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Transformations of the matrices of linear systems to their canonical form with desired eigenvalues

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Abstract. A new approach to the transformations of the matrices of linear continuous-time systems to their canonical forms with desired eigenvalues is proposed. Conditions for the existence of solutions to the problems were given and illustrated by simple numerical examples.

Key words: canonical form; desired eigenvalue; linear system; transformation.

1. INTRODUCTION

The concepts of controllability and observability introduced by Kalman [9, 10] are the basic notions of modern control theory. It is well-known that if the linear system is controllable then by the use of state feedback it is possible to modify the dynamical properties of the closed-loop systems [1, 2, 5–8, 11–13, 17]. If the linear system is observable then it is possible to design an observer that reconstructs the state vector of the system [1, 2, 5–8, 11–13, 17]. Descriptor systems of integer and fractional order were analyzed in [6, 14, 16]. The stabilization of positive descriptor fractional linear systems with two different fractional orders by the decentralized controller was investigated in [16]. A survey of the matrix black box algorithms was given in [14]. The eigenvalues assignment in uncontrollable linear continuous-time systems was analyzed in [4].

In this paper, new approaches to the transformations of the linear continuous-time systems to their asymptotically stable canonical controllable (observable) forms with desired eigenvalues are proposed. In Section 2 some basic definitions and theorems concerning linear standard continuous-time systems and systems of algebraic matrix equations are recalled. A new approach to the transformations of the linear systems to their asymptotically stable controllable and observable canonical forms with desired eigenvalues is proposed in Sections 3 and 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, I_n – the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input, and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Theorem 1. [1, 8–13] The solution of equation (1a) has the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau, \quad x_0 = x(0). \tag{2}$$

Definition 1. [1, 8–13] The linear system (1) is called controllable in time $[0, t_f]$ if there exists an input $u(t) \in \Re^m$ for $t \in [0, t_f]$ which steers the state of the system from the zero initial condition $x(0) = 0$ to the final state $x_f = x(t_f)$.

Theorem 2. [1, 8–13] The linear system (1a) is controllable if and only if

1) $$\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n, \tag{3a}$$

2) $$\operatorname{rank}\begin{bmatrix} I_n s - A & B \end{bmatrix} = n \quad \text{for} \quad s \in \mathrm{W}, \tag{3b}$$

where W is the field of complex numbers.

Definition 2. [6, 8] The continuous-time linear system (1) is called observable if knowing its input $u(t)$ and output $y(t)$ in some given interval $[0, t_f]$ it is possible to find its unique initial condition $x(0)$.

Theorem 3. [1, 8–13] The continuous-time linear system (1) is observable if and only if one of the following conditions is satisfied:

1) $$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{4a}$$

2) $$\operatorname{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = n \quad \text{for} \quad s \in \mathrm{W}, \tag{4b}$$

where W is the field of complex numbers.

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Theorem 4. [3] (Kronecker–Cappelli). Matrix equation

$$PX = Q, \quad P \in \Re^{n\times p}, \quad Q \in \Re^{n\times q} \tag{5}$$

has a solution X if and only if

$$\operatorname{rank}\begin{bmatrix} P & Q \end{bmatrix} = \operatorname{rank} P. \tag{6}$$

Theorem 5. [3] If condition (6) is satisfied then the solution $X \in \Re^{p\times q}$ of matrix equation (5) for $P \in \Re^{n\times p}$ is given by

$$X = \left\{P^T[PP^T]^{-1} + (I_q - P^T[PP^T]^{-1}P)K_1\right\}Q, \tag{7a}$$

or

$$X = K_2[PK_2]^{-1}Q, \tag{7b}$$

where K_1, K_2 are real matrices, $\operatorname{rank} P = n$ and $\det[PK_2] \neq 0$.

3. TRANSFORMATIONS OF THE PAIRS (A, B) AND (A, C) TO THE DESIRED PAIRS IN CANONICAL FORMS AND WITH GIVEN EIGENVALUES

The following two cases will be considered for nonsingular matrix A ($\det A \neq 0$).

Case 1. $m \geq p$. It is assumed that

$$\operatorname{rank}[CA^{-1}B] = p. \tag{8}$$

In this case matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \in \Re^{(n+p)\times(n+m)} \tag{9}$$

has full row rank equal to $n + p$, since

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ CA^{-1} & I_p \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & -CA^{-1}B \end{bmatrix}\begin{bmatrix} I_n & A^{-1}B \\ 0 & I_p \end{bmatrix}. \tag{10}$$

Note that in this case

$$\lim_{s\to 0} T(s) = \lim_{s\to 0}\left\{C[I_n s - A]^{-1}B\right\} = -CA^{-1}B \neq 0 \tag{11}$$

for nonzero matrices B and C, where $T(s)$ is the transfer matrix of system (1).

To simplify the notation we assume $m = p = 1$.

Consider the equation

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} M = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}, \tag{12}$$

where

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{13}$$

$$\overline{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

and the $\overline{A}$ has the desired eigenvalues $s_1, s_2, \dots, s_n$ satisfying the stability condition

$$\operatorname{Re} s_k < 0 \quad \text{for } k = 1, \dots, n. \tag{14}$$

In this case matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \tag{15}$$

is nonsingular and from (12) we have

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1}\begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}. \tag{16}$$

Therefore, knowing the matrices A, B, C and $\overline{A}$, $\overline{B}$, $\overline{C}$ we may compute the desired nonsingular matrix (16).

Theorem 6. If $\det A \neq 0$, matrix (9) is nonsingular and the matrices $\overline{A}, \overline{B}, \overline{C}$ have the canonical forms (13) then the nonsingular matrix M is given by (16).

Example 1. For the given matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{17}$$

and

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{18}$$

compute matrix $M \in \Re^{3\times 3}$ satisfying (12).

In this case the matrices

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{19}$$

are nonsingular and equation (12) has the form

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} M = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{20}$$

and its solution

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}. \tag{21}$$

Matrix (21) is nonsingular.

Now let us assume that $m > p > 1$ and

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p. \tag{22}$$

In this case, by Theorem 4 equation (12) has many solutions which can be computed using (7). The solutions depend on the matrices K_1 and K_2.

Example 2. For the given matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{23}$$

and

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{24}$$

Compute matrix $M \in \Re^{3\times 3}$ satisfying (12).

In this case the matrices

$$\begin{gathered} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{gathered} \tag{25}$$

have full row ranks and equation (12) has the form

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

Using (7b) for

$$\begin{gathered} P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & k_2 \\ 0 & k_3 & 0 \\ 0 & k_4 & 0 \end{bmatrix}, \\ Q = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{gathered} \tag{27}$$

we obtain

$$\begin{aligned} M &= K_2 \left[PK_2\right]^{-1} Q \\ &= \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & k_2 \\ 0 & k_3 & 0 \\ 0 & k_4 & 0 \end{bmatrix} \begin{bmatrix} k_1 & k_3 & 0 \\ 0 & k_4 & 2k_2 \\ 0 & 0 & k_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4k_3}{k_4} & 1 + \frac{3k_3}{k_4} & 1 & -\frac{k_3}{k_4} \\ 1 & 0 & 0 & 0 \\ -\frac{4k_3}{k_4} & -\frac{3k_3}{k_4} & 0 & \frac{k_3}{k_4} \\ -4 & -3 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{28}$$

Matrix (28) is singular even for nonzero k_1, k_2 and k_4.

Case 2. $p \geq m$.

Consider matrix equation

$$\overline{M} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}, \tag{29}$$

where the pair $(\overline{A}, \overline{B})$ is controllable, the pair $(\overline{A}, \overline{C})$ is observable and matrix $\overline{A}$ has the desired eigenvalues satisfying (14).

In this case, it is assumed that

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + m, \tag{30a}$$

or equivalently

$$\operatorname{rank}\left[CA^{-1}B\right] = m. \tag{30b}$$

To simplify the notation, it is assumed $m = p = 1$. In this particular case the matrices $\overline{A}, \overline{B}, \overline{C}$ have canonical forms (13). Applying the transposition to equation (29) we obtain

$$\begin{bmatrix} A^T & C^T \\ B^T & 0 \end{bmatrix} M^T = \begin{bmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & 0 \end{bmatrix}, \tag{31}$$

where T denotes the transposition.

Therefore, the problem in Case 2 has been reduced to the dual problem analyzed in Case 1, and we have the following Theorem.

Theorem 7. If $m = p = 1$, $\det A \neq 0$,

$$\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \neq 0 \tag{32}$$

and the matrices $\overline{A}, \overline{B}, \overline{C}$ have the canonical forms (13) then the nonsingular matrix M^T is given by

$$M^T = \begin{bmatrix} A^T & C^T \\ B^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & 0 \end{bmatrix}. \tag{33}$$

Example 3. For given matrices

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \end{bmatrix} \tag{34}$$

and

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 2 & 0 \end{bmatrix}. \tag{35}$$

compute matrix $M^T \in \Re^{3\times 3}$ satisfying equation (31).

In this case the matrices

$$\begin{bmatrix} A^T & C^T \\ B^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 2 \\ 1 & -4 & 0 \\ 0 & 3 & 0 \end{bmatrix} \tag{36}$$

are nonsingular and equation (31) has the form

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} M^T = \begin{bmatrix} 0 & -3 & 2 \\ 1 & -4 & 0 \\ 0 & 3 & 0 \end{bmatrix} \tag{37}$$

and its solution is given by

$$M^T = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -3 & 2 \\ 1 & -4 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & -1.5 & 1 \\ 0.5 & -3.5 & 0 \end{bmatrix}. \tag{38}$$

Matrix (38) is nonsingular.

In a similar way as in Case 1 the considerations can be easily extended to $m+p>2$.

4. EXTENSIONS TO LINEAR SYSTEMS WITH SINGULAR STATE MATRICES

In this Section, the considerations of Section 3 will be extended to linear systems (1) with singular state matrices ($\det A = 0$).

Case 1. $m > p$.

To simplify the notation we assume $m = p = 1$.

Consider the equation

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} N = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix}, \tag{39}$$

where

$$\det \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \neq 0, \quad \det \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} \neq 0 \tag{40}$$

and the desired matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ have the forms (13).

If condition (40) is satisfied then from (39) we have

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} \tag{41}$$

and $\det N \neq 0$.

Theorem 8. If $\det A = 0$, the condition (40) is satisfied, and desired matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ have the canonical forms (13) then nonsingular matrix N is given by (41).

Example 4. For given matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{42}$$

and

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{43}$$

compute matrix N satisfying equation (39).

Note that matrix A given by (42) is singular, the pair (A, B) is not controllable, and the pair (A, C) is observable.

In this case the matrices

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{44}$$

are nonsingular and equation (39) has the form

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} N = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{45}$$

The solution of (45) has the form

$$N = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1.5 & 0 \\ 1 & 2.5 & 1 \end{bmatrix} \tag{46}$$

and it is nonsingular.

The considerations can be easily extended to the case $n+m > n+p$.

Case 2. $p > m$.

Consider matrix equation

$$\overline{N} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{BC} & 0 \end{bmatrix}, \tag{47}$$

where the pair $(\overline{A}, \overline{B})$ is controllable, the pair $(\overline{A}, \overline{C})$ is observable and matrix $\overline{A}$ has the desired eigenvalues satisfying (14).

It is assumed that

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n+m. \tag{48}$$

To simplify the notation it is assumed that $m = p = 1$ and the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ have the canonical forms (13). Applying the transposition to equation (47) we obtain

$$\begin{bmatrix} A^T & C^T \\ B^T & 0 \end{bmatrix} \overline{N}^T = \begin{bmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & 0 \end{bmatrix}. \tag{49}$$

Therefore, the problem has been reduced to the dual problem analyzed in Case 1.

Theorem 9. If $m = p = 1$, conditions (40) are satisfied and the desired matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ have the canonical forms (13) then the nonsingular matrix N is given

$$\overline{N}^T = \begin{bmatrix} A^T & C^T \\ B^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & 0 \end{bmatrix}. \tag{50}$$

The proof is similar to the proof of Theorem 7.

The considerations can be easily extended to the case $n + m > n + p$.

5. CONCLUDING REMARKS

A new approach to the transformations of the matrices of linear continuous-time systems to their canonical forms with desired eigenvalues is proposed. Conditions for the existence of solutions to the problems are given (Theorems 6–9) and illustrated by simple numerical examples. The considerations can be easily extended to linear discrete-time systems. An open problem is an extension of the considerations to fractional orders linear systems.

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