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Multi–server loss queueing system with random volume customers, non–identical servers and a limited sectorized memory buffer

Marcin ZIÓŁKOWSKI *

Institute of Information Technology, Warsaw University of Life Sciences – SGGW, Poland

Abstract. In the present paper, the model of multi–server queueing system with random volume customers, non–identical (heterogeneous) servers and a sectorized memory buffer has been investigated. In such system, the arriving customers deliver some portions of information of a different type which means that they are additionally characterized by some random volume vector. This multidimensional information is stored in some specific sectors of a limited memory buffer until customer ends his service. In analyzed model, the arrival flow is assumed to be Poissonian, customers' service times are independent of their volume vectors and exponentially distributed but the service parameters may be different for every server. Obtained results include general formulae for the steady–state number of customers distribution and loss probability. Special cases analysis and some numerical computations are attached as well.

Key words: multi–server queueing system with heterogeneous servers; queueing systems with random volume customers; sectorized memory buffer; loss probability; Stieltjes convolution.

1. INTRODUCTION

Queueing theory (QT) is the field of applied mathematics that has been facing the problem of modeling real–life telecommunication and computer systems since early years of the 20th century. At the beginning, it helped to design telecommunication centers (mainly telephone exchanges) and gave first results connected with investigations of simple models of the $M/M/n/m$, $M/G/n/0$ and $M/G/1/\infty$–types (in this paper modified Kendall's notation of the queueing system of the $M/G/n/m$–type is used, where m denotes maximal number of customers waiting in the queue (waiting room) – without those that are on service) which were basic for future research [1]. Almost all first models assumed that servers are identical taking into consideration their service time distribution function. Later on, some works analyzed also systems with non–identical servers [2–5].

After years, together with the headway in telecommunication and computer science, more and more complicated models were analyzed and QT itself became the basic discipline used during analysis and designing process of computer systems, networks and so on. In the 70s of the twentieth century first investigations appeared that tried to analyze systems in which customers transport information that is stored in some memory buffer (so they are non–homogeneous in some sense as having random volume (size)). The practical aim of these works was to calculate or approximate needed size of such memory buffers analyzing customers' total volume characteristics (the sum of the volumes of all customers present in the queueing system). Many of first papers usually used only classical models of QT and authors did not take into account that the character of dependency between customer volume and his service time has an important meaning for customers' summary (total) volume characteristics [6, 7]. Moreover, it was often assumed that analyzed random variables are independent, which is sometimes possible in real systems but it is not the only case, e.g. in computer networks service time of a data packet is usually proportional to its size so they are strongly dependent. In many papers published both in the past and in recent years, customer volume is assumed to be exponentially distributed (say with parameter $f > 0$) and service time of a customer is assumed to be proportional to his volume (say with coefficient $c > 0$) which means that customer service time is also exponentially distributed with parameter f/c. Many authors of investigations from the above-mentioned area assumes that dependence between customer volume and his service time is then considered in the parameter of customer service time distribution function. In fact, it is not true from the mathematical point of view as we take into account only projection on one–dimensional distribution function of the customer service time and still treat these random variables as independent. It can be easily shown that customers total volume characteristics depend strictly on two–dimensional joint distribution function of the customer volume and his service time (see e.g. [8]). But this simplification (assuming independence in some partial form) let obtain approximate performance characteristics of the system.

In some works, it was noticed that such system analysis needs extensions (compared to classical methods) that assume exis-

*e-mail: marcin_ziolkowski@sggw.edu.pl

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tence of a limited memory buffer and dependence between customer volume and his service time [9, 10]. It started a new direction in QT called theory of queueing systems with random volume customers. In future investigations, new models were analyzed and they were divided into four classes according to their level of mathematical complexity (the number of classes is connected with the fact that we analyze systems with limited or unlimited memory buffer and also systems with dependence or independence between customer service time and his volume). In monograph [8] one of the main chapters is dedicated to such models, the most important results can also be found in [11]. Unfortunately, the most difficult to analyze are those models in which memory buffer is limited and service time of a customer is dependent on his volume (which is very often in practice) and exact analysis was possible only in cases where there was no queue [12, 13].

In last years the number of publications from this area has been increasing and there is no problem finding scientists from different countries that deal with analogous anlyses (see e.g. [14–18]) but in most of these papers, the analyzed models still assume the above-mentioned independence between random characteristics of the arriving customers. Authors relatively seldom discuss models with random volume customers and non–identical servers. The examples of such papers are [19, 20]. In the second of them, very interesting investigation of a complicated model belonging to the most difficult class was presented, together with analysis of some special cases showing that the character of dependency between customer service time and his volume has a substantial influence on main characteristics of the system even on the level of steady–state number of customers distribution and the value of loss probability.

The newest, novel approach is concentrated on the fact that customers should be sometimes treated as multidimensional (in the sense of their volume) because they transport information of a different type. For example, in computer networks packets may contain separate text parts, audio parts, video parts and parts consisting of data connected with description of the packet (its volume, header and time of the beginning of its transmission) that are stored independently in sectors of memory buffer dedicated to them, which is schematically presented in Fig. 1. These parts may be both dependent and independent. Technical

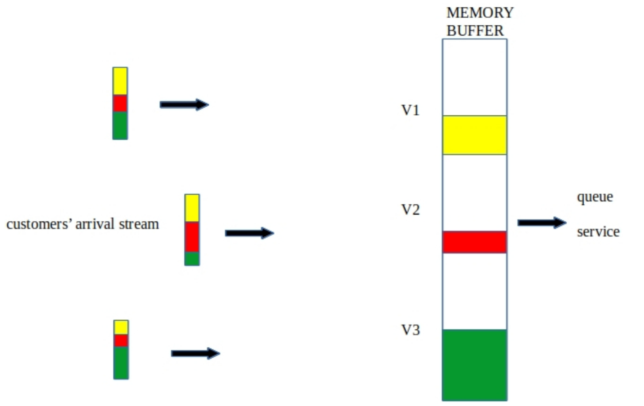

Fig. 1. Scheme of a queueing system with random volume customers and a sectorized memory buffer

realizations of such models are discussed e.g. in [21, 22]. These patents confirm that new investigations in the area of queueing systems with random volume customers are still necessary. In this case, analyses of models that take into account the above-mentioned customer volume sectorization are more interesting but more complex from the mathematical point of view as many of the calculated characteristics are multidimensional (e.g. total volume) and their obtaining needs advanced techniques from the theory of functions of many variables. Moreover, during analysis, new interesting computational problems often appear. Their solutions demand introducing extensions of the well known (in the case of one–variable function) mathematical concepts (integral transforms, convolutions, etc.). First papers facing that problem for some exemplary simple models were [23, 24] whereas in [25, 26] the general results for models of $M/G/n/0$, $M/G/1/\infty$, $M/G/1/m$-types and model with egalitarian processor sharing have been obtained.

In this paper, the exact investigation for the novel modification of a multi–server queueing model of the $M/M/n/m$–type is presented. It is assumed here both that servers are non–identical and customers are additionally characterized by some random volume vectors but service time of a customer is independent of his volume vector. Of course, the last assumption makes analysis simpler (if these random variables were dependent it would not be possible to obtain exact results, only approximate ones) but on the other hand one may show the presence of this assumption in some real–life situations: e.g. mail servers store billions of e-mails whose service time (until user deletes them) does not depend on their size (volume vector). Users delete e-mails based on other reasons (importance of an e-mail, its sender, important attachments). We have analogous situation in FTP servers that store information presented on Internet sites until administrator deletes contents that are no longer needed. The other examples of real–life systems for which we can assume the above-mentioned independence are those that serve customers characterized by approximately the same small volume vector (data packets often have similar sizes, they transport some small constant portions of information) and service time of such customers is independent of their volume vectors. Our investigations can be also applied to model the process of magazines working. In this case, service time of storing some goods in a magazine can also be treated as independent of their volumes (weights).

The multidimensional information delivered by the arriving customers is stored (during their service) in dedicated memory buffer sectors that are limited (so arriving customer may be lost if there is no free server and waiting position for him or at least one indication of his volume vector is too big to accept this customer to the system). The main aim of this work is to obtain formulae for important steady–state characteristics of the system connected with the number of customer distribution and loss probability, as well as numerical analysis of some special cases of the model.

The rest of the paper is organized as follows. Section 2 contains detailed description of the model, discusses mathematical concepts used in the process of analysis and introduces necessary notations. In Section 3 the main results are presented;

the system of equations characterizing the system behavior is obtained and its solution is proved. In this section loss probability formula is also derived. Section 4 investigates some special cases of the model showing symbolic formulae and numerical computations, describing additionally some computational problems appearing. The last Section 5 presents conclusions and final remarks.

2. THE DESCRIPTION OF THE MODEL, MATHEMATICAL BACKGROUND AND BASIC NOTATIONS

Assume that we investigate queueing model of the $M/M/n/m$–type, where n is the number of servers and m is the number of waiting positions in the queue (maximal number of customers that can wait for a service if all servers are busy). It means that only $n+m$ customers can be present in the system at the same time. Servers are non–identical, each of them may have different parameter μ_i ($i=\overline{1,n}$) of exponentially distributed service time that is independent of customer volume vector. Arriving customers form Poisson flow with parameter a and choose free servers in random order (they have no knowledge about service parameters) i.e. before the beginning of the service they may choose every of the p free servers with the same probability $1/p$ (or wait in the waiting room if all servers are busy and there is at least one free waiting position).

In addition, customers are characterized by non–negative l-dimensional random vectors $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_l)$, $l\geq 1$ that are defined by distribution function $L(\boldsymbol{x})=L(x_1,\ldots,x_l)$. System has also l-dimensional limited memory buffer $\boldsymbol{V}=(V_1,\ldots,V_l)$, $l\geq 1$ in which the proper indications of the customer volume vector are stored. By the above-mentioned assumptions, there is an important difference in mechanism of accepting new customers coming to the system compared to classical $M/M/n/m$ model. Assume that, at fixed time moment t, customer having volume vector $\boldsymbol{y}$ is arriving to the system. If at least one indication y_i, $i=\overline{1,l}$ is too big at this moment, i.e. $\sigma_i(t)+y_i>V_i$, where $\sigma_i(t)$ is total volume of i-th indications of all customers present in the system at time t, then it will not be accepted for servicing (it is lost), even when there are free servers or waiting positions. Moreover, in this case nothing changes in the system behavior. In opposite case (if the number of customers present in the system during new customer's arrival $\eta(t)$ is less than $n+m$ and every indication of his volume vector is small enough: $\sigma_i(t)+y_i\leq V_i$), it is accepted to be served and we have obvious relations: $\eta(t^+)=\eta(t)+1$ and $\sigma_i(t^+)=\sigma_i(t)+y_i$, $i=\overline{1,l}$.

The mechanism connected with the end of service is almost exactly the same like in classical $M/M/n/m$ system but takes into account that customers ending their service release memory buffer. If at time instant τ one of the servers finishes service of the customer characterized by the volume vector $\boldsymbol{y}$, then $\eta(\tau^+)=\eta(\tau)-1$ but, in addition, $\sigma_i(\tau^+)=\sigma_i(\tau)-y_i$, $i=\overline{1,l}$, where y_i is the proper indication of this customer volume vector. Investigated model will be denoted as $M/M/n/(m,\boldsymbol{V})$.

Let $A(t)$ be the set of numbers of busy servers at time t. The behavior of the above-mentioned system can be modeled by the following Markovian processes:

$$\left(A(t),\sigma_i(t),i=\overline{1,l}\right),\ \text{if}\ \eta(t)=\overline{0,n} \tag{1a}$$

and

$$\left(\eta(t),\sigma_i(t),i=\overline{1,l}\right),\ \text{if}\ \eta(t)=\overline{n+1,n+m}. \tag{1b}$$

These processes are characterized by the following functions:

$$P_0(t)=\mathbf{P}\{\eta(t)=0\}, \tag{2}$$

$$G_{\{i_1,\ldots,i_k\}}(\boldsymbol{x},t)=\mathbf{P}\{A(t)=\{i_1,\ldots,i_k\},\sigma_i(t)<x_i,i=\overline{1,l}\},\quad k=\overline{1,n}, \tag{3}$$

$$P_{\{i_1,\ldots,i_k\}}(t)=\mathbf{P}\{A(t)=\{i_1,\ldots,i_k\}\},\quad k=\overline{1,n}, \tag{4}$$

$$G_k(\boldsymbol{x},t)=\mathbf{P}\{\eta(t)=k,\sigma_i(t)<x_i,i=\overline{1,l}\},\quad k=\overline{n+1,n+m}, \tag{5}$$

$$P_k(t)=\mathbf{P}\{\eta(t)=k\},\quad k=\overline{n+1,n+m}. \tag{6}$$

In steady state ($t\to\infty$), which exists if $\rho=\dfrac{a}{\sum_{i=1}^{n}\mu_i}<\infty$ (this condition is almost always satisfied, especially in real–life systems), we can consider limits (equivalents) of the functions (2)–(6) that are independent of time variable t: p_0, $g_{\{i_1,\ldots,i_k\}}(\boldsymbol{x})$, $p_{\{i_1,\ldots,i_k\}}$, $g_k(\boldsymbol{x})$ and p_k [8].

The main purpose of investigations is to obtain formulae defining these functions in exact form and show their possible applications in loss probability computations as well as present some interesting results for special cases of analyzed model.

3. THE MAIN RESULTS

3.1. Number of customers distribution

If we analyze the behavior of our system characterized by processes (1a) and (1b), we can write down steady–state equations containing functions p_0, $g_{\{i_1,\ldots,i_k\}}(\boldsymbol{x})$, $p_{\{i_1,\ldots,i_k\}}$, $g_k(\boldsymbol{x})$ and p_k. These equations are some modifications of those obtained in the case of classical $M/M/n/m$ queueing model with non–identical servers – see e.g. [27], but take into account the above-mentioned limitation of memory buffer sectors and random choice of a free server. We will obtain equations using well–known classical method investigating state changes of the stochastic process describing of system's behavior. This method is based on the fact that in the steady state the intensity of quitting from every state equals the sum of all intensities of entering from the other states to this state [1, 8, 27]. In addition, transitions are possible only from neighboring states like in classical birth–death process.

At the beginning, let us analyze state $\mathbf{0}$ of the process. It means that there are no customers in the system. The process can quit from this state only to one of the states $\{j\}$ (then j-th server starts servicing the arriving customer) if a new customer arrives in the system and he has volume vector $\boldsymbol{x}$ less than $\boldsymbol{V}$ (all indications x_i of a volume vector $\boldsymbol{x}$ are less than the proper memory buffer indications V_i), whereas process can enter the investigated state also only from states $\{j\}$ which means that there was one customer served by j-th server and his service was finished. So we obtain the following relation:

$$ap_0 L(\boldsymbol{V}) = \sum_{j=1}^{n} \mu_j p_{\{j\}}. \tag{7}$$

Assume now that the process is in state $\{\boldsymbol{j}\}$ (it means that we have only one customer served by j-th server and, of course, his volume vector is less than $\boldsymbol{V}$). It can quit from this state in two situations. First of them is connected with arrival of a new customer whose volume vector $\boldsymbol{x}$ is small enough to be accepted on service (the sum of the volume vector of served customer and volume vector of the arriving one is less than $\boldsymbol{V}$). Then it enters the state $\{i, j\}$. The second situation describes service termination and then the process enters to state 0. Moreover, analyzed process enters the state $\{j\}$ also in two situations: first means the arrival of a new customer in an empty system (his volume vector must be less than $\boldsymbol{V}$, the j-th server is chosen with probability $1/n$), and second situation is connected with service finishing of the customer served by i-th server (then we enter this state quitting from $\{i, j\}$). The analysis leads to the following equation:

$$a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{j\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \mu_j p_{\{j\}} = \frac{ap_0}{n} L(\boldsymbol{V}) + \sum_{i=1, i\neq j}^{n} \mu_i p_{\{i,j\}}, \; j = \overline{1,n}. \tag{8}$$

Investigate now the presence of the process in state $\{\boldsymbol{i_1, \ldots, i_k}\}$ $(2 \leq k < n)$ (we have k customers in the system, all of them are served by servers $i_1, \ldots, i_k$, respectively). Our process quits from this state in two situations. First of them means again the arrival of a new customer whose volume vector size lets him be accepted on service. Then process enters to state $\{i_1, \ldots, i_k, j\}$. The second situation describes the end of a service on one of the busy servers $\{i_1, \ldots, i_k\}$, then process enters the state $\{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k\}$, whereas process enters the state $\{i_1, \ldots, i_k\}$ also only in two situations: first is connected with the new customer's arrival (the process quits from the state $\{i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k\}$, arriving customer volume vector size must be small enough and the j-th server is chosen with probability $\dfrac{1}{n-k+1}$), the second means service termination on j-th server (then we enter this state quitting from $\{i_1 \ldots, i_k, j\}$). Here is the next obtained relation:

$$\begin{aligned} a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{i_1,\ldots,i_k\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{k} \mu_{i_j}\right) p_{\{i_1,\ldots,i_k\}} \\ = \frac{a}{n-k+1} \sum_{j=1}^{k} \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\ + \sum_{j\notin\{i_1,\ldots,i_k\}} \mu_j p_{\{i_1,\ldots,i_k,j\}}, \\ k = \overline{2,n-1}, \; \{i_1,\ldots,i_k\} \in C_k^n, \end{aligned} \tag{9}$$

where C_k^n denotes the set of all k-element combinations of n-element set.

Consider now state $\{\boldsymbol{1, \ldots, n}\}$ (we have n customers in the system, all servers are busy). Our process can quit from this state to states $\{1, \ldots, j-1, j+1, \ldots, n\}$ or $n+1$. First situation means the end of a service on j-th server and second one – arrival of a new customer having the proper size of his volume vector (of course, the arriving customer waits in a waiting room as all servers are busy). Process can enter this state only from states $\{1, \ldots, j-1, j+1, n\}$ or $n+1$. First situation is connected with an arrival of a new customer (having small enough size of his volume vector) that goes on service to j-th server and second one is connected with the end of the service on one of the n busy servers (then waiting customer leaves waiting room and starts his service). The obtained equation is very similar to the previous one:

$$\begin{aligned} a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{1,\ldots,n\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n} \mu_j\right) p_{\{1,\ldots,n\}} \\ = a \sum_{j=1}^{n} \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{1,\ldots,j-1,j+1,\ldots,n\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\ + \left(\sum_{j=1}^{n} \mu_j\right) p_{n+1}. \end{aligned} \tag{10}$$

If we analyze transitions from and to $\boldsymbol{k}$ state $(k = \overline{n+1, n+m-1})$ then we easily notice that we have analogous situation. The process enters the state k when there were $k+1$ customers in the system and one of these customers ended his service on one of the n busy servers or there were $k-1$ customers in the system and one new customer having the proper size of his volume vector arrived in the system. Transition from this state is possible only to the same states $k-1$ and $k+1$ which means that we consider service termination or arriving of a new customer that can be accepted to the system as having proper size of his volume vector. This analysis leads to the following equation:

$$\begin{aligned} a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_k(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n} \mu_j\right) p_k \\ = a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n} \mu_j\right) p_{k+1}, \\ k = \overline{n+1, n+m-1}. \end{aligned} \tag{11}$$

In the end, we consider the last possible state $\boldsymbol{n+m}$. Entering this state is possible only from state $n+m-1$ because in our queueing system the maximal number of present customers equals $n+m$. This transition is connected again with the new customer's arrival, whereas the only transition from state $n+m$ also leads to the same state $n+m-1$ and means service termination. So we have the final relation:

$$a \int_{\boldsymbol{0}}^{\boldsymbol{V}} g_{n+m-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) = \left(\sum_{j=1}^{n} \mu_j\right) p_{n+m}. \tag{12}$$

It can be easily proved that system of equations (7)–(12) has the following solution:

$$g_{\{i_1,\dots,i_k\}}(\boldsymbol{x}) = \frac{a^k(n-k)!p_0L_k(\boldsymbol{x})}{n!\prod_{j=1}^{k}\mu_{i_j}}, \quad k=\overline{1,n}, \tag{13}$$

$$g_k(\boldsymbol{x}) = \frac{a^n p_0 L_k(\boldsymbol{x})}{n!\prod_{j=1}^{n}\mu_j}\rho^{k-n}, \quad k=\overline{n+1,n+m}, \tag{14}$$

$$p_{\{i_1,\dots,i_k\}} = \frac{a^k(n-k)!p_0L_k(\boldsymbol{V})}{n!\prod_{j=1}^{k}\mu_{i_j}}, \quad k=\overline{1,n}, \tag{15}$$

$$p_k = \frac{a^n p_0 L_k(\boldsymbol{V})}{n!\prod_{j=1}^{n}\mu_j}\rho^{k-n}, \quad k=\overline{n+1,n+m}, \tag{16}$$

where $L_k(\boldsymbol{x})$ is the k-th order Stieltjes convolution of functions $L(\boldsymbol{x})$ defined recurrently as follows:

$$L_0(\boldsymbol{x}) \equiv 1, \qquad L_k(\boldsymbol{x}) = \int\limits_{\boldsymbol{0}}^{\boldsymbol{x}} L_{k-1}(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}L(\boldsymbol{u}), \quad k \geq 1.$$

The value of p_0 can be determined from the normalization condition:

$$p_0 + \sum_{k=1}^{n}\sum_{\{i_1,\dots,i_k\}\in C_k^n} p_{\{i_1,\dots,i_k\}} + \sum_{k=n+1}^{n+m} p_k = 1. \tag{17}$$

Now, using direct substitution of formulae (13)–(16) into equations (7)–(12), the proof of the correctness of obtained results will be presented (note that in the steady state there exists only one solution of previously written system of equations).

Proof.

- Ad (7):

$$\sum_{j=1}^{n}\mu_j p_{\{j\}} = \sum_{j=1}^{n}\mu_j\frac{a(n-1)!p_0L(\boldsymbol{V})}{n!\mu_j} = \frac{an(n-1)!p_0L(\boldsymbol{V})}{n!} = ap_0L(\boldsymbol{V}).$$

- Ad (8):

$$\begin{aligned}
\frac{ap_0}{n}L(\boldsymbol{V}) &+ \sum_{i=1,i\neq j}^{n}\mu_i p_{\{i,j\}} = \mu_j\frac{a(n-1)!p_0L(\boldsymbol{V})}{n!\mu_j} \\
&+ \sum_{i=1,i\neq j}^{n}\mu_i\frac{a^2(n-2)!p_0L_2(\boldsymbol{V})}{n!\mu_i\mu_j} \\
&= \mu_j p_{\{j\}} + (n-1)\frac{a^2p_0L_2(\boldsymbol{V})}{n(n-1)\mu_j} \\
&= \mu_j p_{\{j\}} + a\cdot\frac{ap_0L(\boldsymbol{V})}{n\mu_j} \\
&= \mu_j p_{\{j\}} + a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}}\frac{ap_0}{n\mu_j}L(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&= \mu_j p_{\{j\}} + a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_{\{j\}}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}).
\end{aligned}$$

- Ad (11):

$$\begin{aligned}
&a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_k(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n}\mu_j\right)p_k \\
&= a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}}\frac{a^n p_0 L_k(\boldsymbol{V}-\boldsymbol{x})}{n!\prod\limits_{j=1}^{n}\mu_j}\rho^{k-n}\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n}\mu_j\right)\frac{a^n p_0 L_k(\boldsymbol{V})}{n!\prod\limits_{j=1}^{n}\mu_j}\rho^{k-n} \\
&= \frac{a^{n+1}p_0\rho^{k-n}}{n!\prod\limits_{j=1}^{n}\mu_j}\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} L_k(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n}\mu_j\right)\frac{a^n p_0\rho^{k-n}}{n!\prod\limits_{j=1}^{n}\mu_j}\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} L_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&= \frac{a^{n+1}p_0\rho^{k-n}}{n!\prod\limits_{j=1}^{n}\mu_j}L_{k+1}(\boldsymbol{V}) + \rho\left(\sum_{j=1}^{n}\mu_j\right)\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}}\frac{a^n p_0\rho^{k-n-1}}{n!\prod\limits_{j=1}^{n}\mu_j}L_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&= a\left(\sum_{j=1}^{n}\mu_j\right)^{-1}\cdot\frac{a^n p_0\rho^{k-n}L_{k+1}(\boldsymbol{V})}{n!\prod\limits_{j=1}^{n}\mu_j}\cdot\sum_{j=1}^{n}\mu_j + a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&= \frac{a^n p_0\rho^{k-n+1}L_{k+1}(\boldsymbol{V})}{n!\prod\limits_{j=1}^{n}\mu_j}\cdot\sum_{j=1}^{n}\mu_j + a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&= a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_{k-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) + \left(\sum_{j=1}^{n}\mu_j\right)p_{k+1}.
\end{aligned}$$

- Ad (12):

$$\begin{aligned}
&a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} g_{n+m-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x}) \\
&\qquad= a\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}}\frac{a^n p_0 L_{n+m-1}(\boldsymbol{V}-\boldsymbol{x})}{n!\prod_{j=1}^{n}\mu_j}\rho^{m-1} \\
&\qquad= \frac{a^{m+1}\rho^{m-1}p_0}{n!\prod\limits_{j=1}^{n}\mu_j}\int\limits_{\boldsymbol{0}}^{\boldsymbol{V}} L_{n+m-1}(\boldsymbol{V}-\boldsymbol{x})\,\mathrm{d}L(\boldsymbol{x})
\end{aligned}$$

cont: Ad (12)

$$= \frac{a^{n+1}\rho^{m-1}p_0}{n!\prod_{j=1}^{n}\mu_j}L_{n+m}(V)$$

$$= \frac{a}{\sum_{j=1}^{n}\mu_j}\cdot\frac{a^n\rho^{m-1}p_0L_{n+m}(V)}{n!\prod_{j=1}^{n}\mu_j}\cdot\sum_{j=1}^{n}\mu_j$$

$$= \frac{a^n\rho^m p_0L_{n+m}(V)}{n!\prod_{j=1}^{n}\mu_j}\left(\sum_{j=1}^{n}\mu_j\right) = \left(\sum_{j=1}^{n}\mu_j\right)p_{n+m}.$$

- Ad (9):

$$\frac{a}{n-k+1}\sum_{j=1}^{k}\int_{\mathbf{0}}^{V} g_{\{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k\}}(V-x)\,\mathrm{d}L(x)$$

$$+\sum_{j\notin\{i_1,\ldots,i_k\}}\mu_j p_{\{i_1,\ldots,i_k,j\}}$$

$$= \frac{a}{n-k+1}\sum_{j=1}^{k}\int_{\mathbf{0}}^{V}\frac{a^{k-1}(n-k+1)!p_0L_{k-1}(V-x)\,\mathrm{d}L(x)}{n!\prod_{l=1,l\neq j}^{k}\mu_{i_l}}$$

$$+\sum_{j\notin\{i_1,\ldots,i_k\}}\frac{a^{k+1}(n-k-1)!p_0L_{k+1}(V)}{n!\prod_{j=1}^{k}\mu_{i_j}}$$

$$= \sum_{j=1}^{k}\int_{\mathbf{0}}^{V}\frac{a^k(n-k)!p_0L_{k-1}(V-x)\,\mathrm{d}L(x)}{n!\prod_{l=1,l\neq j}^{k}\mu_{i_l}}$$

$$+\frac{(n-k)a^{k+1}(n-k-1)!p_0L_{k+1}(V)}{n!\prod_{j=1}^{k}\mu_{i_j}}$$

$$= \sum_{j=1}^{k}\frac{a^k(n-k)!p_0L_k(V)}{n!\prod_{l=1,l\neq j}^{k}\mu_{i_l}}+\frac{a^{k+1}(n-k)!p_0L_{k+1}(V)}{n!\prod_{j=1}^{k}\mu_{i_j}}$$

$$= \sum_{j=1}^{k}\frac{a^k(n-k)!p_0L_k(V)}{n!\prod_{l=1,l\neq j}^{k}\mu_{i_l}}$$

$$+a\int_{\mathbf{0}}^{V}\frac{a^k(n-k)!p_0L_k(V-x)\,\mathrm{d}L(x)}{n!\prod_{j=1}^{k}\mu_{i_j}}$$

$$= \left(\sum_{j=1}^{k}\mu_{i_j}\right)\frac{a^k(n-k)!p_0L_k(V)}{n!\prod_{j=1}^{k}\mu_{i_j}}$$

$$+a\int_{\mathbf{0}}^{V} g_{\{i_1,\ldots,i_k\}}(V-x)\,\mathrm{d}L(x)$$

$$= \left(\sum_{j=1}^{k}\mu_{i_j}\right)p_{\{i_1,\ldots,i_k\}}+a\int_{\mathbf{0}}^{V} g_{\{i_1,\ldots,i_k\}}(V-x)\,\mathrm{d}L(x).$$

- Ad (10):

$$a\sum_{j=1}^{n}\int_{\mathbf{0}}^{V} g_{\{1,\ldots,j-1,j+1,\ldots,n\}}(V-x)\,\mathrm{d}L(x)+\left(\sum_{j=1}^{n}\mu_j\right)p_{n+1}$$

$$= a\sum_{j=1}^{n}\int_{\mathbf{0}}^{V}\frac{a^{n-1}p_0L_{n-1}(V-x)\,\mathrm{d}L(x)}{n!\prod_{i=1,i\neq j}^{n}\mu_i}$$

$$+\left(\sum_{j=1}^{n}\mu_j\right)\cdot\frac{a^np_0L_{n+1}(V)\rho}{n!\prod_{j=1}^{n}\mu_j}$$

$$= a\left(\sum_{j=1}^{n}\mu_j\right)\int_{\mathbf{0}}^{V}\frac{a^{n-1}p_0L_{n-1}(V-x)\,\mathrm{d}L(x)}{n!\prod_{j=1}^{n}\mu_j}$$

$$+\frac{a^{n+1}p_0L_{n+1}(V)}{n!\prod_{j=1}^{n}\mu_j}$$

$$= \left(\sum_{j=1}^{n}\mu_j\right)\frac{a^np_0L_n(V)}{n!\prod_{j=1}^{n}\mu_j}+\frac{a^{n+1}p_0L_{n+1}(V)}{n!\prod_{j=1}^{n}\mu_j}$$

$$= \left(\sum_{j=1}^{n}\mu_j\right)p_{\{1,\ldots,n\}}+a\int_{\mathbf{0}}^{V}\frac{a^np_0L_n(V-x)\,\mathrm{d}L(x)}{n!\prod_{j=1}^{n}\mu_j}$$

$$= \left(\sum_{j=1}^{n}\mu_j\right)p_{\{1,\ldots,n\}}+a\int_{\mathbf{0}}^{V} g_{\{1,\ldots,n\}}(V-x)\,\mathrm{d}L(x). \qquad \square$$

Remark 1. Obtained results let calculate mean value of the steady–state number of customers present in the system. This characteristic can be computed using the following formula:

$$\mathbf{E}\eta = \sum_{k=1}^{n}\sum_{\{i_1,\ldots,i_k\}\in C_k^n} kp_{\{i_1,\ldots,i_k\}}+\sum_{k=n+1}^{n+m}kp_k,$$

where $p_{\{i_1,\ldots,i_k\}}$ and p_k are calculated with the use of formulae (15)–(16). For example, for the system $M/M/3/(m,V)$ we obtain the following relation:

$$\mathbf{E}\eta = p_{\{1\}}+p_{\{2\}}+p_{\{3\}}+2(p_{\{1,2\}}+p_{\{1,3\}}+p_{\{2,3\}})$$

$$+\ 3p_{\{1,2,3\}}+\sum_{k=4}^{3+m}kp_k.$$

Remark 2. Formulae (15)–(16) allow to obtain also characteristics of the utilization of j-th server (calculating the value of steady–state probability q_j that j-th server is busy):

$$q_j = \sum_{k=0}^{n}\sum_{\substack{\{i_1,\ldots,i_k\}\in C_k^n,\\ j\in\{i_1,\ldots,i_k\}}} p_{\{i_1,\ldots,i_k\}}+\sum_{k=n+1}^{n+m}p_k.$$

For example, for the system $M/M/3/(1,V)$ we obtain the following results:

$$q_1 = p_{\{1\}}+p_{\{1,2\}}+p_{\{1,3\}}+p_{\{1,2,3\}}+p_4,$$

$$q_2 = p_{\{2\}}+p_{\{1,2\}}+p_{\{2,3\}}+p_{\{1,2,3\}}+p_4,$$

$$q_3 = p_{\{3\}}+p_{\{1,3\}}+p_{\{2,3\}}+p_{\{1,2,3\}}+p_4.$$

3.2. Loss probability formula

Now we will obtain general formula defining steady-state customer loss probability denoted as P_{LOSS}. It is obvious that loss probability is not equal to p_{n+m} as it was in classical models of the $M/M/n/m$–type because an arriving customer can be additionally lost (even if there are free servers or free waiting positions) if at least one of his volume vector indication is too big to be stored in a limited sector of memory buffer. In analyzed case (when service time of a customer and his volume vector are independent) we can use well-known equilibrium equation (see e.g. [8]) assuming that mean number of customers accepted for service (i.e. arriving and not lost) during some time interval is equal to mean number of customers ending their service within this interval. In our case equilibrium equation has the form:

$$a(1-P_{\text{LOSS}}) = \sum_{k=1}^{n} \sum_{\{i_1,\dots,i_k\}\in C_k^n} \sum_{j=1}^{k} \mu_{i_j} p_{\{i_1,\dots,i_k\}} + \left(\sum_{j=1}^{n}\mu_j\right) \sum_{k=n+1}^{n+m} p_k . \tag{18}$$

From (18) we easily obtain:

$$P_{\text{LOSS}} = 1 - \frac{1}{a}\left[\sum_{k=1}^{n} \sum_{\{i_i,\dots,i_k\}\in C_k^n} \sum_{j=1}^{k} \mu_{i_j} p_{\{i_1,\dots,i_k\}} + \left(\sum_{j=1}^{n}\mu_j\right) \sum_{k=n+1}^{n+m} p_k\right]. \tag{19}$$

4. SPECIAL CASES OF THE ANALYZED MODEL, SOME NUMERICAL EXAMPLES AND COMPUTATIONAL PROBLEMS

4.1. Computational problems

Although formulae (13)–(16) and (19) seem to be relatively easy, they contain computationally inconvenient multidimensional Stieltjes convolutions $L_k(\boldsymbol{x})$. The process of calculating such a convolution based on its recurrent definition is often very complicated and obtaining exact form of convolutions is possible only for some special classes of functions (e.g. exponential distribution or gamma distribution). For the other functions, it is possible only for small values of k. But, with the help of computer algebra systems (e.g. *Mathematica* environment), for fixed k, we can obtain exact formulae of convolutions in many situations. To do this, we usually use the following computational algorithm:

1. We calculate multidimensional Laplace–Stieltjes transform (LST) of a single function $L(\boldsymbol{x}) = L(x_1,\dots,x_l)$:

$$\alpha(s_1,\dots,s_l) = \int_0^\infty \dots \int_0^\infty e^{-s_1x_1-\dots-s_lx_l}\,\mathrm{d}L(x_1,\dots,x_l).$$

2. We calculate LST of a convolution $L_k(x_1,\dots,x_l)$ using the fact that this convolution is a distribution function of the sum of k independent random vectors having distribution functions $L(x_1,\dots,x_l)$ and its LST $\alpha_k(s_1,\dots,s_l)$ has the form of product of $\alpha(s_1,\dots,s_l)$ transforms. So we obtain the following relation:

$$\alpha_k(s_1,\dots,s_l) = [\alpha(s_1,\dots,s_l)]^k .$$

3. We obtain formula for multidimensional Laplace transform

$$\mathscr{L}(s_1,\dots,s_l) = \int_0^\infty \dots \int_0^\infty e^{-s_1x_1-\dots-s_lx_l} L_k(x_1,\dots,x_l)\,\mathrm{d}x_1\dots\mathrm{d}x_l$$

of convolution $L_k(x_1,\dots,x_l)$ using connection between LST and Laplace transform:

$$\mathscr{L}(s_1,\dots,s_l) = \frac{[\alpha(s_1,\dots,s_l)]^k}{s_1\cdot\ldots\cdot s_l}.$$

4. Finally, we use **InverseLaplaceTransform** command from *Mathematica* environment to obtain exact formula of $L_k(x_1,\dots,x_l)$ convolution [28]:

$$L_k(x_1,\dots,x_l) = \mathscr{L}^{-1}(s_1,\dots,s_l).$$

Later in this section, we want to investigate some practical special cases of analyzed model and show numerical computations illustrating obtaining results. Let us notice that in real systems we usually have one of two situations:

- all indications $\zeta_1,\dots,\zeta_l$ of a volume vector are independent or
- first $l-1$ indications are independent and the last one is proportional to their sum: $\zeta_l = c(\zeta_1+\dots+\zeta_{l-1})$, $c>0$.

In first situation LST of a single function $L(x_1,\dots,x_l)$ has the form of product:

$$\alpha(s_1,\dots,s_l) = \varphi_1(s_1)\cdot\ldots\cdot\varphi_l(s_l),$$

where $\varphi_1(s_1),\dots,\varphi_l(s_l)$ are one–dimensional LSTs of single $\zeta_1,\dots,\zeta_l$ indications. So in this case we obtain:

$$\mathscr{L}(s_1,\dots,s_l) = \frac{[\varphi_1(s_1)]^k\cdot\ldots\cdot[\varphi_l(s_l)]^k}{s_1\cdot\ldots\cdot s_l}. \tag{20}$$

It can be easily shown that in second situation we obtain the following relation:

$$\mathscr{L}(s_1,\dots,s_l) = \frac{[\varphi_1(s_1+cs_l)]^k\cdot\ldots\cdot[\varphi_{l-1}(s_{l-1}+cs_l)]^k}{s_1\cdot\ldots\cdot s_l}. \tag{21}$$

4.2. Special cases analysis

In this section, we will present some numerical calculations connected with analysis of some special cases of the model. In whole section, we will consider exemplary model of queueing system $M/M/2/(1,\boldsymbol{V})$ and chosen multidimensional distribution functions of customer volume vector. For such defined queueing system, using (15)–(17), we obtain the following formulae:

$$p_{\{1\}} = \frac{ap_0L(\boldsymbol{V})}{2\mu_1}, \qquad p_{\{2\}} = \frac{ap_0L(\boldsymbol{V})}{2\mu_2};$$

$$p_{\{1,2\}} = \frac{a^2p_0L_2(\boldsymbol{V})}{2\mu_1\mu_2}, \qquad p_3 = \frac{a^3p_0L_3(\boldsymbol{V})}{2\mu_1\mu_2(\mu_1+\mu_2)};$$

$$p_0 = \left[1 + \frac{aL(\boldsymbol{V})}{2\mu_1} + \frac{aL(\boldsymbol{V})}{2\mu_2} + \frac{a^2 L_2(\boldsymbol{V})}{2\mu_1\mu_2} + \frac{a^3 L_3(\boldsymbol{V})}{2\mu_1\mu_2(\mu_1+\mu_2)}\right]^{-1}. \tag{22}$$

Later on, we will analyze two versions of this model.

4.2.1. Memory buffer contains two sectors

Assume that customer volume vector contains only two indications. We will show some numerical results in two situations:

- indications of customer volume vector are independent and exponentially distributed with parameters f_1, f_2, respectively,
- first indication of a customer volume vector is exponentially distributed with parameter f and second one is proportional to the first with coefficient c ($c > 0$) which means that $\zeta_2 = c\zeta_1$.

Formulae defining Stieltjes convolutions **in first situation** are calculated as follows (see formula (20)):

$$\begin{aligned} L(V_1,V_2) &= (1-e^{-f_1V_1})(1-e^{-f_2V_2}),\\ L_2(V_1,V_2) &= \left[1-e^{-f_1V_1}(1+f_1V_1)\right]\left[1-e^{-f_2V_2}(1+f_2V_2)\right],\\ L_3(V_1,V_2) &= \left[1-e^{-f_1V_1}\left(1+f_1V_1+\frac{(f_1V_1)^2}{2}\right)\right]\\ &\quad\times\left[1-e^{-f_2V_2}\left(1+f_2V_2+\frac{(f_2V_2)^2}{2}\right)\right]. \end{aligned} \tag{23}$$

In Table 1 calculations of steady–state number of customers distribution, and in Table 2 calculations of loss probability (on the base of (19) formula) are presented. In all calculations we assume that $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f_1 = 2$, $f_2 = 4$ and the values of V_1 and V_2 are changing from 1 to 4 (in the case of number of customers distribution), and from 0.1 to 5.1 (in the case of loss probability).

Table 1

Number of customers distribution for $M/M/2/(1,V_1,V_2)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f_1 = 2$, $f_2 = 4$ (indications independent)

V_1	V_2	p_0	$p_{\{1\}}$	$p_{\{2\}}$	$p_{\{1,2\}}$	p_3
1	1	0.0637	0.1082	0.1353	0.3439	0.3489
1	2	0.0559	0.0966	0.1207	0.3309	0.3959
1	3	0.0555	0.0960	0.1200	0.3298	0.3987
1	4	0.0555	0.0960	0.1200	0.3297	0.3988
2	1	0.0378	0.0728	0.0910	0.3115	0.4870
2	2	0.0321	0.0630	0.0787	0.2906	0.5357
2	3	0.0318	0.0625	0.0781	0.2891	0.5385
2	4	0.0318	0.0625	0.0781	0.2890	0.5386
3	1	0.0331	0.0648	0.0810	0.2954	0.5256
3	2	0.0279	0.0556	0.0696	0.2733	0.5736
3	3	0.0277	0.0552	0.0690	0.2718	0.5763
3	4	0.0277	0.0552	0.0690	0.2717	0.5765
4	1	0.0321	0.0630	0.0787	0.2906	0.5357
4	2	0.0270	0.0539	0.0674	0.2683	0.5834
4	3	0.0268	0.0535	0.0669	0.2667	0.5861
4	4	0.0267	0.0535	0.0669	0.2667	0.5862

Table 2

Loss probability values for $M/M/2/(1,V_1,V_2)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f_1 = 2$, $f_2 = 4$ (indications independent)

P_{LOSS}	$V_2 = 0.1$	$V_2 = 1.1$	$V_2 = 2.1$	$V_2 = 3.1$	$V_2 = 4.1$	$V_2 = 5.1$
$V_1 = 0.1$	0.9494	0.8683	0.8654	0.8653	0.8653	0.8653
$V_1 = 1.1$	0.8197	0.6258	0.6190	0.6186	0.6186	0.6186
$V_1 = 2.1$	0.7971	0.6012	0.5956	0.5953	0.5953	0.5953
$V_1 = 3.1$	0.7919	0.5960	0.5907	0.5905	0.5905	0.5905
$V_1 = 4.1$	0.7908	0.5948	0.5896	0.5894	0.5894	0.5894
$V_1 = 5.1$	0.7906	0.5945	0.5894	0.5892	0.5892	0.5892

The graph presenting values of loss probability is shown in Fig. 2. It is clear that P_{LOSS} does not converge to 0 (when $V_1, V_2 \to \infty$). Of course, if we take into consideration that we have here limited number of servers and waiting positions, we have obvious relation $P_{\mathrm{LOSS}} \to r_3$, where r_3 is the probability that in classical version of this system (without assumption that customers have random volumes) all servers and waiting positions are busy. In analyzed model we have:

$$r_3 = \frac{a^3}{a^3 + a^2(\mu_1+\mu_2) + a(\mu_1+\mu_2)^2 + 2\mu_1\mu_2(\mu_1+\mu_2)}.$$

Substituting fixed values of $a = 1$, $\mu_1 = 0.25$ and $\mu_2 = 0.2$, we obtain $r_3 \approx 0.589102$ and it is the limitation of loss probability in this case (see again Table 2 and Fig. 2).

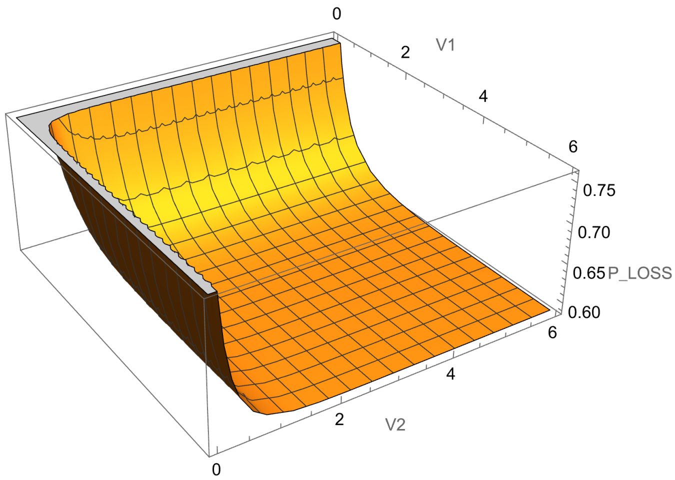

Fig. 2. Graph presenting the values of loss probability in the $M/M/2/(1,V_1,V_2)$ queueing system (indications independent)

If we investigate **second situation**, we have the following relations (using (21)):

$$L(V_1,V_2) = 1 + e^{-\frac{fV_2}{c}}\left[H(V_2 - cV_1) - 1\right] - e^{-fV_1}H(V_2 - cV_1),$$

$$L_2(V_1,V_2) = \frac{e^{-f\left(\frac{V_2}{c}+V_1\right)}}{c}\Big\{e^{fV_1}(c + fV_2)\left[H(V_2 - cV_1) - 1\right] + ce^{\frac{fV_2}{c}}\left(e^{fV_1} - (fV_1+1)H(V_2 - cV_1)\right)\Big\},$$

$$L_3(V_1,V_2) = \frac{e^{-f\left(\frac{V_2}{c}+V_1\right)}}{2c^2} \times \Big\{e^{fV_1}\left(2c^2 + 2cfV_2 + f^2V_2^2\right)\left[H(V_2 - cV_1) - 1\right] + c^2e^{\frac{fV_2}{c}}\left(2e^{fV_1} - \left[fV_1(fV_1+2)+2\right]H(V_2 - cV_1)\right)\Big\}, \quad (24)$$

where $H(x)$ is left–continuous Heaviside's unitstep function ($H(x) = 0$ if $x \le 0$ and $H(x) = 1$, otherwise).

In Table 3 we analogously present calculations of steady–state number of customers distribution, and in Table 4 calculations of loss probability. This time we make the following substitutions: $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f = 4$, $c = 2$ and the values of V_1 and V_2 are changing in the same way as in previously analyzed situation, whereas Fig. 3 presents graph of loss probability values.

Table 3

Number of customers distribution for $M/M/2/(1,V_1,V_2)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f = 4$, $c = 2$ (second indication proportional to the first)

V_1	V_2	p_0	$p_{\{1\}}$	$p_{\{2\}}$	$p_{\{1,2\}}$	p_3
1	1	0.0555	0.0960	0.1200	0.3297	0.3988
1	2	0.0318	0.0625	0.0781	0.2890	0.5386
1	3	0.0318	0.0625	0.0781	0.2890	0.5386
1	4	0.0318	0.0625	0.0781	0.2890	0.5386
2	1	0.0555	0.0960	0.1200	0.3297	0.3988
2	2	0.0318	0.0625	0.0781	0.2890	0.5386
2	3	0.0277	0.0552	0.0690	0.2717	0.5765
2	4	0.0267	0.0535	0.0668	0.2667	0.5862
3	1	0.0555	0.0960	0.1200	0.3297	0.3988
3	2	0.0318	0.0625	0.0781	0.2890	0.5386
3	3	0.0277	0.0552	0.0690	0.2717	0.5765
3	4	0.0267	0.0535	0.0668	0.2667	0.5862
4	1	0.0555	0.0960	0.1200	0.3297	0.3988
4	2	0.0318	0.0625	0.0781	0.2890	0.5386
4	3	0.0277	0.0552	0.0690	0.2717	0.5765
4	4	0.0267	0.0535	0.0668	0.2667	0.5862

Table 4

Loss probability values for $M/M/2/(1,V_1,V_2)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $f = 4$, $c = 2$ (second indication proportional to the first)

P_{LOSS}	$V_2 = 0.1$	$V_2 = 1.1$	$V_2 = 2.1$	$V_2 = 3.1$	$V_2 = 4.1$	$V_2 = 5.1$
$V_1 = 0.1$	0.8653	0.7906	0.7906	0.7906	0.7906	0.7906
$V_1 = 1.1$	0.8653	0.6186	0.5953	0.5945	0.5945	0.5945
$V_1 = 2.1$	0.8653	0.6186	0.5953	0.5905	0.5894	0.5894
$V_1 = 3.1$	0.8653	0.6186	0.5953	0.5905	0.5894	0.5892
$V_1 = 4.1$	0.8653	0.6186	0.5953	0.5905	0.5894	0.5892
$V_1 = 5.1$	0.8653	0.6186	0.5953	0.5905	0.5894	0.5892

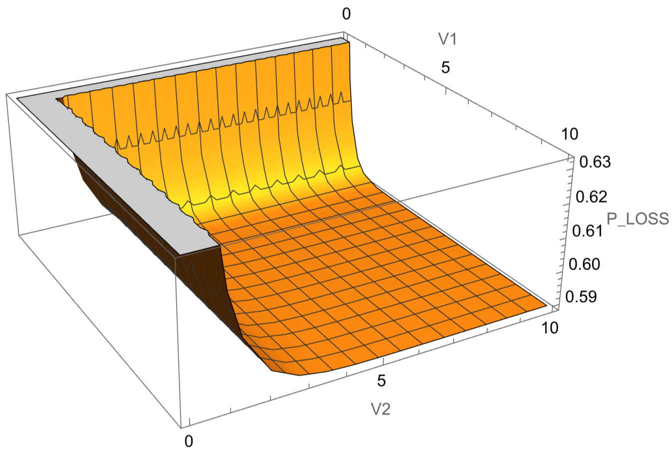

Fig. 3. Graph presenting the values of loss probability in the $M/M/2/(1,V_1,V_2)$ queueing system (second indication proportional to the first)

4.2.2. Memory buffer contains three sectors

Assume now that customer volume vector contains three indications. We will show numerical results in two practical situations:

- indications of customer volume vector are independent and have uniform distribution on the intervals $[0,b_1]$, $[0,b_2]$ and $[0,b_3]$, respectively,
- first two indications of a customer volume vector are independent, have uniform distribution on the intervals $[0,b_1]$ and $[0,b_2]$, respectively, and the third indication is proportional to their sum: $\zeta_3 = c(\zeta_1 + \zeta_2)$, $c > 0$.

In **first situation** we obtain the following formulae:

$$L(V_1,V_2,V_3) = \frac{1}{b_1b_2b_3}\prod_{i=1}^{3}\left[V_i + (b_i - V_i)H(V_i - b_i)\right],$$

$$L_2(V_1,V_2,V_3) = \frac{1}{8b_1^2b_2^2b_3^2}\prod_{i=1}^{3}\left[V_i^2 + (V_i - 2b_i)^2H(V_i - 2b_i) - 2(V_i - b_i)^2H(V_i - b_i)\right],$$

$$L_3(V_1,V_2,V_3) = \frac{1}{216b_1^3b_2^3b_3^3}\prod_{i=1}^{3}\Big[V_i^3 + (3b_i - V_i)^3H(V_i - 3b_i) - 3(2b_i - V_i)^3H(V_i - 2b_i) + 3(b_i - V_i)^3H(V_i - b_i)\Big]. \tag{25}$$

Numerical calculations of number of customers distribution and loss probability are presented in Tables 5–6. We make here the following substitutions: $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$ and $b_3 = 5$, whereas values of V_1, V_2 and V_3 equal 1 or 3 (in the case of customers distribution) and 0.1 or 10.1 or 20.1 (in case of loss probability). Notice that the value of loss probability is also limited by the same value r_3 discussed in the previous subsection.

Table 5

Number of customers distribution for $M/M/2/(1,V_1,V_2,V_3)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$ (indications independent)

V_1	V_2	V_3	p_0	$p_{\{1\}}$	$p_{\{2\}}$	$p_{\{1,2\}}$	p_3
1	1	1	0.6837	0.1367	0.1709	0.0085	0.0001
1	1	3	0.4056	0.2434	0.3042	0.0456	0.0011
1	3	1	0.5025	0.2010	0.2513	0.0440	0.0012
1	3	3	0.2196	0.2635	0.3294	0.1729	0.0146
3	1	1	0.6777	0.1355	0.1694	0.0169	0.0004
3	1	3	0.3859	0.2315	0.2894	0.0868	0.0064
3	3	1	0.4785	0.1914	0.2393	0.0837	0.0071
3	3	3	0.1762	0.2115	0.2643	0.2775	0.0705

Table 6

Loss probability values for $M/M/2/(1,V_1,V_2,V_3)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$ (indications independent)

P_{LOSS}	$V_3 = 0.1$	$V_3 = 10.1$	$V_3 = 20.1$
$V_1 = 0.1, V_2 = 0.1$	0.9999	0.9951	0.9951
$V_1 = 0.1, V_2 = 10.1$	0.9980	0.9176	0.9174
$V_1 = 0.1, V_2 = 20.1$	0.9980	0.9176	0.9174
$V_1 = 10.1, V_2 = 0.1$	0.9990	0.9549	0.9549
$V_1 = 10.1, V_2 = 10.1$	0.9808	0.5931	0.5891
$V_1 = 10.1, V_2 = 20.1$	0.9808	0.5931	0.5891
$V_1 = 20.1, V_2 = 0.1$	0.9990	0.9549	0.9549
$V_1 = 20.1, V_2 = 10.1$	0.9808	0.5931	0.5891
$V_1 = 20.1, V_2 = 20.1$	0.9808	0.5931	0.5891

In **second situation**, exact formulae for $L_k(V_1,V_2,V_3)$ are possible to obtain using *Mathematica* environment but they are very long and complex, we inverse here the following, complicated function:

$$\mathscr{L}(s_1,s_2,s_3) = \frac{\left(1 - e^{-b_1(s_1+cs_3)}\right)^k\left(1 - e^{-b_2(s_2+cs_3)}\right)^k}{b_1^kb_2^ks_1s_2s_3(s_1+cs_3)^k(s_2+cs_3)^k}, \quad k = \overline{1,3}. \tag{26}$$

So we will not present exact relations but only some numerical computations with following substitutions: $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$ and $c = 2$ (see Tables 7–8).

Table 7

Number of customers distribution for $M/M/2/(1,V_1,V_2,V_3)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$, $c = 2$ (third indication proportional to the sum of others)

V_1	V_2	V_3	p_0	$p_{\{1\}}$	$p_{\{2\}}$	$p_{\{1,2\}}$	p_3
1	1	1	0.7765	0.0971	0.1213	0.0051	0.0000
1	1	3	0.2929	0.2563	0.3204	0.1201	0.0103
1	3	1	0.7765	0.0971	0.1213	0.0051	0.0000
1	3	3	0.2661	0.2661	0.3326	0.1247	0.0105
3	1	1	0.7765	0.0971	0.1213	0.0051	0.0000
3	1	3	0.2887	0.2526	0.3158	0.1316	0.0114
3	3	1	0.7765	0.0971	0.1213	0.0051	0.0000
3	3	3	0.2626	0.2626	0.3282	0.1351	0.0115

Table 8

Loss probability values for $M/M/2/(1,V_1,V_2,V_3)$ system, $a = 1$, $\mu_1 = 0.25$, $\mu_2 = 0.2$, $b_1 = 1$, $b_2 = 2$, $c = 2$ (third indication proportional to the sum of others)

P_{LOSS}	$V_3 = 0.1$	$V_3 = 10.1$	$V_3 = 20.1$
$V_1 = 0.1, V_2 = 0.1$	0.9994	0.9951	0.9951
$V_1 = 0.1, V_2 = 10.1$	0.9994	0.9174	0.9174
$V_1 = 0.1, V_2 = 20.1$	0.9994	0.9174	0.9174
$V_1 = 10.1, V_2 = 0.1$	0.9994	0.9549	0.9549
$V_1 = 10.1, V_2 = 10.1$	0.9994	0.5982	0.5891
$V_1 = 10.1, V_2 = 20.1$	0.9994	0.5982	0.5891
$V_1 = 20.1, V_2 = 0.1$	0.9994	0.9549	0.9549
$V_1 = 20.1, V_2 = 10.1$	0.9994	0.5982	0.5891
$V_1 = 20.1, V_2 = 20.1$	0.9994	0.5982	0.5891

All presented symbolic and numerical results were obtained with the use of *Mathematica* environment in version 13.0.1. Times of calculations were not long as the values of k were not too big. The calculations in this case do not have big computational complexity and do not need much size of RAM memory. It is clear that the most complex operation is Laplace transform inversion used in the process of obtaining exact form of multidimensional Stieltjes convolution $L_k(V)$ – the time of obtaining the inversion and needed size of RAM memory increases together with increasing the k value because then Laplace transform formulae are functions having poles of the higher orders and the process of inversion demands calculating derivatives of higher orders (residues method) which is complicated taking into consideration the form of the numerator of Laplace transform (see e.g. formula (26)) and often needs the use of Newton's binomial theorem or the other complex computational techniques (e.g. the use of the theory of Laurent series). In such situations time complexity can increase even exponentially as well as the size of needed RAM memory and calculations become impossible.

5. CONCLUSIONS AND FINAL REMARKS

In the present paper, we have investigated the model of multi–server loss queueing system of the $M/M/n/m$–type in which servers are assumed to be non–identical and arriving customers are additionally characterized by non–negative random vectors that have sense of multidimensional volume, which means that customers transport information of different types measured in bytes. Indications of these vectors (i.e. information) are stored in dedicated sectors of limited memory buffer until customer ends his service. For the analyzed model, we have proved general formulae for the steady–state number of customers distribution and loss probability. We also showed some symbolic and numerical computations for some special cases of the model that are the most interesting from the practical point of view. Moreover, we drew attention to appearing technical problems connected with calculating multidimensional Stieltjes convolutions, presenting some ideas on how to deal with such computations with the help of *Mathematica* environment for the most common situations. Investigated model can be exemplary applied in the process of analysis or designing of real–life computer systems storing various types of information in dedicated for them limited memory buffers or in analysis of magazines centers working process in the case when we can assume that, for analyzed models, customer service time is independent of his volume vector.

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