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An observer design for Takagi-Sugeno fuzzy bilinear control systems

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In this paper, the observer design problem for a T-S fuzzy bilinear control system is investigated. First, an observer of Kalman type is designed to estimate the system states for the linear case. Then, some new sufficient conditions are derived to show the exponential convergence of the solutions of the error equation for fuzzy bilinear systems. Furthermore, we consider some uncertainties of the system that are bounded and satisfy a certain condition where an observer is designed. Moreover, an application to Van de Vusse system is given.

Key words: fuzzy systems, bilinear systems, stability analysis, observer design.

1. Introduction

For the linear control system, the conception of an observer has been solved completely. In the nonlinear case, this problem remains still a difficult task. Therefore, nonlinear state observer design has been an area of constant research for the last three decades and, despite important progress, many outstanding problems still remain unsolved. The observer design problem naturally arises in a system approach. In general, one cannot use as many sensors as signals of interest characterizing the system behavior in presence of parameters or unmeasured external disturbances [14]. In the engineering practice, the real plants often run in a complex and mutative environment with kinds of uncertain factors which inevitably bring certain engineering complexities such as time delays, parameter uncertainties, nonlinearities and sensor/actuator faults ([1,4,5,8,12–25,30–32]). These complex phenomena would directly influence the system state evolution in

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different ways, and largely affect the system performance. Considering the distinct merits of nonlinear T-S fuzzy control schemes, it is a natural idea to use the Takagi-Sugeno fuzzy theory (Takagi and Sugeno [26, 27]) to handle control problems for nonlinear plants subject to engineering-oriented complexities. However, the real physical systems are often nonlinear. The design of state observers for nonlinear systems using Takagi-Sugeno (TS) models has been actively considered during the last decades (see [2, 3]). T-S models are currently being used for a large class of physical and industrial processes, such as electrical machines and robot manipulators. As it is delicate to synthesize an observer for an unspecified nonlinear system, it is preferable to represent this system with the Takagi-Sugeno (T-S) fuzzy model. With a fuzzy observer, the estimated states error system is described as two parts: unknown premise variable caused terms and observer error terms (see [28]). A state observer is a dynamical system which provides the estimation of the internal states of the model. In most practical cases, the physical state of the system cannot be determined by direct observation. The problem of state observation for nonlinear systems is of main importance in automatic control. In recent years many contributions have been presented in literature that solve the control design problem for classes of nonlinear systems (see [7, 9, 21]). Unlike the linear case, the conception of observer is still a difficult task for nonlinear systems.

The disturbance observer for sliding model control has been studied in many papers (see [31] and references therein). Saturation is another important issue in practical applications that needs to be paid attention [29] where in the literature, several methods have been proposed to handle the effects of saturation.

In this paper, we show that the lyapunov approach can be used for the stability analysis concept to solve the observer design problem using the Kalam like configuration. We give some sufficient conditions to ensure that the error fuzzy equation is globally exponentially stable. Under the the condition of observability for any input of a class of bilinear system (see [6]), one can design an observer of Kalman type which converge for any bounded and small input in Takagi-Sugeno fuzzy model sense. Thus, our main contribution is the observer analysis and design methods that can effectively deal with model/plant mismatches. Moreover, as an application we construct an observer for the Van de Vusse system.

2. Preliminarily results

In control theory, a state observer is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented, and provides the basis of many practical applications. Knowing the system state is necessary to solve many control theory problems; for example, stabilizing a system using state feedback.

In most practical cases, the physical state of the system cannot be determined by direct observation. Instead, indirect effects of the internal state are observed by way of the system outputs.

A class of nonlinear systems that has seen much attention in the literature is the class of linear systems:

$$\dot{x}(t) = Ax(t) + Bu, \quad x(0) = x_0, \quad y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $A(n, n)$ constant matrix, $B(n, m)$ matrix control input and $C(p, n)$ is the output matrix. Given a system (1), one can estimate the states by using an observer, whose structure is as follows:

$$\dot{\hat{x}}(t) = O\left(\hat{x}(t), u(t), y(t)\right),$$

where $\hat{x}(t)$ is the state of the observer. It is needed that the estimation error, $e(t) = \hat{x}(t) - x(t)$ has to converge as fast as possible to zero. Most current methods lead to the design of an exponential observer, exponential stability is the most wanted. With the model given in (1), the problem is to design a continuous observer with input $y(t)$ such that the estimates denoted by $\hat{x}(t)$ converge to $x(t)$ exponentially fast. We shall assume that the pair (A, C) is observable. Suppose the observability matrix for the time invariant associated linear system. Then, there exists a gain matrix $L(n \times p)$ such that the matrix $(A - LC)$ is hurwitz. In this condition, one can design an exponential observer for system (1) as:

$$\dot{x}(t) = A\hat{x}(t) + Bu - L(C\hat{x}(t) - y(t)). \tag{2}$$

The system (2) is an exponential observer for system (1), where the matrix L is chosen such that

$$(A - LC)^T P + P(A - LC) = -Q,$$

with P and Q are $(n \times n)$ positive definite symmetric matrices. Here, we just consider the error equation:

$$\dot{e}(t) = (A - LC)e(t),$$

where $e(t) = \hat{x}(t) - x(t)$.

Therefore, using a Kalman like observer, we can design a state observer for (1) as follows (see [10] and [11]):

$$\dot{x}(t) = A\hat{x}(t) + Bu - S_\theta^{-1}C^T(C\hat{x}(t) - y(t)),$$

where S_θ satisfies the following stationary equation:

$$0 = -\theta S_\theta - A^T S_\theta - S_\theta A + C^T C, \quad \theta > 0,$$

$$S_\theta = \lim_{t \longrightarrow +\infty} S_t$$

with $S_t \in \mathcal{S}^+$ the cone of symmetric positive definite matrices on $\mathbb{R}^n$ which satisfies

$$\dot{S}_t = -\theta S_t - A^T S_t - S_t A + C^T C.$$

In the next section, we will consider the case of fuzzy control systems. It is well known that, Takagi-Sugeno fuzzy models are nonlinear systems described by a set of if-then rules which gives local linear approximations of an underlying system. Such models can approximate or describe a wide class of nonlinear systems.

3. T.S fuzzy dynamic model

The mathematical model of a system can be in different forms, such as algebraic equations, differential equations, finite state machines, etc. In the modeling framework considered on rule based fuzzy models, the relationships between variables are described by means of if- then rules, such as: If input is high then output will increase fast. These rules establish logical relations between the system's variables by relating qualitative values of one variable to qualitative values of another variable. The structure of the fuzzy system is composed of a set of if-then rules, where qualitative knowledge can be expressed in the form of rules IF "*condition*" THEN "*action*". The condition part (premise) contains facts in the form of symptoms as inputs and the conclusion part includes events as a logical cause of the facts. In the T-S model, the inference is reduced to a simple algebraic expression, similar to the fuzzy-mean defuzzification formula (Takagi and Sugeno [26, 27]). The algorithm for the development of T-S fuzzy model has the following steps: The optimal number of fuzzy rules is determined; The relevant input variables as antecedents are selected; The membership function parameters are estimated; The consequent structure is selected; The consequent parameters are estimated.

Let now consider the following T.S fuzzy dynamic model:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u\big), \tag{3}$$

$$y = \sum_{i=1}^{r} \mu_i(z) C_i x, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $y \in \mathbb{R}^q$ is the output. The matrices A_i, B_i and C_i are of appropriate dimension, $r \geqslant 2$ is the number of rules, z is the premise vector which may include unmeasurable variables. It is assumed that $\mu_i(z) \geqslant 0$, for all $i = 1, \dots, r$ and $\sum_{i=1}^{r} \mu_i(z) = 1$, for all $t \geqslant 0$.

In many practical control problems, the physical state variables of systems are partially or fully unavailable for measurement, since the state variables are not accessible by sensing devices and transducers are not available or very expensive. In such cases, observer based control schemes should be designed to estimate the state for (3). Taking $\hat{y}$ defined by

$$\hat{y} = \sum_{i=1}^{r} \mu_i(z) C_i \hat{x}.$$

In this case, an observer can be designed which has the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i \hat{x} + B_i u\big) - \sum_{i=1}^{r} \mu_i(z) L_i (\hat{y} - y). \tag{5}$$

Takagi-Sugeno model has proved its effectiveness in the study of nonlinear systems. Indeed, it gives a simpler formulation from the mathematical point of view to represent the behavior of nonlinear systems. Thanks to the convex sum property of the weighing functions, it is possible to generalize some tools developed in the linear domain to the nonlinear systems. This representation is very interesting in the sense that it simplifies the problem of the observer design.

3.1. Fuzzy linear systems

Let consider the following fuzzy model:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + B_i u\big) \tag{6}$$

with the output y defined as in (4).

Let consider an observer design for (3) of the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i \hat{x} + B_i u\big) - \sum_{i=1}^{r} \mu_i(z) L_i (\hat{y} - y), \tag{7}$$

where $\hat{y}$ is given by:

$$\hat{y} = \sum_{i=1}^{r} \mu_i(z) C_i \hat{x}.$$

Taking into account (5) and (7), the system error is given by:

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z)\mu_j(z)(A_i - L_jC_i)e. \tag{8}$$

Thus,

$$\dot{e} = \sum_{i=1}^{r}\mu_i^2\Upsilon_{ij}e + 2\sum_{i<j}\mu_i\mu_j(z)\Upsilon_{ij}e,$$

where

$$\Upsilon_{ii} = A_i - L_iC_i,$$

and

$$\Upsilon_{ij} = \frac{1}{2}\left(A_i - L_jC_i + A_j - L_jC_i\right).$$

Now, we can state the following theorem.

Theorem 1 *Suppose that there exist positive symmetric definite matrices $\widetilde{P}$, $\widetilde{Q}$ and some matrices L_i, $i = 1,\ldots,r$, such that the following inequalities hold,*

$$\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii} < -\widetilde{Q}, \quad i = 1,\ldots,r, \tag{9}$$

and

$$\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij} < -\widetilde{Q}, \quad 1 \leqslant i < j \leqslant r, \tag{10}$$

then the system error (8) is guaranteed to be globally uniformly exponentially stable.

Remark 1 *(9) and (10) can be written as LMIs by a simple congruence as in (Tanaka et al. [26]), with the terms $\widetilde{X} = \widetilde{P}^{-1}$, $L_j = \widetilde{P}N_j$ and $\widetilde{H} = \widetilde{X}\widetilde{Q}\widetilde{X}$.*

Proof. Consider the Lyapunov function candidate $V(e) = e^T\widetilde{P}e$. It's derivative with respect to time is given by:

$$\dot{V}(e) = \sum_{i=1}^{r}\mu_i^2e^T\left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right)e + 2\sum_{i<j}\mu_i\mu_je^T\left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right)e.$$

Let $\widetilde{\lambda}_0 = \lambda_{\min}(\widetilde{Q})$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix. Therefore, we have

$$e^T\left(\Upsilon_{ii}^T\widetilde{P} + \widetilde{P}\Upsilon_{ii}\right)e \leqslant -\widetilde{\lambda}_0\|e\|^2, \quad i = 1,\ldots,r,$$

and

$$e^T\left(\Upsilon_{ij}^T\widetilde{P} + \widetilde{P}\Upsilon_{ij}\right)e \leqslant -\widetilde{\lambda}_0\|e\|^2, \quad 1 < i < j < r.$$

Then, one gets

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0 \|e\|^2 \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \,.$$

Taking into account the above expressions, it follows that

$$\dot{V}(e) \leqslant -\widetilde{\lambda}_0 \|e\|^2.$$

Since

$$V(e) = e^T \widetilde{P} e \leqslant \lambda_{\max}(\widetilde{P}) \|e\|^2,$$

one gets

$$\dot{V}(e) \leqslant -\frac{\widetilde{\lambda}_0}{\lambda_{\max}(\widetilde{P})} V(e).$$

Thus, we obtain the following estimation:

$$\|e(t)\| \leqslant \left(\frac{\lambda_{\max}(\widetilde{P})}{\lambda_{\min}(\widetilde{P})} \right)^{1/2} \|e(0)\| e^{-\frac{1}{2}\widetilde{\lambda}_0 t}.$$

Then, we deduce that (7) is an exponential observer for (5). □

4. Fuzzy bilinear systems

An interesting class on nonlinear systems is the class of bilinear systems:

$$\dot{x}(t) = Ax(t) + uBx(t), \quad x(0) = x_0, \quad y(t) = Cx(t),$$

where x is the state, u is the control and y is the output of the system, one can estimate the states by using an observer, whose structure is as follows:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + uB\hat{x}(t) - L(C\hat{x}(t) - y(t)),$$

where $\hat{x}(t)$ is the state of the observer. It is needed that the estimation error, $e(t) = \hat{x}(t) - x(t)$ has to converge as fast as possible to zero for a suitable choice of the gain matrix L. Most current methods lead to the design of an exponential observer, exponential stability is the most wanted. With the model given below, the problem is to design a continuous observer with input $y(t)$ such that the estimates denoted by $\hat{x}(t)$ converge to $x(t)$ exponentially fast. We shall assume that the pair (A, C) is observable. Suppose the observability matrix for the time invariant associated linear system. Then, there exists a gain matrix $L(n \times p)$ such that the error equation

$$\dot{e}(t) = (A - LC)e(t) \quad \text{where} \quad e(t) = \hat{x}(t) - x(t)$$

is exponentially stable. In this condition, one can design an exponential observer for the bilinear system as:

$$\dot{x}(t) = A\hat{x}(t) + uB\hat{x}(t) - L(C\hat{x}(t) - y(t)).$$

provided the control u is small enough, where the matrix L is chosen such that

$$(A - LC)^T P + P(A - LC) = -Q,$$

with P and Q are $(n \times n)$ positive definite symmetric matrices. Therefore, using a Kalman like observer with the gain matrix is chosen as $L = S_\theta^{-1} C^T$, we can design a state observer as follows (see [6, 7, 12]):

$$\dot{x}(t) = A\hat{x}(t) + uB\hat{x}(t) - S_\theta^{-1} C^T (C\hat{x}(t) - y(t)),$$

where S_θ satisfies the following stationary equation:

$$0 = -\theta S_\theta - A^T S_\theta - S_\theta A + C^T C, \quad \theta > 0,$$

$S_\theta = \lim_{t\to+\infty} S_t$ with $S_t \in \mathcal{S}^+$ the cone of symmetric positive definite matrices on $\mathbb{R}^n$ which satisfies

$$\dot{S}_t = -\theta S_t - A^T S_t - S_t A + C^T C.$$

It is known from [6] that, under the the condition of observability for any input of the bilinear system, this observer converges for any bounded and small input u which is distant from bad inputs or just $u(t)$ is a regularly persistent input. Indeed, since the pair (A, C) is observable, $u = 0$ constitutes an universal input. Thus, for a small input u there exists $\varepsilon > 0$ such that for $|u| < \varepsilon$, u is universal too (i.e. if it distinguishes the points, that is, for all initial conditions $(x_0; \bar{x}_0)$ there exists $T > 0$ such that $C(x^u(T)) \neq C(\bar{x}^u(T))$, where ${}^x u(t)$ is the approximate solution such that $x^u(0) = x_0$). Therefore, the Gramm observability matrix satisfies,

$$W_u(t) = \int_0^t e^{-\theta(t-s)} ({}^T\Phi_{u(s)}(t-s))^{-1} C^T C \Phi^{-1}{}_{u(s)}(t-s)\,\mathrm{d}s \geqslant \alpha I, \quad \alpha > 0.$$

On the one hand, if we consider the error equation:

$$\dot{e}(t) = (A + uB)e(t) - S_\theta^{-1} C^T Ce.$$

The error satisfies the estimate:

$$\|e(t)\| \leqslant k e^{-\frac{1}{2}\theta t}, \quad k > 0,$$

where the constant k depends only on the initial state and u. This implies that the estimate error converges exponentially. On the other hand, for a bounded control $u(t)$ the matrix $S(t)$ is bounded. Therefore, since the bilinear system is observable for any input and the matrix $S(t)$ is bounded, by using some techniques regarding the Ricatti equations as in [6], we can show that the matrix $S^{-1}(t)$ is also bounded.

For the class of fuzzy bilinear systems, under the the condition of observability for any input of the bilinear system, this observer converges for any bounded and small input u which is distant from bad inputs or just $u(t)$ is a regularly persistent input. Let now consider the following T.S fuzzy bilinear model:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)\big(A_i x + uB_i x\big), \tag{11}$$

$$y = \sum_{i=1}^{r} \mu_i(z) C_i x\,, \tag{12}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}^q$ is the output. The matrices A_i, B_i and C_i are of appropriate dimension, then an observer can be designed which has the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i + uB_i\big)\hat{x} - \sum_{i=1}^{r} \mu_i(z) L_i(\hat{y} - y).$$

The system error is given by:

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z)\mu_j(z)\big((A_i + uB_i) - L_j C_i\big)e.$$

Thus,

$$\dot{e} = \sum_{i=1}^{r} \mu_i^2 \widetilde{\Upsilon}_{ij} e + 2\sum_{i<j} \mu_i \mu_j(z) \widetilde{\Upsilon}_{ij} e,$$

where

$$\widetilde{\Upsilon}_{ii} = (A_i + uB_i) - L_i C_i,$$

and

$$\widetilde{\Upsilon}_{ij} = \frac{1}{2}\big((A_i + uB_i) - L_j C_i + (A_j + uB_j) - L_j C_i\big)\,.$$

According to the above analysis, the design procedure for T-S fuzzy systems is summarized as follows:

Step 1: Verify that assumption (A_l, C_l) are observable for $l = 1, \ldots, r$ and then for all $i = 1, \ldots, r$, $(A_i\hat{x} + B_i u)$ are observable for small input u.

Step 2: Solve the Lyapunov equations

$$\widetilde{\Upsilon}_{ii}^T P + P\widetilde{\Upsilon}_{ii} = -Q_i\,, \quad i = 1,\ldots,r,$$

and

$$\frac{1}{2}\widetilde{\Upsilon}_{ij}^T P + \frac{1}{2}P\widetilde{\Upsilon}_{ij} = -Q_i, \quad 1 \leqslant i < j \leqslant r,$$

by using the control toolbox, to determine a common definite positive symmetric matrix P for $i = 1,\ldots,r$. For simplicity, one can take $Q_i = I$.

Step 3: Construct the gain matrices as

$$L_i = S_\theta^{-1}C_i^T, \;\; i = 1,\ldots,r,$$

where S_θ satisfies the following stationary equations:

$$0 = -\theta S_\theta - A_i^T S_\theta - S_\theta A_i + C_i^T C_i, \;\; \theta > 0.$$

Therefore as in the proof of Theorem 1, if we consider the Lyapunov function candidate $V(e) = e^T Pe$, it's derivative with respect to time along the error equation gives $\dot{V}(e) \leqslant -\eta V(e)$ for a certain nonnegative constant η which proves that the error converges exponentially to the origin.

4.1. Uncertain fuzzy bilinear systems

Let now consider the following T.S uncertain fuzzy bilinear model:

$$\dot{x} = \sum_{i=1}^{r}\mu_i(z)\big(A_i x + uB_i x + d_i(t)\big), \tag{13}$$

$$y = \sum_{i=1}^{r}\mu_i(z)C_i x \tag{14}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}^q$ is the output. The matrices A_i, B_i and C_i are of appropriate dimension, $d_i(t)$ are some external disturbances for $i = 1,\ldots,r$, $r \geqslant 2$ is the number of rules, z is the premise vector which may include unmeasurable variables. It is assumed that: $\mu_i(z) \geqslant 0$, for all $i = 1,\ldots,r$ and $\sum_{i=1}^{r}\mu_i(z) = 1$, for all $t \geqslant 0$.

The function d_i represent the uncertain external disturbance of each fuzzy subsystem and are time-varying satisfying the following inequality,

$$\|d_i(t)\| \leqslant \zeta_i(t), \quad i = 1,2,\ldots,r, \tag{15}$$

for all $t \geqslant 0$ and $x \in \mathbb{R}^n$, where $\zeta_i(.)$ are known positive continuous functions satisfying that, there exists κ a nonnegative scalar constant satisfying,

$$\zeta(t) \leqslant \kappa,$$

with

$$\zeta(t) := \left(\sum_{i=1}^{r} \zeta_i(t)^2\right)^{1/2}.$$

One can consider an observer which has the form:

$$\dot{\hat{x}} = \sum_{i=1}^{r} \mu_i(z)\big(A_i + uB_i\big)\hat{x} - \sum_{i=1}^{r} \mu_i(z)L_i(\hat{y} - y). \tag{16}$$

We have the following theorem.

Theorem 2 *Suppose that there exist positive symmetric definite matrices P, Q and some matrices L_i, $i = 1, \ldots, r$, such that the following inequalities hold,*

$$\widetilde{\Upsilon}_{ii}^T P + P\widetilde{\Upsilon}_{ii} < -Q, \; i = 1, \ldots, r, \tag{17}$$

and

$$\widetilde{\Upsilon}_{ij}^T P + P\widetilde{\Upsilon}_{ij} < -Q, \; 1 \leqslant i < j \leqslant r, \tag{18}$$

also the disturbances satisfy (15), then the system (16) is an exponential observer for system (13) in the sense that the error converges exponentially to the compact set

$$\mathcal{S} = \left\{ e \in \mathbb{R}^n / \; \|e\| \leqslant 2\frac{\lambda_{\max}^2(P)}{\lambda_0 \lambda_{\min}^{1/2}(P)}\kappa \right\},$$

where $\lambda_0 = \lambda_{\min}(Q)$.

Remark 2 *(17) and (18) can be written as LMIs by a simple congruence as in (Tanaka et al. [26]), with the terms $X = P^{-1}$, $L_j = PN_j$ and $H = XQX$.*

Proof. The system error is given by:

$$\dot{e} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z)\mu_j(z)\big((A_i + uB_i) - L_jC_i\big)e + \sum_{i=1}^{r} \mu_i(z)d_i(t).$$

Thus,

$$\dot{e} = \sum_{i=1}^{r} \mu_i^2 \widetilde{\Upsilon}_{ij} e + 2\sum_{i<j} \mu_i\mu_j(z)\widetilde{\Upsilon}_{ij} e + \sum_{i=1}^{r} \mu_i d_i(t),$$

where

$$\widetilde{\Upsilon}_{ii} = (A_i + uB_i) - L_iC_i,$$

and

$$\widetilde{\Upsilon}_{ij} = \frac{1}{2}\left((A_i + uB_i) - L_jC_i + (A_j + uB_j) - L_jC_i\right).$$

Consider the Lyapunov function candidate $V(e) = e^T Pe$. It's derivative with respect to time is given by:

$$\dot{V}(e) = \sum_{i=1}^{r} \mu_i^2 e^T \left(\widetilde{\Upsilon}_{ii}^T P + P\widetilde{\Upsilon}_{ii}\right) e + 2\sum_{i<j} \mu_i\mu_j e^T \left(\widetilde{\Upsilon}_{ij}^T P + P\widetilde{\Upsilon}_{ij}\right) e$$
$$+\ 2x^T P \sum_{i=1}^{r} \mu_i d_i(t).$$

Let $\lambda_0 = \lambda_{\min}(Q)$, $\lambda_{\min}$ denoting the smallest eigenvalue of the matrix. Thus, we have

$$e^T(\widetilde{\Upsilon}_{ii}^T P + P\widetilde{\Upsilon}_{ii})e \leqslant -\lambda_0\|e\|^2, \quad i = 1, \ldots, r,$$

and

$$e^T(\widetilde{\Upsilon}_{ij}^T P + P\widetilde{\Upsilon}_{ij})e \leqslant -\lambda_0\|e\|^2, \quad 1 < i < j < r.$$

Then, one gets

$$\dot{V}(e) \leqslant -\lambda_0\|e\|^2 \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j + 2x^T P \sum_{i=1}^{r} \mu_i d_i(t).$$

Remark that,

$$\|\sum_{i=1}^{r} \mu_i d_i(t)\| \leqslant \sum_{i=1}^{r} \mu_i(\zeta_i(t)).$$

Taking into account the above expressions, it follows that

$$\dot{V}(e) \leqslant -\lambda_0\|e\|^2 + 2\|x\|\|P\| \sum_{i=1}^{r} \mu_i(\zeta_i(t)).$$

Then, one gets

$$\dot{V}(e) \leqslant -\lambda_0\|e\|^2 + 2\|x\|\|P\|\left(\left(\sum_{i=1}^{r} \mu_i^2\right)^{1/2} \left(\sum_{i=1}^{r} \zeta_i(t)^2\right)^{1/2}\right).$$

On the one hand, in the case where $d_i(t) = 0$, for all $i = 1, \dots, r$, then using the fact that,

$$V(e) = e^T P e \leqslant \lambda_{\max}(P)\|e\|^2.$$

one obtains,

$$\dot{V}(e) \leqslant -\frac{\lambda_0}{\lambda_{\max}(P)} V(e),$$

and so, we obtain the following estimation:

$$\|e(t)\| \leqslant \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)^{1/2} \|e(0)\| e^{-\frac{1}{2}\lambda_0 t}.$$

Then, we deduce that (16) is an exponential observer for (13).

On the other hand, in the case where $d_i(t)$ are not all zero, it means that in presence of $d_i(t)$ satisfying (15), at least there exists $i = 1, \dots, r$ such that $d_i(t) \neq 0$, then we have

$$\dot{V}(e) \leqslant -\lambda_0 \frac{1}{\lambda_{\max}(P)} V(e) + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}^{1/2}(P)} \zeta(t) V(e)^{1/2}.$$

If follows that,

$$\dot{V}(e) \leqslant -\lambda_0 \frac{1}{\lambda_{\max}(P)} V(e) + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}^{1/2}(P)} \kappa V(e)^{1/2}.$$

Let,

$$r_0 = \lambda_0 \frac{1}{\lambda_{\max}(P)}, \qquad r_1 = 2\frac{\lambda_{\max}(P)}{\lambda_{\min}^{1/2}(P)} \kappa.$$

With the previous notations, it follows that

$$\dot{V}(e) \leqslant -r_0 V(e) + r_1 V(e)^{1/2}.$$

In the last expression, we make the following change of variable, $u(t) = V(e)^{1/2}$. The derivative with respect to time is given by

$$\dot{u}(t) = \frac{\dot{V}(e)}{2V(e)^{1/2}}.$$

This implies that,

$$\dot{u}(t) \leqslant -\frac{1}{2} r_0 u(t) + \frac{1}{2} r_1.$$

Therefore, a simple computation gives:

$$\dot{u}(t) \leqslant \left(u(0) - \frac{r_1}{r_0}\right) e^{-\frac{r_0}{2}t} + \frac{r_1}{r_0}.$$

$$\|e(t)\| \leqslant \frac{1}{\lambda_{\min}^{1/2}(P)} \left(\lambda_{\max}^{1/2}(P)\|e(0)\| - 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa\right) e^{-\frac{\lambda_0}{2\lambda_{\max}(P)}t} + 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa.$$

So, for all initial conditions taken outside

$$\mathcal{S} = \{e \in \mathbb{R}^n / \ \|e\| \leqslant 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa\},$$

one has,

$$\|e(t)\| - 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa \leqslant \frac{1}{\lambda_{\min}^{1/2}(P)} \left(\lambda_{\max}^{1/2}(P)\|e(0)\| - 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa\right) e^{-\frac{\lambda_0}{2\lambda_{\max}(P)}t}.$$

Hence, the error solution converges globally and exponentially to the the compact set $\mathcal{S}$. □

5. Application to Van de Vusse system

Consider the dynamics of an isothermal continuous stirred tank reactor (cstr) for the Van de Vusse example [24], given by:

$$\begin{cases} \dot{x}_1 = -k_1x_1 - k_3x_1^2 + u(C_{AO} - x_1) + d(t), \\ \dot{x}_2 = k_1x_1 - k_2x_2 + u(-x_2), \\ \ y = x_1\,, \end{cases} \tag{19}$$

where the state x_1 represents the concentration of the reactant inside the reactor (mol/L) and the state x_2 is the concentration of the product in the (cstr) output stream (mol/L). The output y determines the grade of the final product. The input-feed stream to the (cstr) consists of a reactant with concentration C_{AO} and the controlled in the dilution rate $u = \dfrac{F}{V}(h^{-1})$ where F is the input flow rate to the reactor (L/h) and V is the constant volume of the (cstr) (litres). In all the following discussions, the kinetic parameters are chosen to be $k_1 = 50h^{-1}$, $k_2 = 100h^{-1}$, $k_3 = 10$ L/(molh), $C_{AO} = 10$ mol/L and $V = 1L$. The system (19) can be written as: with the output $y = Cx$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$,

$$\begin{cases} \dot{x} = Ax + uBx + d(t), \\ y = Cx \end{cases} \tag{20}$$

where $x = (x_1, x_2)^T$ and

$$A(x) = \begin{bmatrix} -k_1 - k_2x_2 & 0 \\ k_1 & -k_2 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and

$$d(t) = \Delta(u(t)) + d(t),$$

where

$$d(t) = \big(\Delta_1(u(t)), \quad \Delta_2(u(t))\big), \qquad d(t) = \big(d_1(t) \quad d_2(t)\big).$$

We define the membership functions as

$$\mu_1(x_1(t)) = \frac{1 - x_1(t)}{2} \quad \text{and} \quad \mu_2(x_1(t)) = \frac{x_1(t) + 1}{2}.$$

The associated fuzzy system, with $x_1(t) \in [-1, 1]$ and $y = Cx(t) = [1, 0](x_1, x_2)^T$, can be represented as:

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_1)\big(A_ix(t) + u(t)B_ix(t) + d_i(t)\big) \tag{21}$$

with

$$A_1 = \begin{bmatrix} -60 & 0 \\ 50 & -100 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -40 & 0 \\ 50 & -100 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The fuzzy output matrices are given by,

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Let $L_i = (l_1, l_2)^T$, $i = 1,\ 2$, one has $\mathcal{R}e\lambda(A_i + L_iC_i) < 0$ for $l_1 = l_2 = 1$, $i = 1,\ 2$. Therefore, taking into account Theorem 2, with $d_1(t) = \dfrac{1}{1+t^2}$ and $d_2(t) = \dfrac{1}{1+t^2}$, one can design an observer which has the following form:

$$\dot{\hat{x}} = \sum_{i=1}^{2} \mu_i(z)\,(A_i + uB_i)\,\hat{x} - \sum_{i=1}^{r} \mu_i(z)L_i\,(\hat{y} - y)\,.$$

Thus, with $e(t) = \hat{x}(t) - x(t)$, one gets the following estimation:

$$\|e(t)\| \leqslant \frac{1}{\lambda_{\min}^{1/2}(P)} \left(\lambda_{\max}^{1/2}(P)\|e(0)\| - 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa\right) e^{-\frac{\lambda_0}{2\lambda_{\max}(P)}t} + 2\frac{\lambda_{\max}^2(P)}{\lambda_0\lambda_{\min}^{1/2}(P)}\kappa.$$

By using the initial conditions $x_1(0) = 1$, $x_2(0) = 1$, $\hat{x}_1(0) = 1$ and $\hat{x}_2(0) = 1$, Figure 1 illustrate the simulation results for the estimated states.

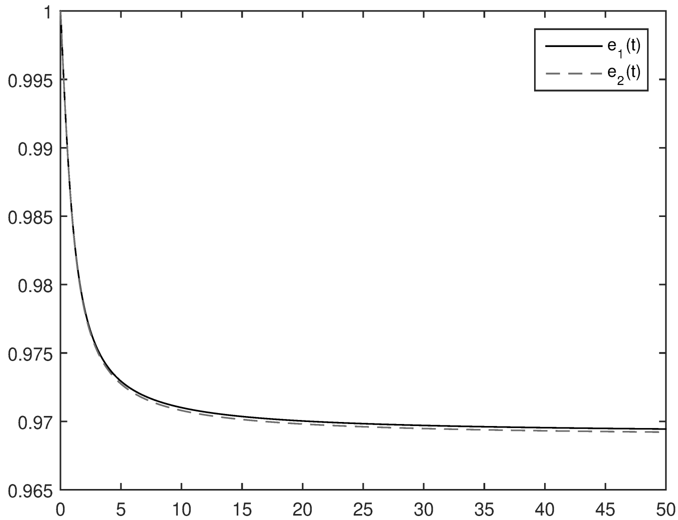

Figure 1: Time evolution of the estimation $e(t) = (e_1(t), e_2(t))$

6. Conclusion

In this paper, a new way to simplify the design of observer for T-S fuzzy bilinear system is presented. Some results are obtained, the observer can therefore be designed under some sufficient conditions. Moreover, an application to Van de Vusse system is given.

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