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The fractional, multi-order, reduced model of the one-dimensional heat transfer process

Krzysztof OPRZĘDKIEWICZ *

AGH University, al. A. Mickiewicza 30, 30-059 Kraków, Poland

Abstract. In the paper a new, fractional, reduced, multi-order model of a one-dimensional heat transfer process is addressed. The proposed model is the generalization of state space models using single fractional order. The use of various orders for each mode of state equation allows to better describe a behaviour of a thermal system. In addition, the analysis of controllability and observability allows to reduce the dimension of the model without loss of its accuracy. Such a model has not been proposed yet. Theoretical considerations are validated using experimental data obtained from the real laboratory system. Results of analysis supported by experiments show that the use of various orders together with eliminating of non-controllable and non-observable modes of the model allows to obtain the accurate and relatively low order model.

Keywords: multi-order fractional system; heat transfer process; multi-order fractional state equation; stability; controllability; observability.

1. INTRODUCTION

Non-integer order or fractional order (FO) models of various physical phenomena have been presented by many Authors for years. Fundamental results can be found, e.g., in books and papers [1–3] (the heat transfer in a one-dimensional beam), [4] (u.a. fractional models of chaotic systems and ionic polymer metal composites). FO models of diffusion processes are proposed u.a. by [5–7]. Results using new Atangana-Baleanu operator are collected in [8]. This paper presents also the FO blood alcohol model, the Christov diffusion equation and fractional advection-dispersion equation for groundwater transport processes.

Recently FO models are employed among others to describe a spread of diseases. This issue is considered, e.g., in the papers given by [9] (the modeling of the dynamics of COVID using Caputo-Fabrizio operator), [10] (the modeling of a transmission of Zika virus with the use of the Atangana-Baleanu operator).

The "classic", single-order state space FO models of the one-dimensional heat transfer have been proposed by author in many papers, e.g., [11–18]. These models used different FO operators: Grünwald-Letnikov, Caputo, Caputo-Fabrizio and Atangana-Baleanu as well as discrete operators: continuous fraction expansion (CFE) and fractional order backward difference (FOBD). Each model has been thoroughly theoretically justified and validated using experimental results. In addition, each of them assures better accuracy in the sense of square cost function than its IO analogue.

The time-continuous, two-dimensional generalization of FO models mentioned above is proposed in the papers [19, 20].

All models mentioned above used single order approach, i.e., the value of the fractional order is the same for all components of the state equation. However the fractional calculus proposes also an alternative, more general approach, called "multi-order". In such a system orders of all components can be various. Of course, the analysis of such a system is generally more difficult than single order. However in some situations it allows to obtain more accurate models.

Theoretical background of multi-order systems can be found, e.g., in the papers: [21–24]. Initial problems of multi-order systems using Caputo operator are discussed e.g. in the paper [25], the stability of this class of systems is discussed, e.g., in [26].

This paper proposes a new, multi-order, fractional, state space model of the one-dimensional heat transfer process. The heat transfer equation is expressed as an infinite dimensional state equation. Next its finite dimensional, multi-order approximation is proposed and analyzed. The proposed model uses a set of various fractional orders to describe a temperature in single place. In addition, the omitting of uncontrollable and unobservable modes allows us to obtain an accurate and low-order model. Such an approach has not been proposed yet. Theoretical considerations are verified by experimental results.

The organization of the paper is following. Firstly elementary ideas and definitions from fractional calculus are given and the construction of the experimental heat system is recalled.

As the main results the new, multi-fractional order state space model is proposed and its basic properties: spectrum decomposition, stability, controllability and observability are discussed. The proposed conditions of controllability and observability are applied to propose the reduced model.

Furthermore orders of the model are numerically identified using data from real experimental system and MSE cost function. Finally the accuracy and numerical complexity of the identified model are compared to the model using single fractional order.

*e-mail: kop@agh.edu.pl

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2. PRELIMINARIES

Theoretical background of the fractional calculus is presented by many books, e.g., in the section "Fractional Systems: Theoretical Foundations" of [27].

2.1. Basic ideas

The non-integer-order, integro-differential operator is defined as follows (see e.g. [1, 2, 27–29]):

Definition 1. (The elementary non-integer-order operator) The non-integer-order integro-differential operator is defined as follows:

$$ {}_aD_t^\alpha f(t) = \begin{cases} \dfrac{d^\alpha f(t)}{dt^\alpha} & \alpha > 0, \\ f(t) & \alpha = 0, \\ \displaystyle\int_a^t f(\tau)(d\tau)^\alpha & \alpha < 0. \end{cases} \tag{1} $$

where a and t are time limits for computing of the operator, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation. If $\alpha \in \mathbb{Z}$, then the operator (1) turns to classic integer order operator.

Next recall an idea of Gamma Euler function (see, e.g., [29]):

Definition 2. (The Gamma function)

$$ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{2} $$

An idea of Mittag-Leffler function needs to be given next. It is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in solution of fractional order state equation. The one-parameter Mittag-Leffler function is defined as follows:

Definition 3. (The one parameter Mittag-Leffler function)

$$ E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}. \tag{3} $$

and the two parameter Mittag-Leffler function is defined as:

Definition 4. (The two parameters Mittag-Leffler function)

$$ E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}. \tag{4} $$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3).

The fractional-order, integro-differential operator can be described by definitions given by Grünwald and Letnikov, Riemann and Liouville (RL) and Caputo (C). In this paper the C definition is used (see e.g. [1, 2, 28, 29]), [30]):

Definition 5. (The Caputo definition of the FO operator)

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(M-\alpha)} \int_0^\infty \frac{f^{(M)}(\tau)}{(t-\tau)^{\alpha+1-M}}\, d\tau, \tag{5} $$

where $M-1 < \alpha < M$ is the fractional order of operation and $\Gamma(..)$ is the Gamma function.

2.2. The multi-order, SISO system

Next the linear, multi-fractional order, Single Input, Single Output (SISO) state equation needs to be given:

$$ \begin{cases} {}_0D_t^\alpha x(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{6} $$

where $x(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in$ is the control, α is the following set of orders:

$$ \alpha = \{\alpha_1, ..., \alpha_n\}, \quad \alpha_1 \neq ... \neq \alpha_n,\ n = 1, ..., N. \tag{7} $$

With respect to (7) the equation (6) can be decomposed as follows:

$$ \begin{cases} D^{\alpha_1} x_1(t) = A_{11}x_1(t) + A_{12}x_2(t) ... + A_{1N}x_N(t) + B_1 u(t), \\ D^{\alpha_2} x_2(t) = A_{21}x_1(t) + A_{22}x_2(t) ... + A_{2N}x_N(t) + B_2 u(t), \\ \quad \vdots \\ D^{\alpha_n} x_N(t) = A_{N1}x_1(t) + A_{N2}x_2(t) ... + A_{NN}x_N(t) \\ \qquad\qquad + B_N u(t), \end{cases} \tag{8} $$

where

$$ A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \dots & A_{NN} \end{bmatrix}_{N\times N}, \tag{9} $$

$$ B = \left[B_1, \ \dots, \ B_N\right]^T. \tag{10} $$

In general, a solution of the state equation (6)–(10) is not a trivial issue. The numerical method to do it is proposed, e.g., in [21], solutions for specific classes of systems are proposed in [24], the representation of solution in the form of series is given in [22].

In particular, a simple form of solution can be given for diagonal form of the state matrix, i.e., $A_{mn} = [0]$ for $m \neq n$. This will be applied in this paper.

For infinite dimensional systems the A and B matrices expand to infinite dimensional, linear operators.

2.3. The stability

The stability analysis for non-commensurate, multi-order systems is more complicated than for single-order or commensurate systems. Results from this area are given, e.g., by [31] or [32]. In this paper the approach using the Mikhailov Theorem proven in [32] and described by Theorems 1 and 2 will be employed. They are recalled below.

Theorem 1. The non-commensurate continuous-time system (6) with fractional orders $\alpha_n \in \mathbb{R}$, $n = 1,\ldots,N$ is asymptotically stable iff the Mikhailov curve $p(j\omega)$ of the system satisfies the following two conditions:

$$\begin{aligned} &p(j\omega) \neq 0 \ \ \forall\omega \in [0;\infty) \\ &\Delta Arg\,(p(j\omega))\big|_0^\infty = \frac{\beta_n \pi}{2}, \end{aligned} \tag{11}$$

where

$$\beta_n = \sum_{n=1}^{N} \alpha_n\,, \tag{12}$$

$$p(j\omega) = \det\left(D^\alpha(j\omega) - A\right), \tag{13}$$

$$D^\alpha(s) = \operatorname{diag}\{s^{\alpha_1}\ s^{\alpha_2}\ ,\ldots s^{\alpha_N}\ \}. \tag{14}$$

Theorem 2. The non-commensurate continuous-time system (6) with fractional orders $\alpha_n \in \mathbb{R}$, $n = 1,\ldots,N$ and the Mikhailov curve $p(j\omega) \neq 0\ \ \forall\omega \in [0;\infty)$ is unstable iff:

$$\Delta Arg\,(p(j\omega))\big|_0^\infty \leq \frac{\beta_n \pi}{2} - \pi. \tag{15}$$

where β_n, p and D_α are described by (12), (13) and (14), respectively.

The use of both theorems to the system with diagonal state operator allows us to obtain stability condition, presented in sequel.

3. THE EXPERIMENTAL HEATING SYSTEM AND ITS TIME-CONTINUOUS, SINGLE-ORDER FRACTIONAL MODEL

The experimental heat system is illustrated by Fig. 1. Its main part is the thin copper rod 260 mm long. To simplify its length is assumed equal 1.0. Thanks to this, the location and length of the heater and RTD-s is expressed relative to 1.0. The rod is heated with use of the electric heater Δx_u long attached at its end. The output signal from the system is the temperature.

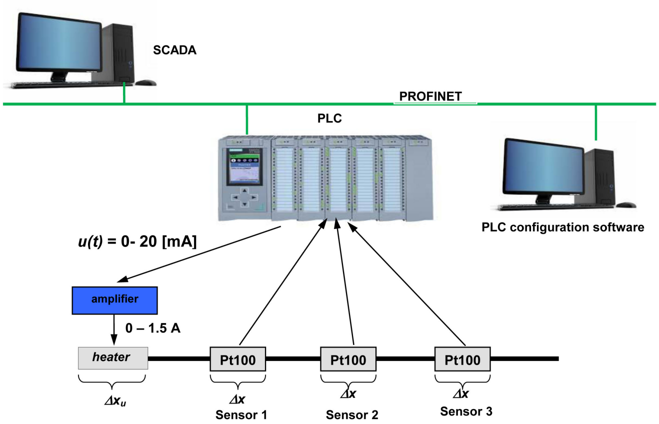

Fig. 1. The experimental system

It is measured using the miniature RTD-s of Pt100 type. The length of each sensor is Δx. Sensors are attached in points $x_{j,1}$, $j = 1,2,3$, $0.0 < x_{j..} < 1.0$. The system is controlled by the standard current from range 0–20 [mA] amplified to the range 0–1.5 [A] and sent to the heater. Signals from the RTDs are read directly by analog input module of the PLC. Data from PLC are collected by SCADA application. The whole system is integrated with the use of PROFINET. The step responses measured by all sensors are presented in Fig. 2.

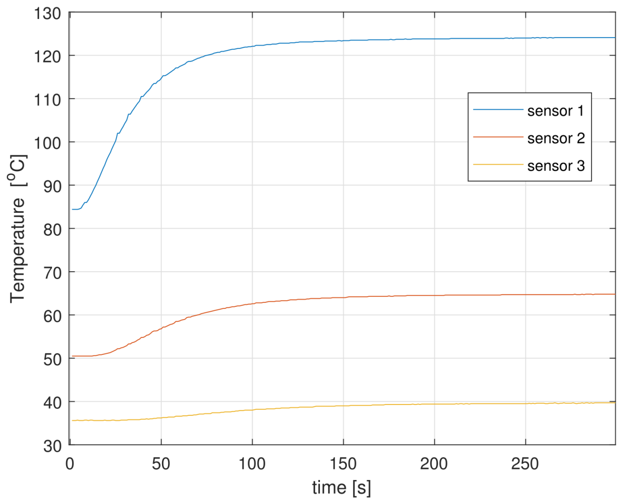

Fig. 2. The step responses from all sensors

The single fractional order model of this thermal system is presented with details in the papers [11, 12]. In this paper, its version with integer order along the length is used. It is as below:

$$\begin{cases} {}^C D_t^\alpha Q(x,t) = a_w \dfrac{\partial^2 Q(x,t)}{\partial x^2} - R_a Q(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, \quad t \geq 0, \\ \dfrac{\partial Q(1,t)}{dx} = 0, \quad t \geq 0, \\ Q(x,0) = Q_0, \quad 0 \leq x \leq 1, \\ y(t) = k_0 \displaystyle\int_0^1 Q(x,t)c(x)\,\mathrm{d}x. \end{cases} \tag{16}$$

In (16) $0 < x < 1$ is the length of the rod, $a_w > 0$ is the coefficient of the heat conduction along the rod, $R_a > 0$ is the coefficient of the heat transfer from rod to environment.

The heat transfer equation (16) can be expressed as an infinte dimensional state equation (see [11]):

$$\begin{cases} {}^C D_t^\alpha Q(t) = AQ(t) + Bu(t), \\ Q(0) = 0, \\ y(t) = y_0 CQ(t). \end{cases} \tag{17}$$

where

$$\begin{cases} AQ(x) = a_w \dfrac{\partial^2 Q(x)}{\partial x^2} - R_a Q(x), \\ a_w,\ R_a > 0, \\ D(A) = \left\{ Q \in H^2(0,1) : \left.\dfrac{\partial Q(x)}{\partial x}\right|_{x=0} = 0, \right. \\ \qquad \left. \left.\dfrac{\partial Q(x)}{\partial x}\right|_{x=1} = 0 \right\}, \\ H^2(0,1) = \left\{ u \in L^2(0,1) : \dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2} \in L^2(0,1) \right\}, \\ CQ(t) = \langle c, Q(t) \rangle, \quad Bu(t) = bu(t), \\ Q(t) = [q_1(t), q_2(t)\,..]^T. \end{cases} \tag{18}$$

The orthonormal basis of the state space is built by the following set of the eigenvectors of the state operator A:

$$h_n = \begin{cases} 0, & n = 0, \\ \sqrt{2}\cos(n\pi x), & n = 1, 2, \ldots \end{cases}. \tag{19}$$

Eigenvalues of the state operator take the following form:

$$\lambda_n = -a_w (n\pi)^2 - R_a, \quad n = 0, 1, 2, \ldots \tag{20}$$

and the state operator is as below:

$$A = \operatorname{diag}\{\lambda_0, \lambda_1, \lambda_2, \ldots\}. \tag{21}$$

The input operator B is as follows:

$$B = [b_0, b_1, b_2, \ldots]^T. \tag{22}$$

Each element $b_n = \langle b(x), h_n \rangle$, where $\langle .. \rangle$ is the inner product:

$$\langle b(x), h_n \rangle = \int_0^1 b(x) h_n(x)\, \mathrm{d}x. \tag{23}$$

In (23) $b(x)$ is the shaping function of the heater:

$$b(x) = \begin{cases} 1, & x \in [0, x_u], \\ 0, & x \notin [0, x_u]. \end{cases} \tag{24}$$

After taking into account (19), (23) and (24) each element b_n is equal:

$$b_n = \begin{cases} \Delta x_u, & n = 0, \\ \dfrac{\sqrt{2}\sin(n\pi\Delta x_u)}{n\pi}, & n = 1, 2, \ldots \end{cases} \tag{25}$$

The output operator C describes the size and location of RTD-s. It is as below:

$$C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}, \tag{26}$$

Each row of output operator C takes the following form:

$$C_j = \left[c_{j,0}, c_{j,1}, c_{j,2}, \ldots \right] \quad j = 1, 2, 3 \ldots, \tag{27}$$

where $c_{j,n} = \langle c(x), h_n \rangle$, $\langle .. \rangle$ is the scalar product analogically as (23), $c(x)$ is the output sensor function:

$$c_j(x) = \begin{cases} 1, & x \in [x_{j,1}, x_{j,2}], \\ 0, & x \notin [x_{j,1}, x_{j,2}], \end{cases} \quad j = 1, 2, 3. \tag{28}$$

In (28) coordinates $x_{j,1}$ and $x_{j,2}$ describe the place of the sensor attachment ($x_{j,2} = x_{j,1} + \Delta x$ or equivalently: $x_{j,1} = x - 0.5\Delta x$, $x_{j,2} = x + 0.5\Delta x$).

With respect to (19), (23) and (28) each element c_{jn} is as follows:

$$c_{j,n} = \begin{cases} \Delta x, & n = 0, \\ \dfrac{\sqrt{2}\left(\sin(n\pi x_{j,2}) - \sin(n\pi x_{j,1})\right)}{n\pi}, \\ n = 1, 2, \ldots, j = 1, 2, 3. \end{cases} \tag{29}$$

The step response of the model read by the j-th sensor (16)–(29) is as follows:

$$y_j(t) = k_0 \sum_{n=0}^{\infty} \frac{(E_\alpha(\lambda_n t^\alpha) - 1(t))}{\lambda_n} b_n c_{jn}, \quad j = 1, 2, 3. \tag{30}$$

In (30) $E_{\alpha_{l,h}}(..)$ is the one parameter Mittag-Leffler function, k_0 is the steady-state gain of the model, necessary to fit a step response of model to experimental one, λ_n, b_n and c_n are described by (20), (25) and (29), respectively.

The non-integer order model (17)–(30) is infinite dimensional. Its use to modeling requires us to apply its finite dimensional approximant, obtained by truncation of further modes in the state equation (17). The dimension of such a finite dimensional model N is the minimum value assuring its good accuracy in the sense of a selected cost function. Simultaneously further increasing of N should no longer improve of the cost function. Looking for suitable value of N can be done numerically with the use of MATLAB. This has been presented in [12]. The estimated value is equal $N = 22$. For a finite dimensional model the operators: A, B and C are interpreted as matrices.

Consequently the step response (30) turns to the finite sum:

$$y_j(t) = k_0 \sum_{n=0}^{N} \frac{(E_\alpha(\lambda_n t^\alpha) - 1(t))}{\lambda_n} b_n c_{jn}, \quad j = 1, 2, 3. \tag{31}$$

In (31) N is the dimension of the finite dimensional approximation of the model (17)–(30). It can be estimated numerically (see [12]).

4. MAIN RESULTS

4.1. The multi-fractional order system

Consider the infinite-dimensional state equation (17) with the state operator (21). It can be decomposed to single, separated modes, as it was shown in [11].

Assume that the fractional order α_n of the n-th decomposed mode can be different from others:

$$\alpha_0 \neq \alpha_1 \neq \ldots \neq \alpha_n \ldots, \quad n = 0, 1, 2, \ldots \tag{32}$$

All orders belong to the following, infinite, countable set:

$$\{\alpha\} = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\} \subset (0.0; 2.0), \quad n = 0, 1, 2, \ldots. \tag{33}$$

This yields the following form of the state equation (17):

$$\begin{cases} {}^C D_t^{\{\alpha\}} Q(t) = AQ(t) + Bu(t), \\ Q(0) = 0, \\ y(t) = y_0 C Q(t). \end{cases} \tag{34}$$

Each n-th order from the set (33) is associated to n-th, scalar mode of the decomposed system, described as below:

$${}^C D_t^{\alpha_n} q_n(t) = \lambda_n q_n(t) + b_n u(t), \quad n = 0, 1, 2, \ldots \tag{35}$$

The impulse response of the single mode (35) takes the following form:

$$g_{jn}(t) = b_n c_{jn} t^{\alpha_n - 1} E_{\alpha_n, \alpha_n}(\lambda_n t^{\alpha_n}). \tag{36}$$

And consequently the impulse response of the system at the j-th output is as follows:

$$g_j(t) = \sum_{n=0}^{\infty} g_n(t), \quad j = 1, 2, 3. \tag{37}$$

For the control being the Heaviside function $u(t) = 1(t)$ and homogenous initial condition the step response of the single mode is as follows:

$$y_{jn}(t) = b_n c_{jn} \frac{\left(E_{\alpha_n}(\lambda_n t^{\alpha_n}) - 1(t)\right)}{\lambda_n}, \tag{38}$$

and consequently the step response of the system (30) takes the following form:

$$y_j(t) = k_0 \sum_{n=0}^{\infty} y_{jn}(t), \quad j = 1, 2, 3, \tag{39}$$

where k_0 is the steady state gain allowing to fit the response of the model to the experimental result.

The multi-order system described by (32)–(39) is infinite dimensional. Analogically as for single-order system discussed previously, possible to apply in practice is its finite dimensional approximation. It is obtained by truncation of further modes of inifnite dimensional system. Consequently the operators A, B and C are interpreted as matrices and the set of orders (33) reduces to the finite set:

$$\{\alpha\} = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}, \quad n = 0, 1, 2, .., N \tag{40}$$

and impulse and step responses take the form of finite sums:

$$g_j(t) = \sum_{n=0}^{N} g_n(t), \quad j = 1, 2, 3. \tag{41}$$

$$y_j(t) = k_0 \sum_{n=0}^{N} y_{jn}(t), \quad j = 1, 2, 3, \tag{42}$$

where $g_n(t)$ and $y_n(t)$ are expressed by (36) and (36), respectively.

Each mode of response (36) or (38) is different from zero iff suitable elements of control and observation operators b_n and c_{jn} are nonzero too. This is equivalent to the requirement of the controllability and observability of particular mode and is associated to the construction of the experimental system. This is discussed with details in the next subsection.

Next, the knowledge about non-controllable and non-observable modes of the system allows to construct a reduced model, containing only controllable and observable modes. Such a model will be as accurate, as full but its dimension will be smaller. This is presented in sequel too.

4.2. The controllability and the observability of the system

The controllability and observability of the considered system can be examined for each decomposed mode separately and some modes can be controllable and observable and other can be not. To describe such a situation ideas of partial controllability and observability are proposed. An idea of partial controllability appears, e.g., in [33], but here it is a little bit different.

Definition 6. (The partial controllability).
Consider the decomposed system (34), (35). It is partially controllable if there exist its non-controllable modes, i.e., $\exists\, b_n = 0$, $n = 0, 1, 2, \ldots$.

Definition 7. (The partial observability).
Consider the decomposed system (34), (35). It is partially observable if there exist its non-observable modes, i.e., $\exists\, c_{jn} = 0$, $j = 1, 2, 3, \; n = 0, 1, 2, \ldots$.

The controllability and observability are determined by the construction of the real experimental system. The controllability is determined by the length of the heater and the observability is determined by the location and size of sensors.

The existence of non-observable or non-controllable modes is described by the following propositions.

Proposition 1. (The non-controllability of the n-th mode)
Consider the infinite dimensional, multi-order system (34). Assume that the heater is $0.0 < \Delta x_u < 1.0$. The n-th mode $q_n(t)$ of the system is non-controllable iff:

$$\begin{gathered} \Delta x_u = \frac{1}{n}, \; n = 2, 3, \ldots \\ \vee \\ \Delta x_u = \frac{2}{n}, \; n = 3, 4, .. \end{gathered} \tag{43}$$

Proof. To prove the condition (43) recall the form of n-th element of the control operator (25). From it we obtain that:

$\sin(n\pi\Delta x_u) = 0$. For $0 < \Delta x_u < 1$ this is equivalent to:

$$\begin{gathered} n\pi\Delta x_u = \pi,\ n = 1,2... \\ \vee \\ n\pi\Delta x_u = 2\pi,\ n = 2,3,.. \end{gathered} \tag{44}$$

Condition (44) yields directly (43) and the proof is completed. □

Next define the set of indices of non-controllable modes:

Definition 8. (The set of indices of non-controllable modes) Consider the control operator of the system, expressed by (22) and (25). The indices of non-controllable modes meet the condition (43):

$$N_{nc} = \{n_{nc} = 1,2,\dots:\ b_{n_{nc}} = 0\}. \tag{45}$$

Analogically the non-observability can be described.

Proposition 2. (The non-observability of the jn-th mode) Consider the infinite dimensional, multi-order system (34). Assume that the sensor is $0.0 < \Delta x < 1.0$ long and attached in the place $0.0 < x_{j,1} + \Delta x < 1.0$. The jn-th mode $q_j, n(t)$ of the system is non-observable iff:

$$\begin{gathered} x_{j,1} = \frac{\frac{1}{n} - \Delta x}{2},\ n = 1,2,3,\dots \\ \vee \\ x_{j,1} = \frac{\frac{3}{n} - \Delta x}{2},\ n = 2,3,\dots \end{gathered} \tag{46}$$

Proof. The jn-th element (29) of the observation operator C for $n = 1,2,\dots$ is as follows:

$$\begin{aligned} &\frac{\sqrt{2}\left(\sin\left(n\pi x_{j,2}\right) - \sin(n\pi x_{j,1})\right)}{n\pi} \\ &\quad = 2\frac{\sqrt{2}}{n\pi}\sin\left(\frac{\Delta x}{2}\right)\cos\left(\frac{2x_{j,1} + \Delta x}{2}\right). \end{aligned} \tag{47}$$

The expression (47) is equal zero iff:

$$\begin{gathered} \frac{n\pi(2x_{j,1} + \Delta x)}{2} = \frac{\pi}{2} \\ \vee \\ \frac{n\pi(2x_{j,1} + \Delta x)}{2} = \frac{3\pi}{2}. \end{gathered} \tag{48}$$

Computing $x_{j,1}$ from (48) gives directly (46) and the proof is completed. □

The set of indices of non-observable modes can be defined analogically:

Definition 9. (The set of indices of non-observable modes) Consider the output operator of the system, expressed by (26), (27) and (29). The indices of non-controllable modes meet the condition (46):

$$N_{no} = \{n_{no} = 1,2,\dots:\ c_{jn_{no}} = 0\}. \tag{49}$$

For the inifinite dimensional model both sets of indices (45) and (49) are infinite and countable sets.

Next, the system will be called fully controllable or fully observable, if all its modes are controllable or observable. This can be examined for infinite dimensional system or its finite dimensional approximation.

The controllability and observability of the infinite dimensional system implies these properties for its finite dimensional approximation, but the inverse implication is not a true, because non-controllable or non-observable modes can appear in the truncated part of the approximated, finite dimensional system.

Proposition 3. (The full controllability of the multi-order fractional system)
Consider the infinite dimensional, multi-order system (34). For it the following sentences are equivalent:

- The system is fully controllable,
- all modes of the system are controllable,
- $b_n \neq 0\ \forall n = 1,2,\dots.,$
- $N_{nc} = \emptyset$.

Proposition 4. (The full observability of the multi-order fractional system)
Consider the infinite dimensional, multi-order system (34). For it the following sentences are equivalent:

- The system is fully observable,
- all modes of the system are observable,
- $c_{jn} \neq 0,\ j = 1,2,3,\ \forall n = 1,2,\dots.,$
- $N_{no} = \emptyset$.

Next criteria of partial controllability and partial observability can be proposed.

Proposition 5. (The partial controllability)
Consider the infinite dimensional, multi-order system (34). It is partially controllable iff:

$$N_{nc} \neq \emptyset \iff \exists b_n = 0,\ n = 1,2,\dots. \tag{50}$$

Proposition 6. (The partial controllability)
Consider the infinite dimensional, multi-order system (34). It is partially observable iff:

$$N_{no} \neq \emptyset \iff \exists c_{j,n} = 0,\ j = 1,2,3,\ n = 1,2,\dots. \tag{51}$$

Both propositions follow directly from above considerations.

The above analysis was run for infinite dimensional system. Its finite dimensional approximation requires dealing with finite number of modes and in general it is a little bit simplier.

The conditions (43), (46), (45) and (49) should be tested for $n \leq N$, where N is the size of finite dimensional model. For such a situation it is also possible to estimate a suitable dimension of approximation assuring keeping its full controllability or full observability for fixed parameters of heater and sensor. It is described by the propositions given below.

Proposition 7. (The maximum size N_c of the finite dimensional approximation assuring the full controllability of the model)
Consider the model of the system, being the finite dimensional

approximation of the model (34). Assume that the heater is Δx_u long.

The maximum size of the finite dimensional approximation assuring its full controllability meets the following inequality:

$$N_c < \frac{1}{\Delta x_u}. \tag{52}$$

Proof. To proove the condition (52) recall the condition of non-controllability (43). For finite amount of modes N_c it is as below:

$$\begin{array}{lcl} \Delta x_u = \frac{1}{n},\ n = 2,3,..,N_c & & \Delta x_u = \frac{1}{N_c},\dots,\frac{1}{2} \\ \vee & \Longleftrightarrow & \vee \\ \Delta x_u = \frac{2}{n},\ n = 3,4,..,N_c & & \Delta x_u = \frac{2}{N_c},\dots,\frac{2}{3}. \end{array} \tag{53}$$

From (53) it can be noted that the minimum "non-controllable" length of heater Δx_u is achieved for maximum size of model N_c. This means that for Δx smaller than this border value all modes will be controllable. This is expressed as follows:

$$\Delta x_u < \frac{1}{N_c} \iff N_c < \frac{1}{\Delta x_u}. \tag{54}$$

This completes the proof. □

Analogical condition describes the maximum size of the approximated model from point of view of observability.

Proposition 8. (The maximum size N_o of the finite dimensional approximation assuring the full observability of the model)
Consider the model of the system, being the finite dimensional approximation of the model (34). Assume that the sensor is equal Δx long and it is attached in the place $x_{j,1}$ where $\Delta x + x_{j,1} < 1.0$.

The maximum size of the finite dimensional approximation assuring its full observability meets the following inequality:

$$N_o < \frac{1}{2x_{j,1} + \Delta x}. \tag{55}$$

The proof is analogical as for condition (52) and it can be omitted.

A quick analysis of the condition (55) shows that its keeping can be difficult for a real system, because it requires attaching small sensors closely to heater.

On the other hand, non-controllable and non-observable modes of a system can be omitted in the impulse and step responses. This allows to reduce the dimension of a model without decreasing its accuracy. Such a reduced model is presented in the next subsection.

4.3. The reduced, finite dimensional model

The analysis of controllability and observability given in the previous section shows that not all modes of the system impact to its input-output behaviour. The non-controllable and non-observable modes described by indices $n \in N_{nc}$ and $n \in N_{no}$ respectively can be omitted during computation of impulse and step responses of the finite dimensional model (42) without loss of its accuracy.

To simplify the further considerations let us define the indices of controllable and observable modes of the finite dimensional model.

Definition 10. (The set of indices of controllable and observable modes N_{co})
Assume that the size of the finite dimensional model is equal N. The set of indices of controllable and observable modes of this model is defined as follows:

$$\begin{aligned} N_{co} &= \{n = 0, \dots N : n \notin N_{nc}, n \notin N_{no}\} \\ &= \{0, \dots, N\} \setminus N_{nc} \setminus N_{no}. \end{aligned} \tag{56}$$

Using the definition (56) the reduced step and impulse responses can be described as below:

$$g_j(t) = \sum_{n \in N_{co}} g_{jn}(t), \qquad j = 1,2,3, \tag{57}$$

$$y_j(t) = k_0 \sum_{n \in N_{co}} y_{jn}(t), \qquad j = 1,2,3, \tag{58}$$

where g_{jn} and y_{jn} are expressed by (36) and (38) and N_{co} is described by (56).

4.4. The stability

At the beginning a sense of stability analysis for the considered system should be shortly explained. The modeled physical processes are heat conduction and dissipation. They are from their nature stable. However, a numerical identification of model parameters with the use of experimental data can lead to obtain of "hidden" unstable parameters. They can assure a good performance of model for single data set used to identification, but of course such a model is useless in general.

The stability is described by the following Propositions. The first one describes the stability of the finite dimensional model, the next one – the infinite dimensional.

Proposition 9. (The stability of the multi-fractional order, finite dimensional system with diagonal state matrix)
Consider the multi-fractional order, finite dimensional system of size N, being finite dimensional approximation of the system described by (34) with operators expressed by (21)–(29) and fractional orders described by (32) and (40).

The system is asymptotically stable for each order $\alpha_n \in \{\alpha\} \subset (0;2)$, $n = 0,1,\dots,N$.

This proposition will be proven using Theorems 1 and 2.

Proof. The polynomial (13) with respect to (21) is as below:

$$p(j\omega) = \prod_{n=0}^{N} p_n(j\omega), \tag{59}$$

where

$$\begin{aligned} p_n(j\omega) &= ((j\omega)^{\alpha_n} - \lambda_n) \\ &= \left(\omega^{\alpha_n}\left(\cos\frac{\alpha_n \pi}{2} + j\sin\frac{\alpha_n \pi}{2}\right) - \lambda_n\right). \end{aligned} \tag{60}$$

The phase of $p(j\omega)$ is equal:

$$\phi(\omega) = \sum_{n=0}^{N} \phi_n(\omega), \tag{61}$$

where

$$\phi_n(\omega) = \arctan\left(\frac{\sin\dfrac{\alpha_n \pi}{2}}{\cos\dfrac{\alpha_n \pi}{2} - \dfrac{\lambda_n}{\omega^{\alpha_n}}}\right). \tag{62}$$

From (59) and (60) it turns out that the 1st condition from (11) is met.

The phase (61) for $\omega = 0$ equals to zero. Its limit value for $\omega \to \infty$ is a sum of limit values of all components (62). For $0.0 < \alpha_n < 2.0$, $n = 0, 1, \ldots, N$ they are equal:

$$\begin{aligned}\lim_{\omega\to\infty} \phi_n(\omega) &= \lim_{\omega\to\infty} \arctan\left(\frac{\sin\dfrac{\alpha_n \pi}{2}}{\cos\dfrac{\alpha_n \pi}{2} - \dfrac{\lambda_n}{\omega^{\alpha_n}}}\right)\\ &\approx \lim_{\omega\to\infty} \arctan\left(\frac{\sin\dfrac{\alpha_n \pi}{2}}{\cos\dfrac{\alpha_n \pi}{2}}\right) = \frac{\alpha_n \pi}{2}\end{aligned} \tag{63}$$

and consequently:

$$\Delta(Arg(p(j\omega))) = \frac{\pi}{2}\sum_{n=0}^{N} \alpha_n . \tag{64}$$

This means that the 2nd condition in (11) is met for $0.0 < \alpha_n < 2.0$.

Next consider $\alpha_n \geq 2.0$, $n = 0, 1, \ldots, N$. It can be expressed as: $\alpha_n = \pi + \alpha_{nr}$, $0 < \alpha_{nr} < 1.0$. This implies that

$$\lim_{\omega\to\infty} \phi_n(\omega) = \frac{\alpha_{nr}\pi}{2} < \frac{\alpha_n \pi}{2}. \tag{65}$$

This yields that the summarized increment of phase for all modes is smaller than reqired to assure the stability and the condition of unstability (15) is met. This completes the proof. □

5. EXPERIMENTS AND SIMULATIONS

5.1. Parameters of the real system

Experiments were done with the use of the system shown in Fig. 1. The relative length of the heater is equal: $\Delta x_u = 0.14$, the sensors are $\Delta x = 0.06$ long and they attached in the following places:

$$\begin{cases} x = 0.29: & x_{1,1} = 0.26, \quad x_{1,2} = 0.32,\\ x = 0.50: & x_{2,1} = 0.47, \quad x_{2,2} = 0.53,\\ x = 0.73: & x_{3,1} = 0.70, \quad x_{3,2} = 0.76.\end{cases}$$

The coefficient of heat conduction a_w and the coefficient of heat exchange R_a are known (see [12]). They are equal: $a_w = 0.000410$, $R_a = 0.0677066$.

The analysis of the finite dimensional model will be done for its two sizes: $N = 8$ and $N = 20$. This is helpful to compare the proposed model vs model using single fractional order.

5.2. Controllability and observability

Firstly the controllability of the model for fixed location and size of heater was examined. The “non-controllable” lengths of the heater are given by (43). They are as follows:

$$\Delta x_u = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \ldots$$

For the length of the sensor $\Delta x_u = 0.14$ the maximum size of model assuring its full controllability can be computed using (52). It equals to:

$$N_c < \frac{1}{0.14} = 7.1429.$$

This yields the order $N_c \leq 7$. The sets N_{nc} for $N = 7$ and $N = 20$ are given in Table 1.

Table 1
The sets N_{nc} of the model

N_c	$N = 7$	$N = 20$
7.1429	$\emptyset$	$\{8, 16\}$

Next the observability needs to be analyzed. This should done for each sensor separately using the conditions (55) and (46). Results are completed in Table 2.

Table 2
The sets N_{no} of the model

Sensor	N_o	N_{no} for $N = 7$	N_{no} for $N = 20$
1	1.56	$\emptyset$	$\emptyset$
2	1.00	$\{1, 3, 5, 7\}$	$\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$
3	0.68	$\emptyset$	$\emptyset$

The 1st conclusion from Table 2 is that the condition of observability (49) for the considered location and size of sensors is impossible to meet in reality. Next, for sensors 1 and 3 the system is fully observable for both tested orders, but for sensor 2 there are non-observable modes for both tested sizes of the model N.

Finally, the set (56) of the system for both tested dimensions N and all sensors can be constructed. It is presented in Table 3.

Table 3
The sets N_{co} for all sensors and $N = 7$, $N = 20$

Sensor	N_{co} for $N = 7$	N_{co} for $N = 20$
1	$\{n = 0 : 7\}$	$\{n = 0 : 20\} \setminus \{8, 16\}$
2	$\{0, 2, 4, 6\}$	$\{0, 2, 4, 6, 10, 12, 14, 18, 20\}$
3	$\{n = 0 : 7\}$	$\{n = 0 : 20\} \setminus \{8, 16\}$

The profit from reduction of the order is illustrated by Table 4 describing the amount of modes of model for each sensor and both considered dimensions N. It can be interpreted as the real order of the proposed model and it will be denoted by N_r.

Table 4
The amount of modes of model necessary to compute the reduced step response (58) (the real order N_r) for all sensors and $N = 7$, $N = 20$

Sensor	$N = 8$	$N = 20$
1	8	18
2	4	8
3	8	18

The sets N_{co} shown in Table 3 are applied to construct of the reduced models with respect to (58). The orders identification and accuracy of this model are presented in the next subsection.

5.3. Identification of orders α_n and accuracy

The accuracy of the model can be estimated with the use of typical mean square error (MSE) cost function. For single j-th sensor it is as below:

$$MSE_j = \frac{1}{K}\sum_{k=1}^{K}\left(y_j(kh) - y_{je}(kh)\right)^2, \tag{66}$$

and its mean value for all sensors is following:

$$\sum_{j=1}^{3}\frac{MSE_j}{3}, \tag{67}$$

where $k = 1,\ldots,K$ are the time instants, h is the sample time, $y_{je}(kh)$ and $y_j(kh)$ are the step responses of plant and model (31), measured and computed at the same time grid. During experiments the number of samples was equal: $K = 300$ and sample time was equal $h = 1$ s.

The cost function (66) is identical as applied in [12]. This allows to compare the proposed, multi-order, reduced model to the model using the single order.

The identification of orders was done via minimization of the cost function (66) with the use of the MATLAB function *fminsearch* for each sensor separately. Results are presented in Tables 5 and 6 and illustrated by Figs. 3 and 4.

Table 5
The orders α for all sensors and $N = 8$

j	α	MSE_j
1	$\{0.9916, 0.7823, 0.6958, 0.9865,$ $0.6772, 1.1430, 0.7787, 1.3753\}$	0.0314
2	$\{0.9064, 0.8950, 0.7594, 0.2825\}$	0.0130
3	$\{0.9730, 0.9556, 0.9426, 0.9170,$ $0.9276, 0.9049, 0.9962, 0.6950\}$	0.0316
	The cost function (67)	0.0253

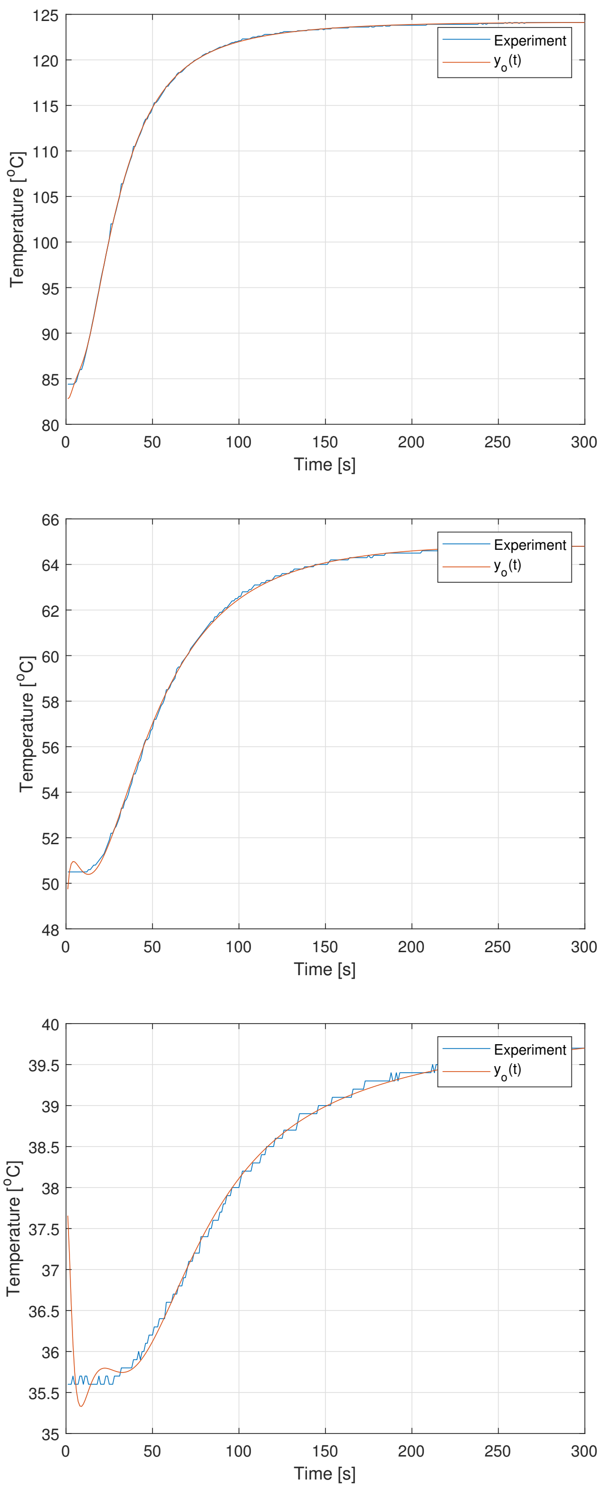

Fig. 3. The comparison of the step responses model vs experiment for $N = 8$. Sensor 1 – top, sensor 2 – middle, sensor 3 – bottom

Furthermore the stability condition was examined. To do it the impulse responses for all sensors and orders given in Table 5 were computed using (57). They are shown in Fig. 5.

Next, the order of the 3rd mode was changed to: $\alpha_3 = 2.3707$. The set of impulse responses for this situation is illustrated by Fig. 6. It is important to note that the response of the 2nd

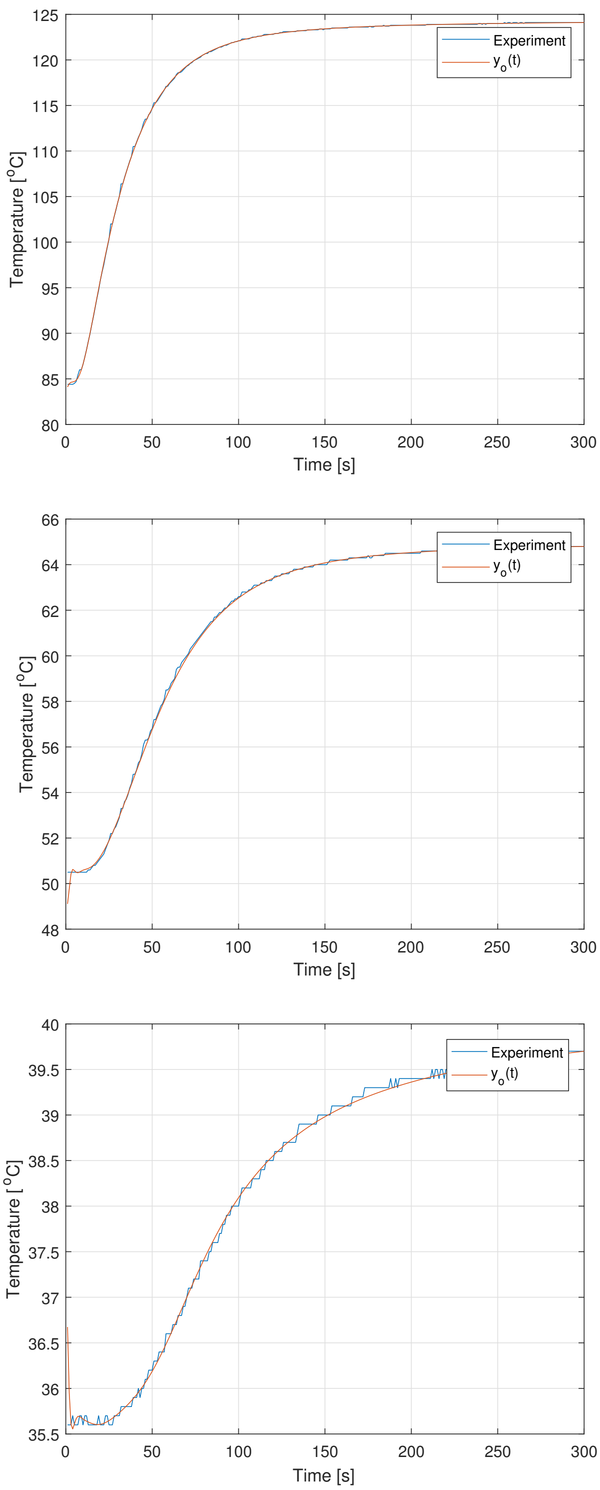

Fig. 4. The comparison of the step responses model vs experiment for $N = 20$. Sensor 1 – top, sensor 2 – middle, sensor 3 – bottom

sensor is stable, because the unstable mode is not observable (see Table 2).

Finally the proposed, multi-order, reduced model should be compared to single-order model discussed in [12], Table I. The values of the cost function (67) for both models are presented in Table 7.

Table 6
The orders α for all sensors and $N = 20$

j	α	MSE_j
1	$\{0.9092, 0.9400, 0.9271, 0.8831, 0.9882, 0.9272, 0.9221, 0.9173, 0.8929, 0.9602, 0.9855, 0.8959, 0.8234, 0.0457, 0.9534, 0.8383, 0.8536, 0.7841\}$	0.0114
2	$\{0.9094, 0.9012, 0.8863, 0.8723, 1.0552, 1.1264, 1.3524, 1.2985\}$	0.0136
3	$\{0.9199, 0.9613, 1.2834, 1.1333, 1.0599, 0.8525, 0.7347, 0.6018, 0.4752, 0.8341, 1.0758, 0.4358, 0.6494, 1.4482, 0.8098, 1.0222, 1.2556, 0.7470\}$	0.0066
	The cost function (67)	0.0105

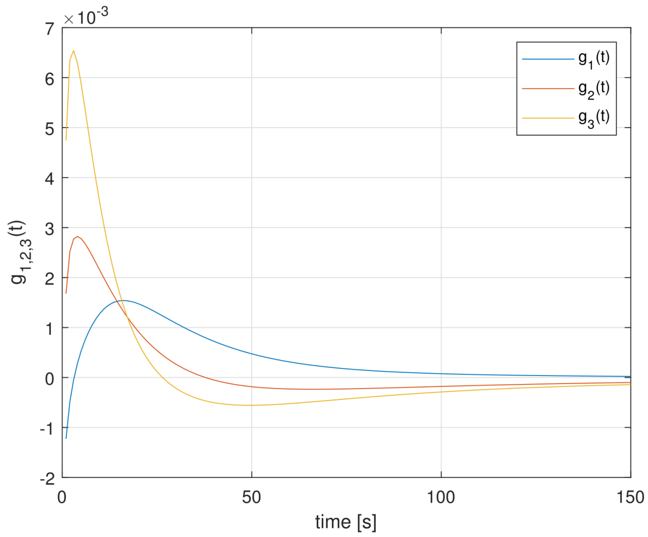

Fig. 5. The impulse responses of the stable model

Table 7
The cost function (67) for single-order model [12] and multi-order reduced model

Model	$N = 8$	$N = 20$
single-order	0.1434	0.0504
multi-order reduced	0.0253	0.0105
N_r for sensors 1, 3	8	18
N_r for sensor 2	4	10

Table 7 shows that the proposed, multi-order model is more accurate in the sense of the MSE cost function than the single-order model. This good accuracy is achieved for relatively low order of model, additionally decreased by omitting of non-controllable and non-observable modes in the step response.

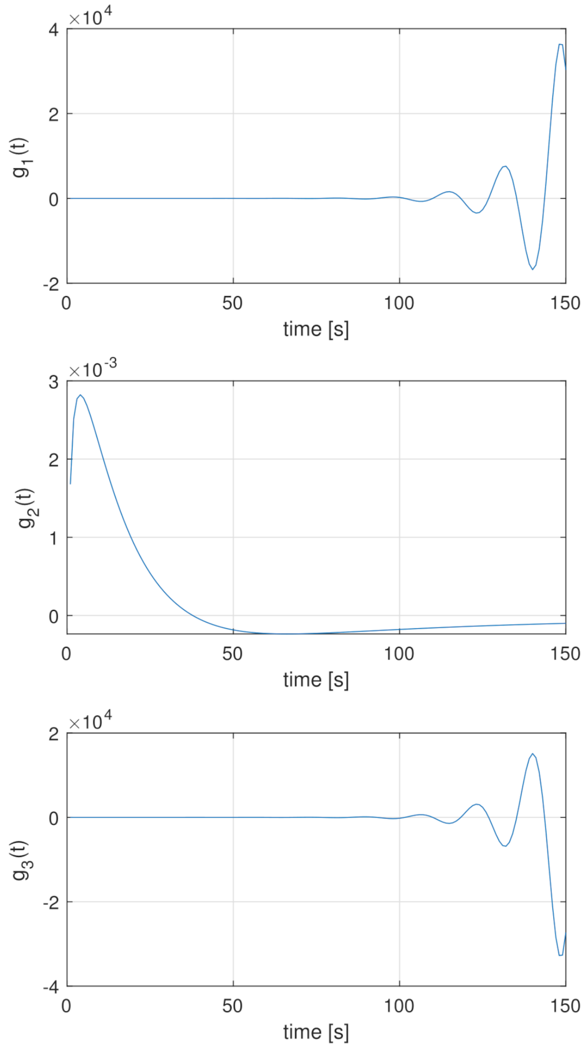

Fig. 6. The impulse responses of the unstable model

6. FINAL CONCLUSIONS

The main conclusion from the paper is that the proposed model using various orders allows to more accurately describe a fractional behaviour of high order system. This is confirmed by results presented in other papers. For example in the paper [34] the fractional order transfer function using two orders more accurately describes real temperature than simplier one, employing only one fractional order.

Next, the analysis of the controllability and observability of the model allows the reduction of the dimension of the model without loss of its accuracy. This is particularly important during implementation of thermal models at bounded digital platforms.

The main disadvantage of the proposed model is the need to identify its many orders. Definitely, the use of the MATLAB function *fminsearch* is not the best solution.

The area of further investigation covers among others a proposition of an effective identification algorithm of orders. Here a biologically inspired approach appears to be promising.

Next, the considered model should be proposed also in the discrete version, ready for digital implementation.

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