimes $T_{x_0}^M$ and $T_{y_0}^W$. However, in the life insurance analyses, we mainly use the survival functions: $S(w,z) = P(T_{x_0}^M > w, T_{y_0}^W > z)$, $S^M(w) = P(T_{x_0}^M > w)$, $S^W(z) = P(T_{y_0}^W > z)$ and the survival copula $C^*(u,v)$, which satisfies the condition $S(w,z) = C^*(S^M(w), S^W(z))$.

There is a following relation between classical $C$ and survival copula $C^*$ (Nelsen 1999):

$$C^*(u,v) = u + v - 1 + C(1-u, 1-v). \tag{2}$$

We can compute the marginal survival functions $S^M$ and $S^W$ using the values of $l_{x_0}^M$ and $l_{y_0}^W$ from life tables:

$$S^M(t) = P(T_{x_0}^M > t) = P(T_0^M > x_0 + t|T_0^M > x_0) = l_{x_0+t}^M / l_{x_0}^M, \tag{3}$$

$$S^W(s) = P(T_{y_0}^W > s) = P(T_0^W > y_0 + s|T_0^W > y_0) = l_{y_0+s}^W / l_{y_0}^W. \tag{4}$$

Hence, the join survival function of the lifetimes $T_x^M$ and $T_y^W$ are equal

$$P(T_x^M > w, T_y^W > z) = P(T_{x_0}^M > t + w, T_{y_0}^W > s + z | T_{x_0}^M > t, T_{y_0}^W > s) = \\ = \frac{S(t+w, s+z)}{S(t,s)} = \frac{C^*(S^M(t+w), S^W(s+z))}{C^*(S^M(t), S^W(s))}. \tag{5}$$

In our later analyses, we will also use the following probability

$$\begin{aligned} P(T_x^M > w, v < T_y^W \leq z) = \\ &= P(T_{x_0}^M > t + w, s + v < T_{y_0}^W \leq s + z | T_{x_0}^M > t, T_{y_0}^W > s) = \\ &= \frac{P(T_{x_0}^M > t + w, T_{y_0}^W > s + v)}{P(T_{x_0}^M > t, T_{y_0}^W > s)} - \frac{P(T_{x_0}^M > t + w, T_{y_0}^W > s + z)}{P(T_{x_0}^M > t, T_{y_0}^W > s)} = \\ &= \frac{C^*(S^M(t+w), S^W(s+v)) - C^*(S^M(t+w), S^W(s+z))}{C^*(S^M(t), S^W(s))}. \end{aligned} \tag{6}$$

In a similar way, we obtain

$$\begin{aligned} P(u < T_x^M \leq w, v < T_y^W \leq z) = \\ &= \frac{C^*(S^M(t+u), S^W(s+v)) - C^*(S^M(t+u), S^W(s+z))}{C^*(S^M(t), S^W(s))} + \\ &+ \frac{C^*(S^M(t+w), S^W(s+z)) - C^*(S^M(t+w), S^W(s+v))}{C^*(S^M(t), S^W(s))}. \end{aligned} \tag{7}$$

When the random variables $T_x^M$ and $T_y^W$ are independent, then we have

$$C^*\left(S^M(w), S^W(z)\right) = S^M(w) \cdot S^W(z). \tag{8}$$

Moreover, for $x \geq \omega$ or $y \geq \omega$ we obtain

$$C^*\left(S^M(t), S^W(s)\right) = 0. \tag{9}$$

**Theorem 1.** *Assume that the copula $C^*(x_0, y_0)$ is the link between the joint and marginal survival functions for $x_0$-year-old men and $y_0$-year-old women. In addition, $\{X(t)\}$ is a nonhomogeneous Markov chain describing the evolution of the insured risk in the multistate model for the marriage insurance contract with the last surviving status, i.e. $(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\})$. Then the elements of the chain of the transition matrices $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$ have the following form:*

*a) for $k \in \{0,1,2,\ldots,n-1\}$ and $x + k < \omega$ and $y + k < \omega$*

$$q_{11}(k) = \frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)}, \tag{10}$$

$$q_{12}(k) = \frac{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k+1)\right) - C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(y+k)\right)}, \tag{11}$$

$$q_{13}(k) = \frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k)\right) - C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)}, \tag{12}$$

$$q_{14}(k) = 1 - \sum_{j=1}^{3} q_{1j}(k), \tag{13}$$

$$q_{22}(k) = \frac{C^*\left(S_{x_0}^M(t), S_{y_0}^W(s+k+1)\right) - C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t), S_{y_0}^W(s+k)\right) - C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)}, \tag{14}$$

$$q_{24}(k) = 1 - q_{22}(k), \tag{15}$$

$$q_{33}(k) = \frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s)\right) - C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s)\right) - C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)}, \tag{16}$$

$$q_{34}(k) = 1 - q_{33}(k); \tag{17}$$

b) *for* $k \in \{1, 2, \dots, n-1\}$ *and* $x + k \geq \omega$ *and* $y + k < \omega$*, elements in the first and third row of the matrix* (1) *equal zero and* $q_{22}(k)$*,* $q_{24}(k)$ *are defined by* (14) *and* (15)*, respectively;*

c) *for* $k \in \{1, 2, \dots, n-1\}$ *and* $x + k < \omega$ *and* $y + k \geq \omega$ *elements in the first and second row of the matrix* (1) *equal zero and* $q_{33}(k)$*,* $q_{34}(k)$ *are defined by* (16) *and* (17)*, respectively.*

*Proof.* For elements in the first row of the matrix (1), under the assumption that $x + k < \omega$ and $y + k < \omega$, we have

$$q_{1i}(k) = P(X(k+1) = i | X(k) = 1) = \frac{P(X(k+1) = i, X(k) = 1)}{P(X(k) = 1)}. \tag{18}$$

Note that $P(X(k) = 1) = P\left(T_x^M > k, T_y^W > k\right)$ and from (5) we straightforwardly obtain

$$P(X(k) = 1) = \frac{C^*\left(S^M(t+k), S^W(s+k)\right)}{C^*\left(S^M(t), S^W(s)\right)}. \tag{19}$$

Note that event $\{X(k+1) = 1\}$ is contained in $\{X(k) = 1\}$, so the joint probability reduces to $P(X(k+1)=1)$. Then for $i=1$

$$\begin{aligned} P(X(k+1)=1, X(k)=1) &= P\left(T_x^M > k+1, T_y^W > k+1, T_x^M > k, T_y^W > k\right) = \\ &= P\left(T_{x_0}^M > t+k+1, T_{y_0}^W > s+k+1 | T_{x_0}^M > t, T_{y_0}^W > s\right) = \\ &= \frac{P\left(T_{x_0}^M > t+k+1, T_{y_0}^W > s+k+1\right)}{P\left(T_{x_0}^M > t, T_{y_0}^W > s\right)}, \end{aligned} \tag{20}$$

again using (5) in (20) we have

$$P(X(k+1)=1, X(k)=1) = \frac{C^*\left(S^M(t+k+1), S^W(s+k+1)\right)}{C^*\left(S^M(t), S^W(s)\right)}. \tag{21}$$

Finally, applying (19) and (21) to (18), we obtain (10). Moreover, (13) results from the property of the transition matrix. $\mathbf{Q}(k)$ is a stochastic matrix; therefore, the sum of its elements in each row equals 1.
Namely, for $i=2$

$$\begin{aligned} &P(X(k+1)=2, X(k)=1) = \\ &\quad = P\left(T_x^M \leq k+1, T_y^W > k+1, T_x^M > k, T_y^W > k\right) = \\ &\quad = P\left(t+k < T_{x_0}^M \leq t+k+1, T_{y_0}^W > s+k+1 | T_{x_0}^M > t, T_{y_0}^W > s\right) = \\ &\quad = \frac{P\left(t+k < T_{x_0}^M \leq t+k+1, T_{y_0}^W > s+k+1\right)}{P\left(T_{x_0}^M > t, T_{y_0}^W > s\right)}, \end{aligned} \tag{22}$$

using (5) and (6) in equation (22), we have

$$\begin{aligned} &P(X(k+1)=2, X(k)=1) = \\ &\quad = \frac{C^*\left(S^M(t+k), S^W(s+k+1)\right) - S^M(t+k+1), S^W(s+k+1)}{C^*\left(S^M(t), S^W(s)\right)}. \end{aligned} \tag{23}$$

Finally, applying (19) and (23) to (18), we obtain (11).
In case of $i=3$

$$\begin{aligned} P(X(k+1)=3, X(k)=1) &= P\left(T_x^M > k+1, T_y^W \leq k+1, T_x^M > k, T_y^W > k\right) = \\ &= \frac{P\left(T_{x_0}^M > t+k+1, s+k < T_{y_0}^W \leq s+k+1\right)}{P\left(T_{x_0}^M > t, T_{y_0}^W > s\right)}, \end{aligned} \tag{24}$$

using (5) and (6) in (24), we have

$$\begin{aligned} &P(X(k+1)=2, X(k)=1) = \\ &\quad = \frac{C^*\left(S^M(t+k+1), S^W(s+k)\right) - S^M(t+k+1), S^W(s+k+1)}{C^*\left(S^M(t), S^W(s)\right)}. \end{aligned} \tag{25}$$

Analogously to (11), putting (19) and (25) to (18), we obtain (12).
For elements in the second row of the matrix (1), under the assumption that $x+k<\omega$ and $y+k<\omega$ we have

$$q_{2i}(k) = P(X(k+1)=i|X(k)=2) = \frac{P(X(k+1)=i, X(k)=2)}{P(X(k)=2)} \tag{26}$$

and according to the model $(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\})$, only for $i=2,4$ transition probabilities are possible and non-zero. Note that $P(X(k)=2) = P\left(T_x^M \le k, T_y^W > k\right)$, and from (6), we straightforwardly obtain

$$P(X(k)=2) = \frac{C^*\left(S^M(t), S^W(s+k)\right) - C^*\left(S^M(t+k), S^W(s+k)\right)}{C^*\left(S^M(t), S^W(s)\right)}. \tag{27}$$

Moreover,

$$\begin{aligned}
&P(X(k+1)=2, X(k)=2) = \\
&= P\left(T_x^M \le k+1, T_y^W > k+1, T_x^M \le k, T_y^W > k\right) = \\
&= P\left(T_{x_0}^M \le t+k+1, T_{y_0}^W > s+k+1, T_{x_0}^M \le t+k, T_{y_0}^W > s+k | T_{x_0}^M > t, T_{y_0}^W > s\right) = \\
&= \frac{P\left(t < T_{x_0}^M \le t+k, T_{y_0}^W > s+k+1\right)}{P\left(T_{x_0}^M > t, T_{y_0}^W > s\right)},
\end{aligned} \tag{28}$$

again using (6) in (28) we have

$$\begin{aligned}
&P(X(k+1)=2, X(k)=2) = \\
&\qquad = \frac{C^*\left(S^M(t), S^W(s+k+1)\right) - C^*\left(S^M(t+k), S^W(s+k+1)\right)}{C^*\left(S^M(t), S^W(s)\right)}.
\end{aligned} \tag{29}$$

Finally, applying (27) and (29) to (26), we obtain (14). Formula (15) is a natural consequence of the property of a transition matrix.
Similarly to proving (14) and (15), we obtain formulas (16) and (17), respectively.
For $k$ such that $x+k \ge \omega$ or $y+k \ge \omega$, matrices $\mathbf{Q}(k)$ consist of rows that have only zero elements. It is so because after the death of one spouse, the model degenerates into model ({2,4},{(2,4)}) for the death of the wife, and model ({3,4},{(3,4)}) for the death of the husband. This explains the form of the transition matrix elements defined in b) and c), respectively. □

**Corollary 2.** *Assume that copula $C^*(x_0, y_0)$ is the link between the joint and marginal survival functions for $x_0$-year-old men and $y_0$-year-old women. In addition, $\{X(t)\}$ is a nonhomogeneous Markov chain describing evolution of the insured risk in the multistate model for the marriage insurance contract with joint-life status i.e.*

$(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\})$. *Then the elements of the chain of the transition matrices* $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$ *have the following form for* $k \in \{0,1,2,\ldots,n-1\}$*:*

$$q_{11}(k) = \frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)},$$

$$q_{12}(k) = \frac{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k+1)\right) - C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(y+k)\right)},$$

$$q_{13}(k) = \frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k)\right) - C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)},$$

$$q_{14}(k) = 1 - \sum_{j=1}^{3} q_{1j}(k),$$

$$q_{ij}(k) = \begin{cases} 1 \text{ for } i=j \text{ and } i,j \in \{2,3,4\}, \\ 0 \text{ for } i \neq j \text{ and } i,j \in \{2,3,4\}. \end{cases} \tag{30}$$

*Proof.* For the marriage insurance contract with joint-life status, we have $n = \min\left\{w_x^M, w_y^W\right\}$. It means that we analyse $\mathbf{Q}(k)$ for $k = 0,1,\ldots,\min\left\{w_x^M, w_y^W\right\} - 1$, and $x + k < \omega$ or $y + k < \omega$. Then the first row of $\mathbf{Q}(k)$ matrix is the same as in case of last survival status i.e. is defined by (10)–(13) in Theorem 1.
Note that in the model $(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\})$ states $2,3,4$ are absorbing states, which explains (30). □

Note that under the assumption that $T_x^M$ and $T_y^W$ are independent, the copula has the form of $C(u,v) = uv$. Then using Theorem 1 and Corollary 2, we may obtain transition matrix (1) straightforwardly under the assumption that future lifetimes of husband and wife are independent.

# 3 Matrix approach to counting actuarial value

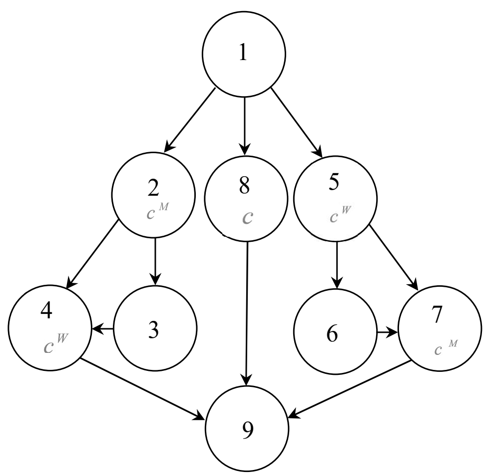
## 3.1 Modified multiple state model and its probabilistic structure

Multiple state modelling is a stochastic tool for designing and implementing insurance products. The multistate methodology is commonly used in the calculation of actuarial values of different types of life and health insurances; hence we also used it to determine premiums and reserves for marriage insurance contracts with mortality dependence.
The total future cash flows $Z$ that arise from insurance contract can be treated as a set of separate streams of cash flows of particular types (e.g. premiums, annuities,

lump sums). This way of valuation of the insurance contract represents an *actuarial* approach to analysing cash flows arising from the insurance contract. From a *financial* point of view, $Z$ is treated as a sum of current values of the total cash flows realised at individual moments of the duration of the insurance contract, whereby the total cash flow at any given time is the sum of premiums and different kinds of benefits realised at a given time in the insurance period. The distinction between these two approaches to the analysis $Z$ is important when the insurance contract guarantees lump sum benefits directly related to the occurrence of a random event covered by the insurance contract. For the lump sum, the information that the insured risk is in a particular state at the moment $k$ is not enough to determine the benefit at the moment $k$ because one needs additional information about the state of the insured risk at time $k-1$. A general financial approach to calculation of two moments of $Z$ arising from a multistate insurance contract can be found in, e.g. Dębicka (2013). This methodology, developed for the discrete-time model (where insurance payments are exercised at the end of time intervals), is based on an modified multiple state model, for which matrix formulas for actuarial values can be derived. This approach to costing contracts not only makes calculations easier, but also enables us to factorise the stochastic nature of the evolution of the insured risk and the interest rate, which can be observed in the derived formulas.

Note that all benefits under the marriage insurance contract correspond to cash flows connected with the transition between states (compare Figure 1). Observe that for such a type of benefit, the information that $X(k)$ is in a particular state is not enough to determine the benefit at time$k$, because we need additional information about where the process $\{X(k)\}$ was at the moment $k-1$. Matrix is a two-dimensional structure; thus, it is not possible to determine the exact moment of realisation of lump sum benefit by using three pieces of information (value of the benefit; $X(k-1), X(k)$). For this purpose, we have to modify the model for the marriage insurance contract with the last surviving status, replacing lump sum benefits with benefits associated with the insured risk staying in particular states. Following the procedure of modification the multiple state model presented in Dębicka (2013), we introduce five reflex states, which are strictly transitional and after one unit of time, the insured risk leaves this state, e.g. states 2, 4, 5, 7, 8. Thus the new state space consists of nine states ($N^* = 9$). As a result, the *modified multiple state model* $(S^*, T^*)$ for marriage insurances assumes the form presented in Figure 2. Note that now the benefits are the cash flows connected with the process $\{X(k)\}$ staying in considered states. The information in a particular state uniquely determines if the benefit is paid at time $k$. In particular, we have state 2 and state 3 corresponding to the situation that the wife lives and the husband is dead (compare state 2 in the diagrams of Figure 1). The difference between these two states (2 and 3) is related to the period that elapses from the death of her husband which directly affects the payment of benefits to his wife. Namely, in the first year after the death of the husband, the wife receives the lump sum benefit (state 2), and in the subsequent years of widowhood, the lump sum benefit is not paid.

Figure 2: Modified multiple state model for marriage insurance contract




State space:

1 - both spouses are alive

2 - death of the husband, wife is alive

3 - husband has been dead for at least one year, wife is alive

4 - husband has been dead for at least one year, death of the wife

5 - death of the wife, husband is alive

6 - wife has been dead for at least one year, husband is alive

7 - wife has been dead for at least one year, death of the husband

8 – death of both spouses within year

9 - both spouses have been dead for at least one year

Let the evolution of the insured risk for $(S^*, T^*)$ be described by a discrete-time stochastic process $\{X^*(k);\ k = 0, 1, 2, \ldots\}$. In Section 2, we assumed that $\{X(k)\}$ is a nonhomogeneous Markov chain, then the process$\{X^*(k)\}$ is also a nonhomogeneous Markov chain with $\{\mathbf{Q}^*(k)\}_{k=0}^{n-1}$, where $\mathbf{Q}^*(k) = \left(q^*_{ij}(k)\right)_{i,j=1}^{N^*}$ and $q^*_{ij}(k) = P(X^*(k+1)=j \mid X^*(k)=i)$. It appears that the transition matrix $\mathbf{Q}^*(k)$ can be described based on elements of the transition matrix (1) (Dębicka 2003). In particular, for $k \in \{0, 1, 2, \ldots, n-1\}$, $x + k < \omega$ and $y + k < \omega$, the transition matrix has the following form

$$\mathbf{Q}^*(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & 0 & 0 & q_{13}(k) & 0 & 0 & q_{14}(k) & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

In the model shown in Figure 2, the stay of process$\{X^*(k)\}$ in state 1 and transitions to other states are the same as in the models shown in Figure 1. Due to the renumbering of states (3 and 4 in Figure 1 to 5 and 8 in Figure 2), we have $q^*_{11}(k) = q_{11}(k)$, $q^*_{12}(k) = q_{12}(k)$, $q^*_{15}(k) = q_{13}(k)$ and $q^*_{18}(k) = q_{14}(k)$ Note that states 2, 4, 5, 7 and 8 are reflex states, which are strictly transitional and after one unit of time, the insured risk leaves this state thus for such states $q^*_{ii}(k) = 0$. In addition, the probabilities of transition from state 2 to state 3 and staying in state 3 by the insured risk concern the survival of the year by the widow, therefore $q^*_{23}(k) = q^*_{33}(k) = q_{22}(k)$ and as a result $q^*_{24}(k) = q^*_{34}(k) = 1 - q_{22}(k) = q_{24}(k)$. We have an analogous situation related to states 5 and 6 regarding the survival of the year by a widower thus $q^*_{56}(k) = q^*_{66}(k) = q_{33}(k)$ and $q^*_{57}(k) = q^*_{67}(k) = 1 - q_{33}(k) = q_{34}(k)$. Analogous to the classical model $(S, T)$, for $k$ such that $x+k \geq \omega$ or $y+k \geq \omega$, matrices $\mathbf{Q}^*(k)$ consists of rows that have only zero elements. After the husband's death model $(S^*, T^*)$ degenerates into the model $(\{2,3,4,9\}, \{(2,3),(2,4),(3,4),(4,9)\})$, where states 2 and 4 are reflex. Then for $k \in \{1, 2, \ldots, n-1\}$ such that $x + k \geq \omega$ and $y + k < \omega$, matrix (31) is reduced to a matrix of the form

$$\mathbf{Q}^*_x(k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

Whereas after the wife's death we analyse model ({5,6,7,9},{(5,6),(5,7),(6,7),(7,9)}) with two reflex states 5 and 7. Thus for $x + k < \omega$ and $y + k \geq \omega$ we obtain

$$\mathbf{Q}^*_y(k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{33}$$

Note that the matrices given by formula (32) and (33) are not typical transition matrices for the model shown in Figure 2, because from a certain point in the insurance

period the model with 9 states reduces to a 4-state model. For those states where the probability of stay of the process $\{X^*(k)\}$ in particular state is equal to zero we have rows containing only zeros, if we remove them the remaining rows create a transition matrix for the 4-state model. In this sense matrices (32) and (33) can be treated as reduced transition matrices of the 9-state model.
Let matrix $\mathbf{D}$ consist of probabilities of process $\{X^*(k)\}$ staying at states in each moment of the insurance period

$$\mathbf{D} = \begin{pmatrix} \mathbf{P}^T(0) \\ \mathbf{P}^T(1) \\ \vdots \\ \mathbf{P}^T(n) \end{pmatrix} \in \mathfrak{R}^{(n+1)\times N^*}$$

where $\mathbf{P}^T(k) = (P(X^*(k) = 1),\ P(X^*(k) = 2), \ldots,\ P(X^*(k) = N^*))$ and $\mathbf{P}^T(0) = (1, 0, 0, \ldots, 0) \in \mathfrak{R}^{N^*}$ is a vector of the initial distribution. Based on $\{\mathbf{Q}^*(k)\}_{k=0}^{n-1}$ we describe columns of the matrix $\mathbf{D}$ as follows $\mathbf{P}^T(k) = (P(X^*(k) = 1), P(X^*(k) = 2), \ldots, P(X^*(k) = 9))^T$ (note that for multistate model presented in Figure 2 $N^* = 9$) and based on matrices (31)–(33) we obtain

$$\mathbf{P}^T(k) = \begin{cases} \mathbf{P}^T(0) \cdot \prod_{k=0}^{\omega-x-1} \mathbf{Q}^*(k) \prod_{k=\omega-x}^{t-1} \mathbf{Q}^*_x(k) & \text{for } x+t \geq \omega \wedge y+t < \omega, \\ \mathbf{P}^T(0) \cdot \prod_{k=0}^{t-1} \mathbf{Q}^*(k) & \text{for } x+t < \omega \wedge y+t < \omega, \\ \mathbf{P}^T(0) \cdot \prod_{k=0}^{\omega-y-1} \mathbf{Q}^*(k) \prod_{k=\omega-y}^{t-1} \mathbf{Q}^*_y(k) & \text{for } x+t < \omega \wedge y+t \geq \omega. \end{cases}$$

The matrix $\mathbf{D}$ determined for $n = \max\left\{w_x^M, w_y^W\right\}$ (for last surviving status) can be used in actuarial values calculations for both statuses, as shown in the next sections.

## 3.2 Cash flows matrices and statuses

Let $cf_j(k)$ be the future cash flow payable at the time $k$ if $X^*(k) = j$ $(j \in S^*)$ and $\mathbf{C} = (cf_j(k))_{\substack{j=1,2,\ldots,N^* \\ k=0,1,\ldots,n}}$ be the cash flows matrix. Let us note that from the financial point of view, the cash flow $cf_j(k)$ is a sum of inflows (benefits) representing an *in*come to the *insurer total loss fund* and *out*flows (premiums) representing an outgo from this fund, then premiums and benefits take the opposite signs. Hence $\mathbf{C} = \mathbf{C}_{\text{in}} + \mathbf{C}_{\text{out}}$. In the article, we assume that all lump sum benefits are cash flows paid from below. The form of the matrix $\mathbf{C}_{\text{in}}$ depends on the status. Note that for joint-life status, when the insured risk falls in one of the states 2 or 5, the insurance contract will expire; therefore, the benefits from states 4 and 7 cannot be realised. That is for

i) last surviving status

$$\mathbf{C}_{\text{in}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^M & 0 & 0 & c^W & 0 & 0 & c & 0 \\ 0 & c^M & 0 & c^W & c^W & 0 & c^M & c & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c^M & 0 & c^W & c^W & 0 & c^M & c & 0 \end{pmatrix} \in \mathfrak{R}^{(n+1)\times 9}, \tag{34}$$

ii) joint-life status

$$\mathbf{C}_{\text{in}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^M & 0 & 0 & c^W & 0 & 0 & c & 0 \\ 0 & c^M & 0 & 0 & c^W & 0 & 0 & c & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c^M & 0 & 0 & c^W & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{R}^{(n+1)\times 9}. \tag{35}$$

Independently of statuses, the cash flow matrix consisting only of outflow from the insurer loss fund has the following form; for:

i) $\pi$ - net single premium paid at the beginning of the insurance period

$$\mathbf{C}_{\text{out}} = \begin{pmatrix} -\pi & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{R}^{(n+1)\times 9}$$

ii) $p$ - net period premiums paid at the first $m$ units of the insured period

$$\mathbf{C}_{\text{out}} = \begin{pmatrix} -p & 0 & \cdots & 0 \\ -p & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -p & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{R}^{(n+1)\times 9}$$

where $n = \max\left\{w_x^M, w_y^W\right\}$.

### 3.3 Actuarial values

Let $\mathbf{I} \in \mathfrak{R}^{(n+1)\times(n+1)}$ be the identity matrix and $\mathbf{S} = (1, 1, \ldots, 1)^T \in \mathfrak{R}^{n+1}$, $\mathbf{I}_k = \left(0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0\right)^T \in \mathfrak{R}^{n+1}$, $\mathbf{J}_i = \left(0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0\right)^T \in \mathfrak{R}^{N^*}$ for each $k = 0,\ 1,\ 2, \ldots,\ n$ and $j = 1, 2, \ldots, N^*$. Furthermore, for any matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n+1}$, let $Diag(\mathbf{A})$ be a diagonal matrix whose diagonal elements consist of a diagonal of the matrix $\mathbf{A}$.
The following notation is necessary to describe the interest rate. Let $\upsilon(k) = e^{-Y(k)}$ be the discount function, where $Y(k)$ denotes the rate of interest in the time interval $[0,\ k]$. Then $\upsilon(k_1, k_2) = e^{-(Y(k_2)-Y(k_1))}$ is a discount function for a period of time $[k_1, k_2]$. We introduce a matrix $\mathbf{\Lambda} = (\lambda_{k_1k_2})_{k_1,k_2=1}^{n+1}$ where

$$\lambda_{k_1k_2} = \begin{cases} E(\upsilon(k_2, k_1)) & \text{for } k_1 > k_2, \\ 1 & \text{for } k_1 = k_2, \\ E(\upsilon(k_1, k_2)) & \text{for } k_1 < k_2, \end{cases}$$

where $E(\upsilon(k_1, k_2))$ means the expected value of the discount function $\upsilon(k_1, k_2)$ (cf. Dębicka et al. 2022).
If the interest rate is constant and equal to $u$, then $\lambda_{k_1k_2} = \upsilon^{k_2-k_1}$ and $\upsilon = (1+u)^{-1}$ is a discount factor. Namely, if the interest rate is a function of time then $\lambda_{k_1k_2} = e^{-(k_2-k_1)R_{k_1,k_2}}$, where $R_{k_1,k_2}$ is the immediate interest rate.
In order to obtain matrix formulas for premiums and reserves, we make the following standard assumptions (see also Dębicka 2013; Frees 1990; Parker 1994):

***Assumption* A1**. The random variable $X^*(k)$ is independent of $Y(k)$.
***Assumption* A2**. The first moment of the random discounting function $e^{-Y(k)}$ is finite.
The *net single premium* paid in advance (at time 0, when $X^*(0) = 1$) for the insurance modelled by $(S^*, T^*)$ equals (cf. Dębicka 2013, p. 217)

$$\pi = \mathbf{I}_1^T \mathbf{\Lambda}^T Diag(\mathbf{C}_{\text{in}} \mathbf{D}^T)\mathbf{S}. \tag{36}$$

Additionally, the *net period premium* payable in advance at the beginning of the time unit during the first $m$ units ($m \le n$) if $X^*(k) = 1$ equals (cf. Dębicka 2013, p. 217)

$$p = \frac{\mathbf{I}_1^T \mathbf{\Lambda}^T Diag(\mathbf{C}_{\text{in}} \mathbf{D}^T)\mathbf{S}}{\mathbf{I}_1^T \mathbf{\Lambda}^T \left(\mathbf{I} - \sum_{t=m+1}^{n+1} \mathbf{I}_t \mathbf{I}_t^T\right) \mathbf{D}\mathbf{J}_1}, \tag{37}$$

where the denominator in (37)is equal to the actuarial value of a temporary ($m$-year) life annuity-due contract.
In general terms, a *reserve* is a retained part of the generated increase in funds to cover expected costs and losses. The structure of insurance provides for the creation

of insurance reserves, which are a fund created from the surplus of net premiums over the sum of claims in a given year. This fund is intended to cover current and future liabilities arising from concluding an insurance contract. Insurers use this actuarial value, among other things, to monitor the solvency of the company and to determine the (average) profit expected to emerge at the end of each year of the insurance contract.

One of the methods of creating the reserve is the prospective method, which consists in determining the amount needed at time t (of the duration of the insurance contract), which, together with the premiums paid in the future, will allow for the payment of future benefits. Let $L_k$ be the *insurer's prospective loss* at time $k$, defined as the difference between the present value of future benefits and the present value of future net premiums. Thus, the *prospective reserve* $V_i(k)$ is the conditional expected value of the insurer's prospective loss under the condition at moment $k$ process $\{X^*(k)\}$ is at state $i$ i.e. $V_i(k) = E\left(L_k | X^*(k) = i\right)$. Let reserves matrix $\mathbf{V}$ consist of net reserves in each moment of the insurance period

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}^T(0) \\ \mathbf{V}^T(1) \\ \vdots \\ \mathbf{V}^T(n) \end{pmatrix} \in \mathfrak{R}^{(n+1)\times N^*},$$

where (cf. Dębicka 2013, p. 220)

$$\begin{aligned} \mathbf{V}(k) &= \left(V_1(k), V_2(k), \ldots, V_{N^*}(k)\right)^T = \\ &= \left(\mathbf{C}_{\text{out}}^T + \mathbf{C}_{\text{in}}^T + \sum_{t=k+1}^{n} \prod_{u=k}^{t-1} \mathbf{Q}^*(u)\mathbf{C}^T \mathbf{I}_{t+1}\mathbf{I}_{t+1}^T \mathbf{\Lambda}\right)\mathbf{I}_{k+1} \end{aligned} \tag{38}$$

is a vector containing net prospective reserves $V_i(k)$ for each state $i = 1, 2, \ldots, N^*$at the moment $k = 0, 1, 2, \ldots, n$. Note that not all elements of the matrix $\mathbf{V}$ may occur

$$V_i^{\text{real}}(k) = \begin{cases} \mathbf{J}_i^T \mathbf{V}^T \mathbf{I}_{k+1} & \text{for } P(X^*(k)) > 0, \\ - & \text{for } P(X^*(k)) = 0. \end{cases}$$

Finally, we analyse only the prospective insurance reserves, which have a chance to come into being

$$\mathbf{V}^{\text{real}} = \begin{pmatrix} \mathbf{V}^{\text{real}^T}(0) \\ \mathbf{V}^{\text{real}^T}(1) \\ \vdots \\ \mathbf{V}^{\text{real}^T}(n) \end{pmatrix} \in \mathfrak{R}^{(n+1)\times N^*}.$$

Matrix formulas for actuarial values described in this section can be used for both statuses. Note that it is possible to determine matrix $\mathbf{D}$ only for $n = \max\left\{w_x^M, w_y^W\right\}$

(for last surviving status) and use it for counting actuarial values in the case of two statuses. In this case, the matrix $\mathbf{C}_{\text{in}}$ decides on the type of status, i.e. we have (34) for the last surviving status and (35) for the joint-life status.

# 4 Applications

**Insurance contract** For numerical illustrative purpose we consider the last surviving status marriage insurance contract. We assume that the lump sum paid when the husband (wife) dies equals 1 unit ($c^M = c^W = 1$) and the lump sum paid if both spouses die equals 2 units ($c = c^M + c^W = 2$). Moreover, the constant period premium $p$ is paid when both spouses are alive (i.e. if $X^*(k) = 1$).

**Probabilistic structure of the model** We investigate three scenarios. In the first one, valuation of marriage insurance is done under the assumption of independence of future lifetimes of husband and wife. In the pricing insurance contracts, the Czech Statistical Office Life Tables 2017 are used. The next two scenarios are the Markov models based on the stationary Markov chain (see: Wolthuis and Van Hoeck 1986; Norberg 1989; Denuit et al. 2001). These models let us establish the joint distribution of the lifetimes of spouses, i.e. $(T_x^M, T_y^W)$, for fixed ages $x$ and $y$.
Based on the data for the Czech Republic in 2017 obtained from the Czech Statistical Office and Eurostat (2020) the joint cumulative distribution was determined

$$F_{x_0y_0}(w,z) = P(T_{x_0}^M \le w, T_{y_0}^W \le z)$$

for $x_0 = y_0 = 60$ where values $x_0$ and $y_0$are treated as base ages for men and women. The Kendall correlation coefficient was calculated for lifetimes $T_{x_0}^M$ and $T_{y_0}^W$ as following

$$\tau = 4\int_{x_0}^{\infty}\int_{y_0}^{\infty} F_{x_0y_0}(t,s)dF_{x_0y_0}(t,s) - 1,$$

and based on the above-mentioned data, is equal $\tau = 0.1064$.
In the next step, copula $C(u, v)$ was selected to describe the dependency structure between $T_{x_0}^M$ and $T_{y_0}^W$. For this purpose, four copula families were considered: AMH, Clayton, Gumber and Frank (Dębicka et al. manuscript). These are Archimedean copula families, and each family is described by a parameter $\alpha$ reflecting the degree of dependence. Namely, there is a relation $\alpha = g(\tau)$ between the parameter $\alpha$ and the value of Kendall correlation coefficient $\tau$, e.g. for Gumbel copula there is a relation $\alpha = 1/(1-\tau)$ . Then, copula with parameters $\alpha$ satisfying the relation $\alpha = g(\tau)$ were selected in each family. From among these four representatives, copula $C(u, v)$ was selected as the one that best describes the structure of the relationship between $T_{x_0}^M$ and $T_{y_0}^W$ using a criterion based on the distance between the distribution and the

theoretical distribution function induced by copula $C(u,v)$

$$d_2(F_{x_0y_0}, F_C) = \left( \int_0^\infty \int_0^\infty \left| F_{x_0y_0}(t,s) - C(F_{x_0}(t), F_{y_0}(s)) \right|^2 dt ds \right)^{1/2},$$

where $F_{x_0}$ are $F_{y_0}$the marginal distributions of the lifetimes $T^M_{x_0}$ and $T^W_{y_0}$. The technical details are included in Durrleman et al. (2000); Heilpern (2015). The Gumbel copula $C(u,v) = \exp\left(-((-\ln u)^\alpha + (-\ln v)^\alpha)^{1/\alpha}\right)$, where$\alpha = 1.1190$ (scenario II), gives us the smallest value of this criterion while the AMH copula $C(u,v) = uv/(1-\alpha(1-u)(1-v))$, where $\alpha = 0.4240$ (scenario III), took the second place. We must remember that we use the survival copula $C^*$, not the classical copula, in our analyses. In this paper, we will use Gumbel($\alpha$) and AMH($\alpha$) to denote copulas with the parameter $\alpha$.

**Interest rate** We based on real European Union market data related to the yield euro area central government bonds from January the 2$^{\text{nd}}$ 2017 (European Central Bank 2020). It appears, that the best fitted model of short term rate $R_{0,k}$ ($k$ is time to maturity) is Svensson model. In this case, the function $R_{0,k}$ has the following form (cf. Marciniuk 2009)

$$R_{0,k} = \beta_0 + \beta_1 \frac{\tau_1}{k}\left(1 - e^{-\frac{k}{\tau_1}}\right) + \beta_2 \left(\frac{\tau_1}{k}\left(1 - e^{-\frac{k}{\tau_1}}\right) - e^{-\frac{k}{\tau_1}}\right) + \\ + \beta_3 \left(\frac{\tau_2}{k}\left(1 - e^{-\frac{k}{\tau_2}}\right) - e^{-\frac{k}{\tau_2}}\right),$$

where $\beta_0 = 0.01450$, $\beta_1 = -0.02274$, $\beta_2 = 0.11886$, $\beta_3 = -0.016085$, $\tau_1 = 1.33662$, $\tau_2 = 1.57465$. Since $\beta_0$ is interpreted as a long term rate of interest, we assume that the $r = \beta_0 = 1.45\%$.

In the remaining part of this section, we observe the influence of dependence between future lifetimes of spouses on premiums and prospective reserves. Let pair $(x,y)$ denote the age at entry of husband and wife. We start our considerations with an analysis for couples at the following ages: $(60,60)$, $(65,60)$, $(60,65)$. Insurance periods ($n$), periods of premium payment ($m$), net single premiums ($\pi$) and net period premiums ($p$) for such insurance contracts are presented in Table 1. Single and periodical premiums are calculated according to the formulas (36) and (37), respectively.

Let us observe that premiums are lower on the condition that future lifetimes of spouses are dependent. In the age of asymmetric case (for $x \neq y$), the net single premium is higher when the wife is older. But the net period premium is higher when the husband is older, which is directly connected with the actuarial value of a temporary ($m$ - year) life annuity-due contract (see the denominator in (36)). In the case of scenario I, a single premium for a married couple at the age of $(60,65)$ is 0.33% higher than for a pair at the age of $(65,60)$, while for scenario II, these differences

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Table 1: Net premiums for marriage insurance contracts

| $(x,y)$ | $(60,60)$ | $(65,60)$ | $(60,65)$ |
|---|---|---|---|
| $n$ | 45 | 45 | 45 |
| $m$ | 45 | 40 | 40 |
| scenario I | $T_x^M$, $T_y^W$ independent | | |
| $\pi$ | 1.463983328 | 1.502732787 | 1.507648167 |
| $p$ | 0.099801522 | 0.116481266 | 0.112476636 |
| scenario II | $T_x^M$, $T_y^W$ dependent Gumbel(1.1190) | | |
| $\pi$ | 1.463981321 | 1.500960718 | 1.506651958 |
| $p$ | 0.097632138 | 0.113867543 | 0.109420526 |
| scenario III | $T_x^M$, $T_y^W$ dependent AMH(0.4240) | | |
| $\pi$ | 1.463981300 | 1.500544872 | 1.506543419 |
| $p$ | 0.097611342 | 0.11375613 | 0.109363979 |

equal 0.38% and for scenario III 0.40%. Therefore, according to formula (36), the period premium for a married couple at the age of $(60, 65)$ is 3.44% lower than the age of $(65, 60)$ (and, according to scenario II and scenario III of 3.91% and 3.86%, respectively). Note that for certain spouses, the insurance premiums specified for the scenario I are higher than for scenario II, and these in turn are higher than those calculated for scenario III.

Differences between premiums for scenarios I, II and III result from the difference between the elements of matrix $\mathbf{D}$ (i.e. probabilities of the process $\{X^*(k)\}$ being at a given time in a particular state), which is graphically presented in Figure 3.

In order to explain in detail in which state and moment of insurance period the differences between the probabilities are the largest, the relative differences between the elements of matrices $\mathbf{D}$ defined for individual scenarios are presented in the next three figures.

In Figure 4, it is observed that the probability that both spouses are alive (state 1) is greater for scenario II (dependence-Gumbel(1.1190)) than for scenario I (independence), especially in the second half of the insurance period. A higher probability of survival of a married couple means that the premium according to scenario II is lower, and the insurer also has a possibility to create lower net premium reserves (compare Table 1 and Figure 7). Moreover, the biggest relative differences can be observed at the end of the term of insurance. We observe analogous relationships for the situation related to the death of the husband (state 2) and the death of the wife (state 5).

The relative difference between the probabilities of a widower's life (when the wife has been dead for at least a year) grows year by year (state 6) when the husband is not younger than his wife. This means that according to the scenario I, the probability that the widower is alive is higher than in scenario II. We observe a much lower

Figure 3: Probability distributions of $X^{*}(k)$ for the $k$th year of the insurance contract (k$= 0, 1, 2, \ldots, 45$) according to scenarios for couples at the ages (60,60), (65,60) and (60,65)

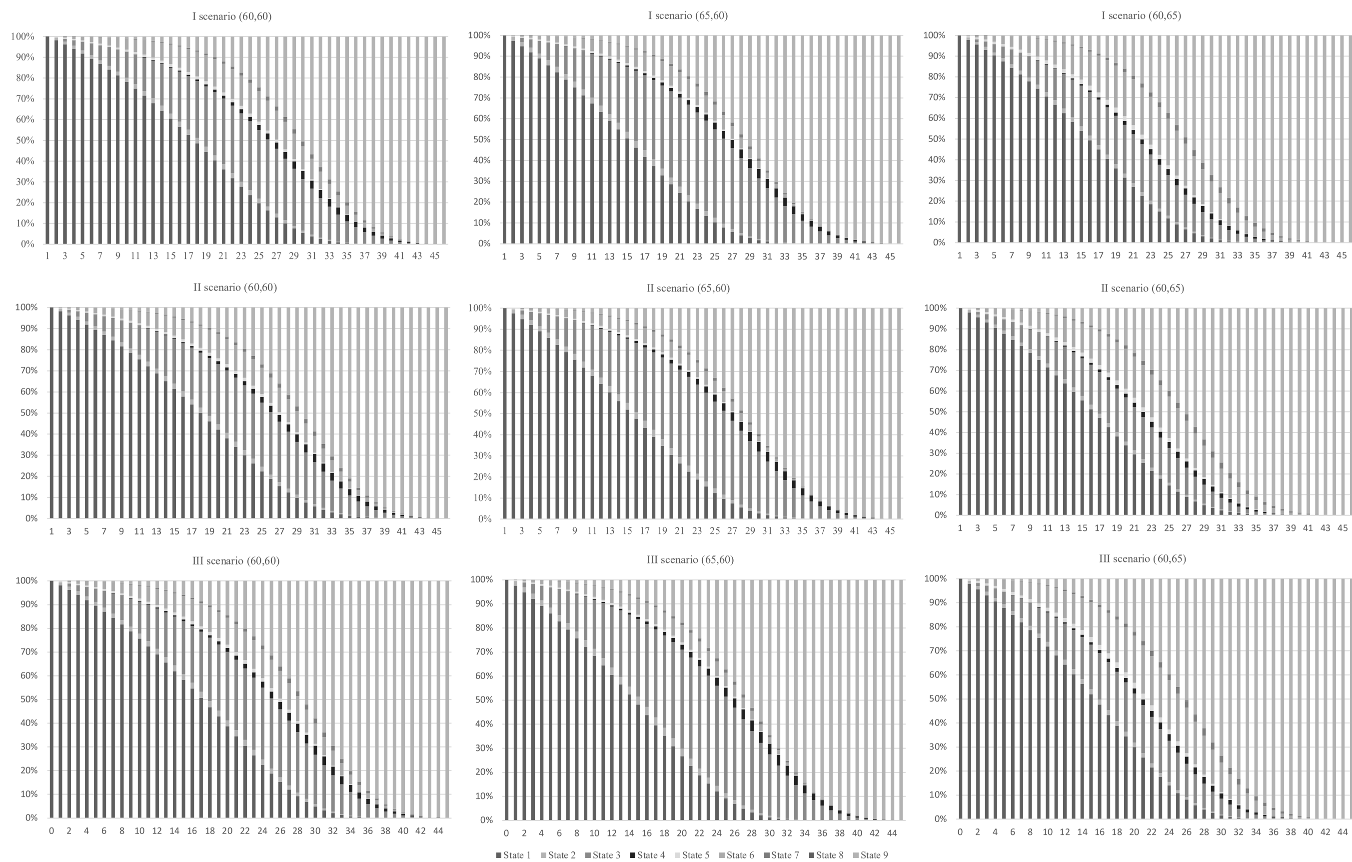
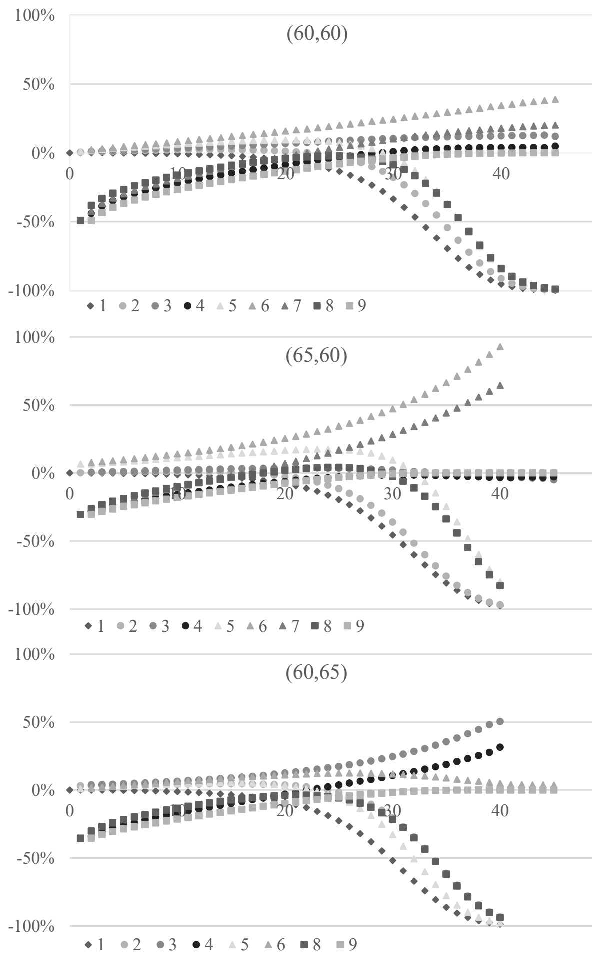


Figure 4: The relative differences between elements of matrices D (for states 1,2,... 9) for scenario I (independent) and scenario II (dependent - Gumbel(1.1190)) of spouses at the ages (60,60), (65,60) and (60,65) for the $k$th year of the insurance contract ($k = 0, 1, 2, \ldots, 45$)




relative difference between the probability of a widow's life (when the husband has been dead for at least a year) for couples in which the husband is not younger than his wife (state 3). The difference in probabilities is significant when the husband is younger than his wife.

Note that in the first half of the insurance period, we observe that the probability of a widow's death (state 4) is always higher, according to scenario II. However, in the second half of the insurance period, this probability is higher for scenario I (when the wife is older), or the probabilities for both scenarios are similar (when the wife is at most as old as the husband). Analysing the relationships concerning the probabilities of the widower's death (state 7), it can be noticed that in the first half of the insurance period, it is visible that scenario I gives a much higher probability than scenario II only if the woman is younger. On the other hand, in the second half of the insurance period, one can observe that regardless of the age of the spouses, the probability that the widower is dead is higher when it is calculated according to scenario II.

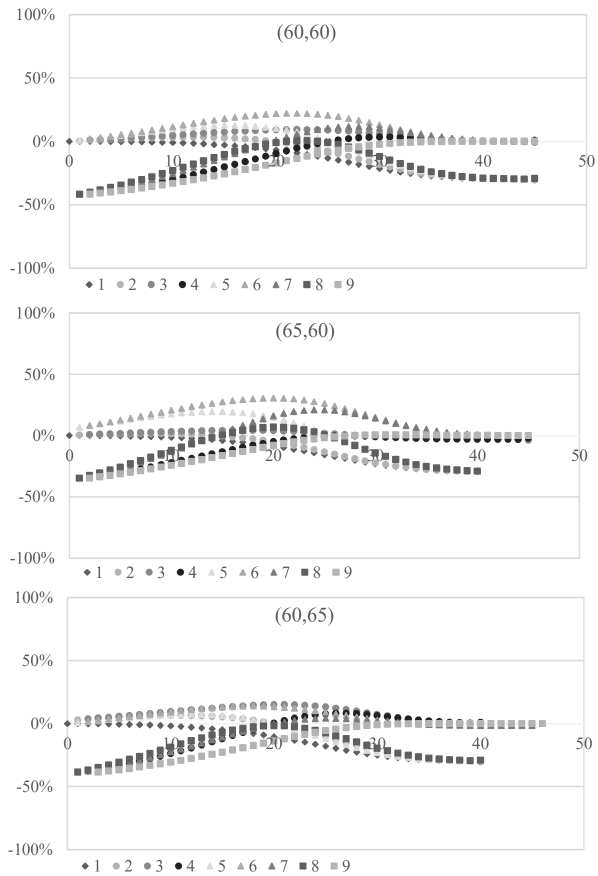
The relative differences between elements of matrices $\mathbf{D}$ for scenario I (independent) and scenario III (dependent - AHM(0.4240)) and for scenario III (dependent - AHM(0.4240)) and scenario II (dependent – Gumbel(1.1190)) are presented in Figure 5 and Figure 6, respectively. The largest range of relative differences (up to 149%) is observed when the probabilities are determined under the assumptions of independence and the Gumbel copula (Figure 4), as well as the Gumbel copula and the AMH copula (Figure 6) are compared. The range of differences between the probabilities specified for independence and the AMH copula (Figure 5) is 65%. Regardless of the compared scenarios, the biggest range of relative differences between the probabilities is observed when the husband is older than the wife. These differences can, among other things, be explained by the different dependence in the upper tail of the copulas present in the considered scenarios. We define the upper dependence in the tail of variables $T^M_{x_0}$ and $T^W_{y_0}$, reflecting the life expectancy dependence of elderly spouses, as a limit (Embrechts et al. 2003; Heilpern 2007)

$$\lambda_U(T^M_{x_0}, T^W_{y_0}) = \lim_{u \to 0} P(T^M_{x_0} \leq F^{-1}_{x_0}(u) | T^W_{y_0} \leq F^{-1}_{y_0}(u)) = \lim_{u \to 0} \frac{C^*(u,u)}{u}.$$

For independence and AMH copula we get $\lambda_U(T^M_{x_0}, T^W_{y_0}) = 0$, i.e. independence in the tail. In contrast, for Gumbel copula we have $\lambda_U(T^M_{x_0}, T^W_{y_0}) = 2 - 2^{1-\tau} \approx 0.142$, i.e. dependence in the tail. This explains the larger differences for older individuals in Figure 4 and Figure 6 and the smaller differences in Figure 5. What is also shown in Table 1 and the impact on insurance premiums is noticeable. The probabilistic structure also has a special impact on the amount of insurance reserves.

In Figure 7, net real prospective reserves for marriage insurance contracts (calculated according to Equation (38)) are presented for a married couple in which at least one spouse is 60 years old. Note that net real prospective reserves for state 2 and state 3 have the same value because they are associated with the same random event (the death of the husband), and every prospective reserve $V_i(k)$ is calculated at the moment $k$ after paying the benefits due at time $k$ for the period $[0, k)$ (but before the insurer receives the premium at the moment $k$ and will pay the benefits payable in advance for the period $[k, k+1)$). The difference between them lies in the fact that $V_3^{\text{real}}(1)$ does not exist because in the first year of the insurance period the transition to state 3 is not possible. A similar situation can be observed for states

Figure 5: The relative differences between elements of matrices D (for states 1,2,. . . ,9) for scenario I (independent) and scenario III (dependent - AMH(0.4240)) of spouses at the ages (60,60), (65,60) and (60,65) for the $k$th year of the insurance contract ($k = 0, 1, 2, \ldots, 45$)




4 and 5. Let us note that net real premium prospective reserves (reserves for state 1) are lower on condition that future lifetimes of spouses are dependent (scenario II). Let us observe that net prospective reserves for other states for scenario II are higher than for scenario I at the end of the insurance contract. The net premium reserve calculated for scenario III in the first half of the insurance period is close to the reserve specified for scenario II, and in the second half of the insurance period for scenario I. Note that if the husband and wife are not the same age, then in the last years of the insurance contract, the net premium reserve calculated for scenario III is greater

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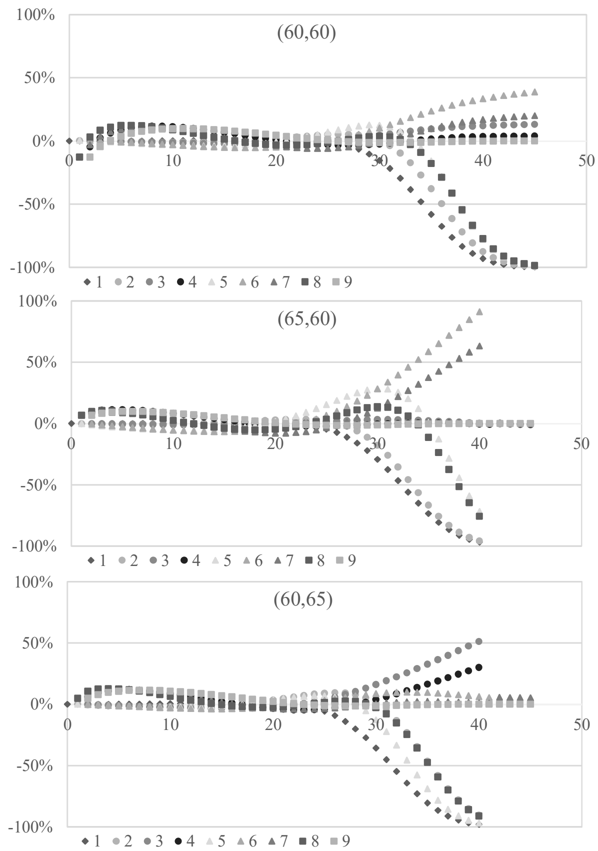
Figure 6: The relative differences between elements of matrices D (for states 1,2,…, 9) for scenario III (dependent - AMH(0.4240)) and scenario II (dependent - Gumbel(1.1190)) of spouses at the ages (60,60), (65,60) and (60,65) for the $k$th year of the insurance contract ($k = 0, 1, 2, \ldots, 45$)




than those calculated for scenario I and scenario II. The prospective net premium reserve calculated assuming that the future lifetimes of the spouses are dependent (scenario II and scenario III) is lower than when assuming that the future lifetimes of the husband and wife are independent (scenario I), which is visible in Figure 8. We observe the greatest relative differences for scenarios I and II, and the range of relative differences between the reserves is the greatest when the spouses are peers. If the joint probability distribution of future lifetimes of the spouses is modelled by the AMH copula, then the largest differences between reserves for state 1 in comparison to other scenarios are observed in the first years of the insurance contract. On the

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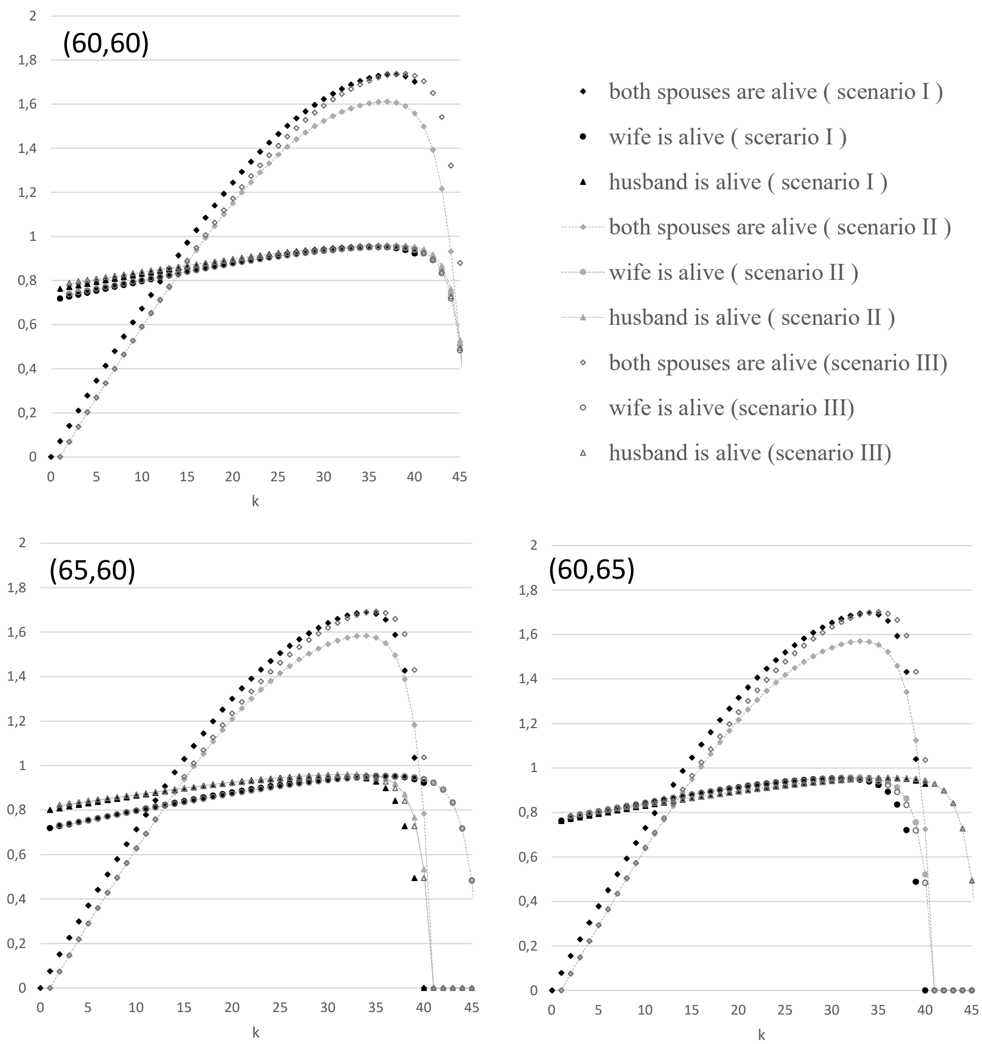
Figure 7: Net real prospective reserves for marriage insurance contracts




other hand, if the joint probability distribution of future lifetimes of the spouses is modelled by the Gumbel copula, then the largest relative differences between reserves for state 1 compared to other scenarios are observed in the last years of the insurance contract.

So far, the comparative analysis concerned married couples in which at least one spouse is 60 years old. Later in this section, we will also deal with older spouses.

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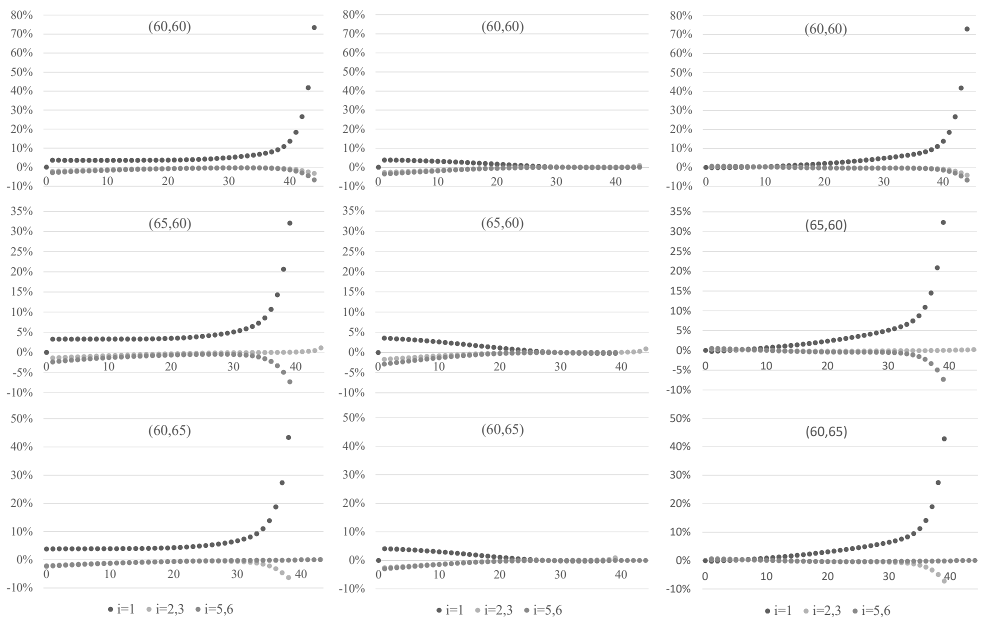
Figure 8: The relative differences between net real prospective reserves for scenarios of spouses at the ages (60,60), (65,60) and (60,65) for the $k$th year of the insurance contract ($k = 0, 1, 2, \ldots, 45$)

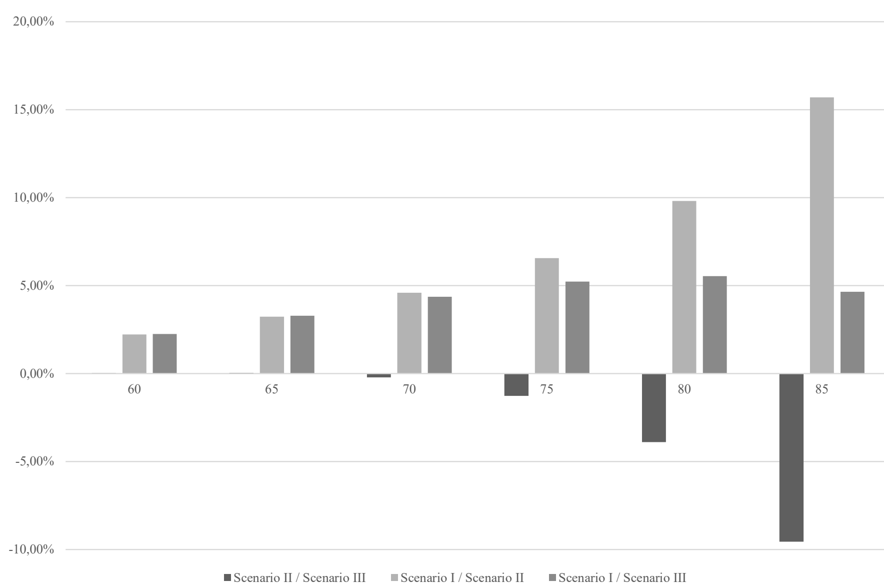



The premium for spouses of the same age presented in Table 2 is always greater in independence cases than in those of Gumbel or AMH, which is also presented in graphs of Figure 9. For the relation between premiums in the cases of Gumbel or AMH we do not observe such regularities. For instance, for 60 and 65 years old spouses, the premium in Gumbel case is greater than for AMH case (compare Table 1), but for spouses older than 65 years we have the opposite situation. The reversal of the relationship between contributions results from the significant differences between the probabilities of process $\{X^*(k)\}$ staying at states in a particular moment of the insurance period (compare the graphs of Figure 6 for $k > 25$). And for spouses aged 70 and over, the period over 85 is a significant part of the insurance period. Note that while for younger spouses, the choice of scenario II or scenario III does not result in greater differences in the amount of premiums, it is nevertheless decisive in case of older spouses. Therefore, choosing the copula that is better suited to the data (in this case Gumbel(1.119)) results in lower premiums.

Table 2: Net period premiums for marriage insurance contracts

| Age at entry x = y | Insurance period n = m | Scenario I Independent | Scenario II Gumbel(1.1190) | Scenario III AMH(0.4240) |
|---|---|---|---|---|
| 60 | 45 | 0.099802 | 0.097632 | 0.097611 |
| 65 | 40 | 0.128599 | 0.124565 | 0.124510 |
| 70 | 35 | 0.170235 | 0.162756 | 0.163112 |
| 75 | 30 | 0.232351 | 0.218038 | 0.220815 |
| 80 | 25 | 0.333625 | 0.303805 | 0.316117 |
| 85 | 20 | 0.493549 | 0.426572 | 0.471591 |

Figure 9: The relative differences between period premiums for marriage peers at 60, 65, ..., 85 years of age




Finally, we would like to test the robustness of premium value to the changing coefficient values, i.e. the degree of dependency between spouses' future lifetimes. The example concerns scenario II (because the Gumbel copula is the best fit for the data) for $\alpha \in \{1.08, 1.10, 1.119, 1.14, 1.16\}$ when both spouses are at the same age $(x, y \in \{60, 65, 70, 75, 80, 85\})$. The premiums are presented in Table 3.
The premium is more sensitive to the value of the correlation coefficient, especially for older people. It is noticeable that the older the spouses are, the more significant

Table 3: The copula's resistance to $\alpha$ coefficient in case of Scenario II

| $\alpha$ | 1.08 | 1.10 | 1.119 | 1.14 | 1.16 |
|---|---|---|---|---|---|
| $\tau$ | 0.0741 | 0.0909 | 0.1063 | 0.1228 | 0.1379 |
| $x = y$ | net period premiums | | | | |
| 60 | 0.098287 | 0.097945 | 0.097632 | 0.097299 | 0.096994 |
| 65 | 0.125766 | 0.125137 | 0.124565 | 0.123960 | 0.123409 |
| 70 | 0.164934 | 0.163788 | 0.162756 | 0.161675 | 0.160700 |
| 75 | 0.222072 | 0.219935 | 0.218038 | 0.216076 | 0.214329 |
| 80 | 0.311760 | 0.307505 | 0.303805 | 0.300056 | 0.296783 |
| 85 | 0.442517 | 0.433834 | 0.426572 | 0.419480 | 0.413495 |

the relative difference between premiums. Let us note that the higher the correlation coefficient, the lower the premium. In the presented example, it can be observed that as the correlation coefficient increases by an average of 1.6%, the premium falls by an average of 0.33% for younger couples ($x = y = 60$) and by as much as 1.68% on average for older couples ($x = y = 85$). Note that, for the oldest spouses, an increase in the correlation coefficient of partners' future lifetimes of about 0.02 can increase premiums by as much as 2%. Such a change can be significant from an insurance company's point of view.

## 5 Conclusions

In this paper, we applied and developed a matrix representation to the analysis of all types of future cash flows arising from the marriage insurance contract. For this purpose, we derived transition probabilities under the assumption that future lifetimes of spouses are dependent and such dependence is modelled by a copula. It allowed for efficient analysis of the stochastic structure of the model and cash flows resulting from the realisation of contracts.

The analysis and comparison (dependence and independence) confirm that the premiums and reserves depend on the probabilistic structure of the model. From the empirical examples, it is obvious that this structure has a greater impact on actuarial values for older spouses. Furthermore, premiums are lower when spouses enter the contract at a younger age and live longer. This is confirmed for both dependent and independent future life of spouses. Moreover, the premium is more sensitive to the value of the correlation coefficient. The results provided in Section 4 indicated that in the case of the last survivor status, the independence assumption overestimates net single and period premiums. The percentage differences are higher for older spouses. To sum up, in pricing life insurance, age plays a crucial role in determining the premiums for the insurance coverage in that the higher one's age, the higher the premium and vice versa when other risk factors are held constant.

In the case of reserves, the conclusions are not as clear as in the case of premiums.

It turns out that net real prospective reserves for state 2 and state 3 have the same value because they are associated with the same random event (the husband's death). A similar situation can be observed for states 4 and 5. Let us note that net real premium prospective reserves (reserves for state 1) are lower on the condition that the future lifetimes of spouses are dependent (scenario II). Let us observe that net prospective reserves for other states for scenario II are higher than for scenario I at the end of the insurance contract. The net premium reserve calculated for scenario III in the first half of the insurance period is close to the reserve specified for scenario II and in the second half of the insurance period for scenario I. Note that if the husband and wife are not the same age, then in the last years of the insurance contract, the net premium reserve calculated for scenario III is greater than those calculated for the scenario I and scenario II. The prospective net premium reserve calculated assuming that the future lifetimes of the spouses are dependent (scenario II and scenario III) is lower than when assuming that the future lifetimes of the husband and wife are independent (scenario I). Because the copula Gumbel(1.119) (scenario II) is the best fit for the data not only results in lower premiums, but we observe the greatest relative differences between reserves for scenarios I and II, and the range of differences is the greatest when the spouses are peers.

Choosing the type of copula for modelling the joint probability distribution of the future lifetimes of the spouses significantly affects the range of differences between the reserves for individual scenarios. Choosing the AMH copula, the largest differences between reserves for state 1 in comparison to other scenarios are observed in the first years of the insurance contract. On the other hand, if the joint probability distribution of future lifetimes of the spouses is modelled by the Gumbel copula, then the largest relative differences between reserves for state 1 compared to other scenarios are observed in the last years of the insurance contract.

We analysed three scenarios, which can be important elements of designing the policy cash flows in the analysis of the expected profit and the expected cash flows for each year of the insurance policy (compare Dębicka et al. 2016). It allows us to determine the range of profit and is an element of profit testing. In particular, net premiums are calculated securely. Then both the interest rate and the transition probabilities in the model should be carefully chosen. This set of assumptions is often called the first-order basis (scenario I can be used to describe the probabilistic structure of the model). The second-order basis is the determination of the insurer's profit and loss on the basis of realistic parameters cf. Haberman and Pitacco (1999). In this situation, the assumption that future lifetimes of spouses are dependent random variables and the distribution of their future lifetime is modelled by copulas seems to be the best. Especially scenario II, which gives a smaller value of the distance between the distribution and the theoretical distribution function induced by copula C(u,v) is better fitted than scenario III. In addition, a higher interest rate value is usually included than in the first-order basis. Regardless of the compared scenarios, the biggest range of relative differences between the probabilities is observed when the husband is older than the wife. These differences can, among other things, be explained by the different dependence in the upper tail of the copulas present in the considered scenarios. The analysis of the results, presented in Tables 1 and 2, shows that the premium is higher when there is no correlation between the spouses' future lifetime. It is a more secure option from the insurer's financial point of view. However, if the insurer were to consider the relationship between the spouses' future lifetimes, its insurance offering could become more competitive in the market. It is a common actuarial practice to calculate premiums and reserves under a set of assumptions that represent a worst-case scenario for the insurer, which corresponds to the application of the assumptions defining the first-order base in the calculations (in the context of our considerations this brings us to the scenario I - in particular the assumptions that the lifetimes of the spouses are independent - because it gives higher premiums). The analysis performed in this article may also be useful for the insurer in determining the expected profit or solvency margin. The new solvency regime of the European Union (Solvency II) uses worst-case scenarios for the calculation of solvency capital requirements for life insurance business. Research involving profit testing with the results obtained in the article may be an interesting direction for future research.

In summary, the most accurate determination of the probability structure of the multistate model is very important from the point of view of the valuation and profitability assessment of marriage insurance contracts. It is possible to continue the research to define this structure as accurately as possible. One possible alternative is to use a time-varying copula, which is especially important for long-term contracts. The broken heart syndrome can also be taken into account (cf., e.g. Henshaw et al. 2020) but, for this purpose, it would be necessary to expand the multistate model presented in Figure 2. In particular, state 3 should be divided into two states 31 (widow is alive during the second year after bereavement) and 32+ (widow is alive beyond the second year after bereavement) and similarly state 6, which would allow for increased widow (widower) mortality; see Ji et al. (2011). Note that increasing the number of states would increase the number of columns of the cash flow matrices and the matrices related to the probability structure of such an extended multistate model but would not complicate the calculation of actuarial values (matrix formulas (36)–(38) would remain unchanged).

# References

[1] Carriere J. F., (2000), Bivariate Survival Models for Coupled Lives, *Scandinavian Actuarial Journal* 1, 17–31.

[2] Denuit M., Dhaene J., Le Bailly de Tillegheim C., Teghem S., (2001), Measuring the impact of a dependence among insured lifelengths, *Belgian Actuarial Bulletin* 1(1), 18–39.

[3] Dębicka J. (2003), Moments of the cash value of future payment streams arising from life insurance contracts, *Insurance: Mathematics and Economics* 33, 533–550.

[4] Dębicka J. (2013), An approach to the study of multistate insurance contracts, *Applied Stochastic Models in Business and Industry* 29(3), 224–240.

[5] Dębicka J., Heilpern S., Marciniuk A., (2016), Profitability of marriage insurance contract, [in:] 19-th AMSE. Applications of Mathematics in Economics (Conference Proceedings). International Scientific Conference: Banská Štiavnica, Slovakia, 31 Aug.–4 Sep. 2016.

[6] Dębicka J., Heilpern S., and Marciniuk A., (2022), Modelling Marital Reverse Annuity Contract in a Stochastic Economic Environment, *Statistika* 102(3), 261–281.

[7] Durrleman V., Nikeghbali, A., Roncalli, T., (2000), Which copula is right now?, Groupe de Recherche Operationnelle, Credit Lyonnais, Working Papers.

[8] Embrechts P., Lindskog F., McNeil A. J., (2003), Modeling Dependence with Copulas and Applications to Risk Management, [in:] S. Rachev (ed.), *Handbook of Heavy Tailed Distributions in Finance*, 329–384, Elsevier, Rotterdam.

[9] European Central Bank, (2020), access 17-07-2020, available at: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`.

[10] Eurostat, (2020), access 16-03-2020, available at: `https://ec.europa.eu/eurostat/data/database`.

[11] Frees E. W., (1990), Stochastic life contingencies with solvency considerations, *Transactions of the Society of Actuaries* XLII, 91–148.

[12] Genest C., Mackay J., (1986), The joy of copula: bivariate distributions with uniform margins, *The American Statistician* 40, 280–283.

[13] Genest C., Rivest L. P., (1993), Statistical inference procedures for bivariate Archimedean copulas, *Journal of the American Statistical Association* 88, 1034–1043.

[14] Gourieroux C., Lu Y., (2015), Love and death: A Freund model with frailty, *Insurance: Mathematics and Economics* 63, 191–203.

[15] Haberman S., Pitacco E., (1999), *Actuarial Models for Disability Insurance*, Chapman & Hall CRC, Londyn.

[16] Heilpern S., (2007), *Funkcje łączące*. Wydawnictwo Akademii Ekonomicznej we Wrocławiu, Wrocław.

[17] Heilpern S., (2015), Dependent structure induced by Markov chain in the multiple life insurance, [in:] *18-th International Scientific Conference Applications of Mathematics in Economics*, Jindrichuv Hradec.

[18] Henshaw K., Constantinescu C., Menoukeu Pamen O., (2020), Stochastic Mortality Modelling for Dependent Coupled Lives, *Risks* 8, 17.

[19] Hoem J. M., (1969), Markov Chain Models in Life Insurance, *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* IX, 91–107.

[20] Hoem J. M., (1988), The Versality of the Markov Chain as a Tool in the Mathematics of Life Insurance, *Transactions of the 23rd International Congress of Actuaries*, Helsinki, Vol. R, 171–202.

[21] Ji M., Hardy M., Siu-Hang Li J., (2011), Markovian Approaches to Joint-Life Mortality, *North American Actuarial Journal* 15(3), 357–376.

[22] Luciano E., Spreeuw J., Vigna E., (2008), Modeling stochastic mortality for dependent lives, *Insurance: Mathematics and Economics* 43, 234–244.

[23] Luciano E., Spreeuw J., Vigna E., (2016), Spouses' Dependence across Generations and Pricing Impact on Reversionary Annuities, *Risks* 4(2), 1–18.

[24] Marciniuk A., (2009), Nielosowe modele natychmiastowej stopy procentowej i ich zastosowanie w klasycznych ubezpieczeniach życiowych, *Prace Naukowe Uniwersytetu Ekonomicznego we Wrocławiu. Ekonometria* 27, 112–127.

[25] Nelsen R. B., (1999), *An Introduction to Copulas*, Springer, New York.

[26] Norberg R., (1989), Actuarial analysis of dependent lives. *Bulletin de l'Association Suisse des Actuaries* 40, 243–254.

[27] Parker G., (1994), Stochastic Analysis of Portfolio of Endowment Insurance Policies, *Scandinavian Actuarial Journal* 77(2), 119–130.

[28] Spreeuw J., (2006), Types of dependence and time-dependent association between two times in single parameter copula models, *Scandinavian Actuarial Journal* 5, 286–309.

[29] The Czech Statistical Office Life Tables, access: 16-03-2020, available at: `https://www.czso.cz/csu/czso/life-tables-for-the-czech-republic-cohesion-regions-and-regions-2016-2017`.

[30] Wolthuis H., (1994), *Life insurance mathematics (The Markovian model)*, CAIRE education Series, No. 2, Bruxelles.

[31] Wolthuis H., Van Hoek I., (1986), Stochastic models for life contingencies, *Insurance: Mathematics and Economics* 5, 217–254.