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MEASUREMENT MODELS – THEORY AND PRACTICE OF UNCERTAINTY EVALUATION OF GEOMETRICAL DEVIATION MEASUREMENTS

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Abstract

The publication provides a critical analysis of fundamental documents concerning the determination of measurement uncertainty from the perspective of the machinery industry. The requirements contained in the documents JCGM 104, JCGM 100, and JCGM 101 were compared with important documents used in geometrical measurements, particularly with EA-4/02, ISO 14253-2, ISO/TS 15530-1, ISO 15530-3, ISO/TS 15530-4, and VDI/VDE 2617-11. Significant differences between the documents analysed, both terminological and interpretative, were highlighted. The analysis was performed in the sequence of stages for determining measurement uncertainty: formulation, propagation, and summarizing. Special attention was paid to the problem of defining the measurement model and the insufficient reference to the measurement model in the analysed documents. Attention was drawn to the wide range of characteristics measured in the machinery industry, such as linear and angular dimensions and form, orientation, position, and runout deviations, as well as the wide range of measurement equipment used, from simple instruments like callipers, micrometers, and mechanical dial gauges, to coordinate measuring machines and measurement systems. The current approach to the uncertainty of coordinate measurements, including the new possibility of modelling coordinate measurement, was discussed.

Keywords: uncertainty, law of propagation of uncertainty, methods of uncertainty propagation, sensitivity analysis, Monte Carlo method.

1. Introduction

A measurement result is generally expressed as a single measured quantity value and a measurement uncertainty. The measurement uncertainty is defined, among others, in [1] as a non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used. Measurement uncertainty may be presented as a standard deviation (called standard measurement uncertainty) or as a specified multiple of it (called expanded measurement uncertainty). Measurement uncertainty can be presented as a standard deviation (standard measurement uncertainty) or as a specified multiple of it (expanded measurement uncertainty).

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Providing measurement uncertainty is mandatory for measurements and calibrations performed by accredited laboratories, as required by ISO 17025 [2]. Increasingly, the provision of measurement uncertainty is also expected for measurements performed in industry (especially automotive), particularly for measurements used to assess the conformity with the requirements. This is directly derived from the provisions of ISO 14253-1 [3] and IATF 16949 [4].

Measurement in the mechanical engineering industry involves not only the manufactured parts but also the equipment, such as ring gauges [5], thread gauges [6], or taper gauges [7] (tapered parts are the gripping parts of cutting tools such as drills or reamers). The measured characteristics include linear and angular dimensions (see ISO 14405) [8–10] and geometrical deviations, such as form deviations (straightness, flatness, roundness, and cylindricity), orientation (parallelism, perpendicularity, and angularity), location (position, coaxiality, concentricity, and symmetry), and runout (see [11]).

Measurement equipment mainly consists of coordinate measuring machines and other coordinate measuring systems such as measuring arms or optical machines. In addition, simple measuring instruments such as callipers, micrometers, mechanical dial gauges, and dial test indicators are still used. Specialized instruments are also utilized, such as roundness testers, gear measurement instruments, and surface roughness measurement devices.

The above information indicates the extensive knowledge required by people dealing with measurement uncertainty in mechanical engineering. Therefore, the industry anticipates publications that contain examples facilitating the development of procedures/instructions for determining the uncertainty of various types of measurement. A fundamental element of such an instruction should be the measurement model.

Measurement uncertainty should be a component of every measurement result. However, it is difficult to imagine that, under industrial conditions, a full uncertainty analysis is conducted for every measurement.

In measurements of geometric quantities, the most crucial components of measurement uncertainty are often repeatability (especially in measurements performed under challenging conditions, such as difficult access to the measured surface or poor lighting), the thermal effects component, and the instrument-related component. The latter can be calculated on the basis of the formula for maximum permissible error (MPE), provided by the instrument manufacturer and verified during periodic checks/calibrations. Therefore, a single analysis should suffice to obtain a valid uncertainty value over a more extended period, or at least to obtain a calculation scheme (*e.g.* in the form of a spreadsheet) within which only changing data, such as the temperature at which the measurement was performed, need to be provided to obtain the uncertainty value.

The Guide to the expression of uncertainty in measurement (GUM) together with supplements organises largely the issues regarding measurement uncertainty determination. However, some ambiguity remains. This paper presents the evaluation of the current situation from the point of view of measurements conducted in mechanical engineering. In particular, requirements contained in JCGM 200 (ISO/IEC Guide 99, VIM) [1], JCGM 104 [12], JCGM 100 (GUM) [13], JCGM 101 [14], JCGM 102 [15] and JCGM GUM-6 [16] (theory) were compared with other documents applied in geometrical measurements, and in particular with EA-4/02M [17], ISO 14253-2 [18], ISO/TS 15530-1 [19], ISO 15530-3 [20], ISO/TS 15530-4 [21] and VDI/VDE 2617-11 [22] (practice). Attention was given, *inter alia*, to terminology, and, in particular, to terms such as: *GUM uncertainty framework* (GUF), law of propagation of uncertainty (LPU), type A and type B evaluations, three stages of uncertainty evaluation: formulation, propagation and summarising, explicit, implicit, extended, nested and multistage measurement models, propagation of distributions, three methods of propagation of uncertainty: analytical, GUF and Monte Carlo, application of the central limit theorem, model linearization (Taylor expansion), correlation of input quantities, and others.

In the cited passages of the quoted documents, symbols applied therein were kept and, in most cases, their meaning was not explained; it was assumed that these documents could be easily consulted. The purpose of this paper is to evaluate the consistency and completeness of existing standards and documents on measurement uncertainty, especially in relation to geometrical measurements. Particular attention is paid to the availability of the measurement model as it forms the basis for verifying the uncertainty budget.

The authors participate in ISO standardization work and in the development of a simple method to determine the uncertainty of coordinate measurements n [23]. They also strive to spread more interesting examples of uncertainty analyses performed for the mechanical industry.

2. Stages of measurement uncertainty evaluation

The main stages of uncertainty evaluation constitute formulation, propagation, and summation [14], Ch. 5.1, [12], Ch. 5.

The formulation includes:

- defining the output quantity Y, the quantity intended to be measured (the measurand),
- determining the input quantity $X_1, \ldots, X_N$ upon which Y depends,
- developing a model that relates Y and X,
- assigning (on the basis of available knowledge) *probability distribution functions* (PDFs) to particular X_i; assigning a joint PDFs to those X_i that are not independent.

Propagation consists in the determination of the output quantity distribution on the basis of the model and the input quantity PDFs.

Summarising: using the PDF for Y to obtain (depending on the need) a calculation (from the PDF of the quantity Y) of all or some of the following:

- the expectation of Y, taken as an estimate y of the quantity,
- the standard deviation of Y, taken as the standard uncertainty $u(y)$ associated with y,
- a coverage interval containing Y with a specified probability (the coverage probability).

In measurements of geometric quantities, the output quantity Y is the measured characteristic of the object, such as a dimension or geometric deviation. Most often, direct measurement involves an input quantity X_1, which is the one being measured (in the measurement of the diameter of the shaft with a micrometer, both the output and the input quantities are the diameter of the shaft, which means $X_1 = X_1$). However, if we want to take into account the fact that the measurement result is affected by elastic deformation caused by the micrometer's measuring force, a second input quantity X_2 appears – the correction related to the deformation. Since the correction should be added to the raw (uncorrected) measurement result, the relationship between Y and X_1 and X_2 (the measurement model) will take the form $Y = X_1 + X_2$.

The diameter of the shaft can be measured using a coordinate measurement technique, such as that with a coordinate measuring machine (CMM). From the user's perspective, this measurement can be treated similarly to a micrometer measurement (after completing the measurement program, the diameter value is obtained). However, upon closer inspection (analysing the part program), it can be observed that the measurement is, in fact, an indirect measurement: the coordinates of a certain number of points on the shaft's surface are "measured", and from the obtained values, the diameter is calculated. This means that the measurement involves a certain number of input quantities $X_1, \ldots, X_n$, and the measurement model takes the form of an unknown (non-linear) function: $Y = f(X_1, \ldots, X_n)$.

Let us imagine another case. The aim of the measurement is the diameter of a hole in a flat object. We have a measurement microscope that is not equipped with a computer. However, we decide to perform the measurement using the coordinate technique by "measuring" the coordinates of 3 points: x_A, y_A, x_B, y_B, x_C, and y_C (Fig. 1).

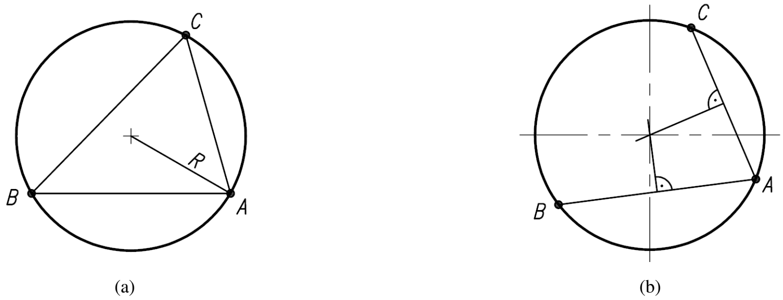

Fig. 1. Illustration of different models for coordinate measurement of a circle radius: a) radius calculated as the radius of a circle circumscribed on a triangle, b) centre of the circle determined as the intersection point of the perpendicular bisectors of the triangle's sides.

There are several possibilities for calculating the radius of a circle, which is equivalent to saying that there are several possible measurement models. Two of them are presented. One is to use the well-known mathematical formula for the radius of a circle circumscribed around a triangle using Heron's formula for the area of the triangle. The input quantities here are the lengths of the sides a, b, and c calculated from the measured differences in the coordinates of the pairs of points $B - C$, $A - C$, and $B - A$:

$$\begin{aligned} R &= \frac{abc}{2\sqrt{(a+b+c)\,(a+b-c)\,(a-b+c)\,(-a+b+c)}}, \\ a &= \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2}, \\ b &= \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2}, \\ c &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}. \end{aligned} \tag{1}$$

The other possibility is to determine the centre of the circle as the intersection point of its perpendicular bisectors.

$$\begin{aligned} R &= \sqrt{[0.5\,(x_B - x_A) + t\,(y_B - y_A)]^2 + [0.5\,(y_B - y_A) - t\,(x_B - x_A)]^2}, \\ t &= \frac{1}{2}\,\frac{(x_C - x_B)\,(x_C - x_A) + (y_C - y_B)\,(y_C - y_A)}{(y_B - y_A)\,(x_C - x_A) - (y_C - y_A)\,(x_B - x_A)}. \end{aligned} \tag{2}$$

It can be assumed that both measurement models involve six input quantities: x_A, y_A, x_B, y_B, x_C, and y_C (coordinates of points A, B, and C). However, it will turn out that instead of coordinates, it is better to assume certain differences of coordinates as input quantities (also six). In the first case, these are: $x_B - x_C$, $y_B - y_C$, $x_A - x_C$, $y_A - y_C$, $x_B - x_A$, and $y_B - y_A$, in the second: $x_B - x_A$, $y_B - y_A$,$x_C - x_B$, $y_C - y_B$, $x_C - x_A$, and $y_C - y_A$. This is because for length measuring devices, MPE applies to the length value regardless of the location of this length in the measuring range.

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In the above, different example measurement models of the same output quantity: the diameter or radius of the circle are presented. It is already evident that the development of the measurement model is an important and at the same time difficult element of the measurement uncertainty evaluation procedure.

3. General notation of the measurement model

A measurement model, as an essential element that enables the determination of the measurement uncertainty, is discussed in many documents. In the GUM, the term "*measurement model*" does not appear; instead, the term "*mathematical model*" [13], Ch. 3.1.6, Ch. 3.4.1 is used as a shorthand for "*mathematical measurement model*" [13], Ch. 3.4.2. The term "*measurement model*" (or its shortened form "*model*") is commonly used in JCGM 104 [14], Ch. 3.10, referring to its definition from VIM [1], Ch. 2.48: "*mathematical relation among all quantities known to be involved in a measurement*". It is worth mentioning that VIM contains many other terms related to measurement uncertainty.

The general formula for the measurement model has the following form [12], Ch. 3.16 $h\left(Y, X_1, \ldots, X_N\right) = 0$, and it can be most frequently presented in the form of a measurement function [12], Ch. 3.15 $Y = f\left(X_1, \ldots, X_N\right)$. In JCGM 101 [14] the same measurement models have vector notation [14], Ch. 4.1 $h\left(Y, \boldsymbol{X}\right) = 0$, $Y = f(\boldsymbol{X})$.

In JCGM 102 [15] a possibility of the output quantity being a vector quantity is considered, and then we talk about a multivariate measurement model, which has a general form [15], Ch. 3.8 $h\left(\boldsymbol{Y}, \boldsymbol{X}\right) = 0$ and, in the case where it can be presented as a measurement function, $\boldsymbol{Y} = f(\boldsymbol{X})$.

The basic application of this measurement model is related to complex quantities. In geometric measurements such approach can be used as regards geometrical elements which are defined by a particular number of parameters (vector). For example, to define a 2D circle, 3 parameters are needed: two coordinates of the centre and a radius.

The simplest form of the measurement model (function) is $Y = X$ where the output quantity is simply equal to the reading of a measuring instrument. Frequently, when it is possible to indicate additive sources of error, *e.g.* when it is possible (although not necessarily applied) to correct systematic errors, the measurement function takes the form $Y = X + \delta_1 + \ldots + \delta_n$. Then we talk about the extended measurement model [16], Ch. 10. In general, when the output quantity is a sum of input quantities $Y = X_1 + \ldots + X_N$, we talk about the additive model [14], Ch. 9.2. In practice, linear measurement functions act frequently as models, that is, functions of the form [12], Ch. 4.14 $Y = c_1 X_1 + \ldots + c_N X_N$.

Moreover, nonlinear models can be linearized [16], Annex F by, for example, expanding the function into a Taylor series and rejecting higher-order terms. In some cases, to obtain proper accuracy, it is necessary to leave some higher-order terms. This situation occurs, among others, in thread measurements.

4. Measurement model in other important documents

From the area of geometrical measurements, we find in JCGM 100 a measurement model of the gauge block length taking into account the effect of the thermal expansion phenomenon [13], Ch. H.1.2 in the form

$$l = \frac{l_s\left(1 + \alpha_s \theta_s\right) + d}{1 + \alpha\theta} = l_s + d + l_s\left(\alpha_s \theta_s - \alpha\theta\right) + \ldots, \tag{3}$$

where: d is the difference in the lengths of the gauge block being calibrated and the standard, l is the measurand, that is, the length at 20°C of the gauge block being calibrated, $l_S$is the length of the standard at 20°C as given in its calibration certificate, α and α_S are the coefficients of thermal expansion, respectively, of the gauge being calibrated and the standard, θ and θ_S are the deviations in temperature from the 20°C reference temperature, respectively, of the gauge block and the standard.

The first part of the above formula is an exact formula, and the second part is its linearization (a fragment of the Taylor series expansion). The formula clearly indicates the significant importance of temperature differences and differences in expansion coefficients (the two error components from thermal effects are more visible in it).

The other form of the model is the following:

$$l = f\left(l_s, d, \alpha_s, \theta, \delta\alpha, \delta\theta\right) = l_s + d - l_s\left(\delta\alpha \cdot \vartheta - \alpha_s \cdot \delta\theta\right), \tag{4}$$

where $\delta\alpha = \alpha - \alpha_s$ i $\delta\theta = \theta - \theta_s$.

Gauge blocks are among the most used length standards. Their measurements (calibration) are of interest to calibration laboratories and have been the subject of numerous publications *e.g.*, [24].

In document EA-4/02M [17] the measurement model is most often referred to as "*model function*" or "*relation*". We find there three examples of models from the area of gauges and geometrical measurement instrument calibration:

– calibration of a gauge block of nominal length of 50 mm [17], Ch. S4:

$$l_X = l_S + \delta l_D + \delta l + \delta l_C - L\left(\overline{\alpha} \cdot \delta t + \delta\alpha \cdot \Delta\bar{t}\right) - \delta l_V, \tag{5}$$

where: l_X is the length of the unknown gauge block, l_S is the length of the reference gauge block at the reference temperature $t_0 = 20$°C according to its calibration certificate, δl_D is the change of the length of the reference gauge block since its last calibration due to drift, δl is the observed difference in length between the unknown and the reference gauge block, δl_C is the correction for non-linearity and offset of the comparator, L is the nominal length of the gauge blocks considered; $\overline{\alpha}$ is the average of the thermal expansion coefficients of the unknown and reference gauge blocks, δt is the temperature difference between the unknown and reference gauge blocks, $\delta\alpha$ is the difference in the thermal expansion coefficients between the unknown and the reference gauge blocks, $\Delta\bar{t}$ is the deviation of the average temperature of the unknown and the reference gauge blocks from the reference temperature, δl_V is the correction for non-central contacting of the measuring faces of the unknown gauge block;

– calibration of a vernier calliper [17], Ch. S10:

$$E_X = l_{iX} - l_S + L_S \cdot \overline{\alpha} \cdot \Delta t + \delta l_{iX} + \delta l_M, \tag{6}$$

where: E_X is the error of indication l_{iX} is the indication of the calliper, l_S is the length of the actual gauge block, L_S is the nominal length of the actual gauge block, $\overline{\alpha}$ is the average thermal expansion coefficient of the calliper and the gauge block, Δt is the difference in temperature between the calliper and the gauge block, δl_{iX} is the correction due to the finite resolution of the calliper, δl_M is the correction due to mechanical effects, such as applied measurement force, Abbe errors, flatness and parallelism errors of the measurement faces;

– calibration of a ring gauge of nominal diameter of 90 mm [17], Ch. S13:

$$d_X = d_S + \Delta l + \delta l_i + \delta l_T + \delta l_P + \delta l_E + \delta l_A \tag{7}$$

where: d_X is the diameter of the ring, d_S is the diameter of the reference setting ring at the reference temperature, Δl is the observed difference in displacement of the measuring spindle when the contact tips touch the inner surface of the rings at two diametrically apart points, δl_i is the correction for the errors of indication of the comparator, δl_T is the correction due to the temperature effects of the ring to be calibrated, the reference setting ring and the comparator line scale, δl_P is the correction due to coaxial misalignment of the probes with respect to the measuring line, δl_E is the correction due to the difference in elastic deformations of the ring to be calibrated and the reference setting ring, δl_A is the correction due to the difference of the Abbe errors of the comparator when the diameters of the ring to be calibrated and the reference setting ring are measured.

Let us look at those models in more detail. Significant inconsistency is noticeable in the treatment of the temperature error. In the first example there are two components (types) of this error: the first one ($L \cdot \overline{\alpha} \cdot \delta t$) is derived from the difference in temperature of the workpiece and the instrument (δt), and the other one ($L \cdot \delta\alpha \cdot \Delta\bar{t}$) from the difference of expansion coefficients ($\delta\alpha$) and deviation from the reference temperature ($\Delta\bar{t}$). In the second example only the first of these two components is present ($L_S \cdot \overline{\alpha} \cdot \Delta t$), that is, the difference in temperature of the gauge block and the vernier calliper (Δt) (one can guess that the other component was omitted as insignificant). The third example differs from the previous ones in that two rings and a comparator participate in the measurement. In the model, the temperature error is jointly present as δl_T ("*correction due to the temperature effects of the ring to be calibrated, the reference setting ring and the comparator line scale*"), four components of this error were provided, and a separate analysis was used to determine the uncertainty component related to this error ("uncertainty *sub-budget*").

The term "measurement model" is not used in ISO 14253-2 [18]. Instead, the term "model of uncertainty estimation" is commonly used, within which a distinction is made between the black box method and the transparent box method. In the black box method of uncertainty estimation, the result of the measurement is the reading corrected by an eventually known correction (that is, the measurement model has the form):

$$Y = X + C. \tag{8}$$

In the transparent box method of uncertainty estimation, the value of the measurand is modelled as a function of several measured values X_i, which themselves could be functions (transparent box models) or black box models, or both: (that is, the measurement model has the form):

$$Y = G\left(X_1, X_2, \ldots, X_{p+r}\right). \tag{9}$$

As a side note: the $p + r$ index was applied to emphasize that among input quantities there are p uncorrelated quantities and r correlated quantities.

In the standard, there are the following four examples of uncertainty estimation for:

– calibration of a setting ring,
– measurement of local diameter with an external micrometer,
– calibration of error of indication of an external micrometer,
– measurement of roundness.

The second example mentioned, with some completions, can serve as a basis for developing a guideline to determine the uncertainty of direct measurements performed in industry.

The following formula for the measurement uncertainty was provided:

$$u_c = \sqrt{u_{ML}^2 + u_{MF}^2 + u_{MF}^2 + u_{MP}^2 + u_{RR}^2 + u_{NP}^2 + u_{TD}^2 + u_{TA}^2 + u_{WE}^2}. \tag{10}$$

The individual uncertainty components are: u_{ML} – micrometer – indication error, u_{MF} – micrometer – flatness of measuring anvils, u_{MP} – micrometer – parallelism of measuring anvils,

u_{RR} – resolution u_{RA} or repeatability u_{RE} (the largest of the two), u_{NP} – variation of the zero point between the operators, u_{TD} – temperature difference, u_{TA} – difference in thermal expansion coefficients and the deviations in temperature from the 20°C reference temperature, u_{WE} – workpiece form error.

The given formula for measurement uncertainty emphasizes the fact that individual components of uncertainty are added in a quadratic manner. This indicates the presence of an extended measurement model which can be reconstructed in the following manner:

$$l = l_M + \delta_{MF} + \delta_{MF} + \delta l_{MP} + \delta l_{RR} + \delta_{NP} + \delta l_{TD} + \delta l_{TA} + \delta l_{WE}. \tag{11}$$

It is worth paying attention to the fact that the uncertainty components are grouped consecutively according to their four sources: instrument (4 components), human (2 components), environment (2 components), and measured workpiece (1 component). The relevant uncertainty components come consecutively from the error of indication, the flatness of the measuring anvils (two identical), the parallelism of the measuring anvils, repeatability or resolution, the variation of the zero point, the temperature difference between the micrometer and the workpieces, the deviation from the standard reference temperature and error of form deviation of the workpiece. Two components (the source of which is a metrologist) were determined with the A type method, the remaining were determined with the B type method.

In the case of the B type method, the highest values that could be assumed by particular errors were determined.

The type A standard uncertainty is determined experimentally. For the type B method, the highest values that particular errors can assume are determined (denoted by a), and appropriate probability distribution is identified for this random variable. Each specific probability distribution is associated with a coefficient b which allows the conversion of value a to the value of standard uncertainty u [18], Ch. 8.3.2

$$u = a \cdot b. \tag{12}$$

Usually, one of the following distributions is chosen (the coefficient b value is given in parentheses): normal (usually 0.5), uniform (also called rectangular or even, 0.58), triangular (0.41), U-shaped (typically arcsine distribution, 0.71). The provided (approximate) values for coefficient b result from the relationship between the value of parameter a and the variance of the distribution.

For the components derived from the instrument (micrometer) information on *MPE* (maximum permissible error of indication), flatness and parallelism of anvils provided by the manufacturer was used. Two components derived from the environment (to be more precise, because of thermal influences) were considered. For the first (temperature difference between micrometer and workpieces), the extreme value a_{TD} was calculated based on earlier observations that this difference does not exceed 10 °C. For the other (following from the differences between linear coefficients of thermal expansion and the fact that the measurements are not conducted at a temperature of 20 °C), the extreme value a_{TA} was calculated based on earlier observations that the maximum deviation from the standard reference temperature is 15°C and on the assumption that linear coefficients of thermal expansion differ by a maximum of 10%. The extreme value for the component deriving from the workpiece was calculated based on the observation that cylindricity does not exceed 1.5 µm.

The standard contains a minor error; instead of $a_{TA} = 0.1 \times \Delta T_{20} \times \alpha \times D$ there should be $a_{TA} = \Delta T_{20} \times \Delta\alpha \times D$ (only in the example $\Delta\alpha = 0.1\alpha$)

Attention must be paid to the fact that in the current standard on micrometer requirements [25], MPEMF and MPEMP are no longer present, which results in the need for modification of the example in future revision of the standard.

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The technical specification ISO/TS 15530-1 [19] (*Technique for determining the uncertainty of measurement. Part 1: Overview and metrological characteristics*) mentions compliance with GUM and ISO 14253-2. With reference to the specificity of the coordinate measuring technique, attention was drawn to "*three general uncertainty categories*". The first is the instrumentation factors. These factors are typically the responsibility of the CMM manufacturer and are controlled by establishing permissible limits, *e.g.* temperature ranges, under which the CMM may be used. Some or all of these error sources may be assessed during the acceptance or reverification testing of the CMM. The second category is measurement plan factors that involve how the CMM user decides to perform the measurement. This includes the location and orientation of the workpiece, the probes and styli selected for the measurement, and the particular measurement point sampling strategy. In this category attention is paid to the fact that the quantity being measured shall be unambiguously specified (it relates, among others, to matching criteria: a least-squares, minimum-circumscribed, maximum-inscribed, or minimum-zone). The third category is extrinsic factors such as non-ideal workpiece geometry (surface roughness, form errors, finite stiffness and thermal distortions), contamination, workpiece fixturing problems and variations among operators.

Unfortunately, despite using the term "task specific uncertainty", no explicit attention is paid to the large diversity of geometrical characteristics of the measured workpieces (dimensions, angles, and deviations of form, orientation, location and runout) [11].

The document distinguishes three techniques for determining the uncertainty of coordinate measurements. The first, "*sensitivity analysis*", refers clearly to GUF. The second one, however, "*use of calibrated workpieces or measurement standards*", is treated as a technique not present in GUM, whereas it is fully compliant with GUF. The third technique, "*using simulation*", refers to uncertainty propagation using the Monte Carlo method present in GUM. As a side note, it is worth noting that the Monte Carlo method can also be used to determine uncertainty components. The simulation of the measurement of the circle diameter of the workpiece with the error of the three-lobed form with the application of sampling in six uniformly distributed points, described in [21], Annex F, can serve as an example. The obtained distribution of errors is not a normal distribution and does not even contain the true value. More information can be found in [26].

In ISO 15530-3 [20] (*Technique for determining the uncertainty of measurement. Part 3: Use of calibrated workpieces or measurement standards*) the term "*measurement model*" is not used. From the contents, and, in particular, from the provided formula for the measurement uncertainty calculation:

$$U = k \cdot \sqrt{u_{cal}^2 + u_p^2 + u_b^2 + u_w^2}, \tag{13}$$

where: u_{cal} is the standard uncertainty associated with the uncertainty of the calibration of the calibrated workpiece stated in the calibration certificate, u_p is the standard uncertainty associated with the measurement procedure as assessed below, u_b is the standard uncertainty associated with the systematic error of the measurement process evaluated using the calibrated workpiece, u_w is the standard uncertainty associated with material and manufacturing variations (due to the variation of the expansion coefficient, form errors, roughness, elasticity, and plasticity).

It can, however, be inferred that GUF was applied and that we are dealing with an extended model in the form:

$$Y = X + \delta_{cal} + b + \delta_w, \tag{14}$$

in which the output quantity is encumbered with three errors: (δ_{cal}, b and δ_w). From the provisions of the standard, it also follows that b shall be treated as a systematic (corrected) error with uncertainty u_b, whereas δ_{cal} and δ_w shall not be corrected. Standard deviation calculated from the results of 20 repetitions of the measurement is treated as a component of the measurement uncertainty u_p

determined with the type A evaluation (the standard provides a long list of factors having impact on this uncertainty component). As a side note, it is worth considering whether any other number of measurement repetitions and application of *t* distribution should not be provided for in the standard.

The component u_{cal} is determined with the B method. The standard states that the component u_w can be determined with the type A or type B evaluation. It also states that the component u_w shall be calculated as a geometric sum of two components u_{wt} and u_{wp}, but there is no convincing argument in favour of the wisdom of their application. The general conclusion is that there are significant inconsistencies compared to GUM.

From the title of the technical specification ISO/TS 15530-4 [21] (*Evaluating task-specific measurement uncertainty using simulation*) it can be incorrectly inferred that the document contains recommendations as regards the design of a relevant simulation model, while in reality we only find requirements to be met by simulation software as well as by any other software used to determine coordinate measurements uncertainty, generally referred to as ”*uncertainty evaluating software* (*UES*)”.

The only formula related to the measurement model is the following:

$$u = \sqrt{u_{sim}^2 + \sum u_i^2}, \tag{15}$$

stating that an uncertainty component determined with the simulation technique should be complemented by components obtained otherwise.

The title of the document VDI/VDE 2617:11 [22] (*Determination of the uncertainty of measurement for coordinate measuring machines using uncertainty budgets*) refers to GUF through the words “*uncertainty budget*”. The term “*mathematical model*” [22], Ch. 3.2.2 and other terms present in GUM, such as, for example, type A or type B evaluation, standard uncertainties or sensitivity coefficients can be found in the document. The document indeed describes in detail a method of estimating the uncertainty of coordinate measurements. Two measurement models are also provided. The first refers to the uncertainty of measuring the diameter of a bore and has the following form (symbols according to [22]):

$$D = (D_W - \Delta D_T + \Delta D_C) - \Delta L_{CMM} - \Delta L_t, \tag{16}$$

where: D_W is the diameter of the regression feature on the workpiece, ΔD_T is the error of stylus diameter during stylus qualification, ΔD_C is the error of the calibrated diameter of the reference standard, ΔL_{CMM} is the geometrical error of the CMM, ΔL_t is the length error due to the error of the expansion coefficient of the scale.

The other model refers to the measurement of the distance between a plane and a cylinder axis (symbols according to [22]):

$$\begin{aligned} L = &\left(X_E - W_E \frac{L_{SE}}{L_{ME}} + \Delta X_{TE} - \Delta R_{TE} + \frac{\Delta D_C}{2}\right) + \\ &- \left(X_B - W_B \frac{L_{SB}}{L_{MB}} + \Delta X_{TB} - \Delta R_{TB} + \frac{\Delta D_C}{2}\right) - (-\Delta X_{TR} - \Delta L_{CMM} - \Delta L_t), \end{aligned} \tag{17}$$

where: X_E is the coordinate of the toleranced feature at the centre of gravity, W_E is the angle error of the toleranced feature for the measured length, L_{ME}, ΔX_{TE} is the error of the centre coordinate of the stylus for the toleranced feature during qualification, ΔR_{TE} is the error of the radius of the stylus for the toleranced feature, X_B is the coordinate of the reference feature at the centre of gravity, W_B is the angle error of the reference feature for the measured length, L_{MB}, ΔX_{TB} is the error of the centre coordinate of the stylus for the reference feature during qualification, ΔR_{TB} is the error of the radius of the stylus for the reference feature, ΔD_C is the error of the calibrated diameter of the reference standard, ΔX_{TR} is the error of the centre coordinates of the styli due to rotation errors, ΔL_{CMM} is the geometrical error of the CMM.

For the mentioned models, example uncertainty budgets are also provided. The problem is that this method has not been widely recognised because of complex theoretical principles.

5. Multistage measurement models

Here, it may be worth referring to the term "*multistage model*"" that occurs in JCGM 104 [12]. In the multistage model, output quantities from previous stages become input quantities to subsequent stages. In the first example, two input quantities are measured with the use of various measuring systems of the same instrument, and it cannot be ruled out that each of them is measured with a different uncertainty. In the second example, two input quantities are measured with various measuring instruments. In the third example, we are dealing with two types of quantity: the output quantity is an angle, and the input quantities are lengths. In the fourth example, in the first step (stage), measurement uncertainties for 3 lengths a, b, and c need to be determined. It is worth mentioning here that the information concerning the accuracy of the measurement of these lengths is contained in the formula for maximum permissible error for length measurement $E_{L,MPE}$ provided by the manufacturer [27], Ch. 3.6. Additional measurement models may be needed to evaluate the measurement uncertainty of the input quantities. In the last two examples, some functions of the directly measured quantities were assumed to be input quantities. In all cases, it is possible (or necessary) to use two-stage models. One of the two lower-stage models needed for the measurement shown in Fig. 1b measurement may be a model similar to the model of measuring the shaft diameter with a micrometer from ISO 14253-2 [18].

Two-stage models can also be all previously provided models in which "corrections" were used, and there was no information on whether they are actually applied – if they are, then the first-stage model shall be needed to evaluate their uncertainty.

It should be noted that the first two models as referred to in EA-4/02M [17], formulae 8 and 9 would be more elegant if the components related to the temperature error were relocated to the lower stage, that is, if the highest stage models assumed the following form (see (6)):

$$l_X = l_S + \delta l_D + \delta l + \delta l_C + \delta l_T - \delta l_V, \tag{18}$$

$$E_X = l_{iX} - l_S + \delta l_T + \delta l_{iX} + \delta l_M. \tag{19}$$

6. Measurement models in the coordinate measuring technique

Currently, the basic technique used for geometrical measurements commonly applied in the industry is the coordinate measurement technique. This name means measurements with such measuring instruments as coordinate measuring machines, coordinate measuring arms, *Computed Tomography* (CT) scanners, laser-trackers, and others [28].

In ISO/TS 15530-4 [21] there is a statement: "... *in the case of a CMM, the formulation of a classical uncertainty budget is impractical for the majority measurement tasks due to the complexity of the measuring process*". This probably concerns less an uncertainty budget, since in an extreme case it can contain only one or two components including jointly a significant number of influencing factors, than the fact that it is impossible to demonstrate how individual factors, known in particular to CMM manufactures, influence measurement uncertainty. This concerns especially such factors as individual geometrical errors (and usually 21 are listed) or measuring head errors. The interest in the influence of these errors is related to previous research that aims to rectify their mathematical errors mastered by manufacturers.

A little further down there is a statement that one of the alternative methods (for an uncertainty budget?) uses UES available on the market "... *based on a computer-aided mathematical model of the measuring process. In this model, the measurement process is represented from the measurand to the measurement result, taking important influence quantities into account"*. The best-known measurement model is the measurement model of single point coordinates. This model follows from the CMM design (geometry, kinematics) and assumes the existence (in the simplest case) of 21 geometrical errors, of which only 3 (perpendicularity errors) are single random variables and the remaining 18 are functions of the readings of the three measuring systems of the CMM. In the most general version, the model has the form (symbols according to [29]):

$$\boldsymbol{x}* = \boldsymbol{x}_x(x,\boldsymbol{b}) + \boldsymbol{R}_x(x,\boldsymbol{b}) – \boldsymbol{x}_y(y,\boldsymbol{b}) + \boldsymbol{R}_y(y,\boldsymbol{b})[\boldsymbol{x}_z(z,\boldsymbol{b}) + \boldsymbol{R}_z(z,\boldsymbol{b})]^{\prime\prime} + \varepsilon, \tag{20}$$

where $\boldsymbol{x}_x$, $\boldsymbol{x}_y$ and $\boldsymbol{x}_z$ mean matrices of translation errors, and $\boldsymbol{R}_x$, $\boldsymbol{R}_y$, and $\boldsymbol{R}_z$ matrices of rotation errors (nesting results from the CMM design; it was assumed that the unit connected with the axis z moves together with the unit connected with the axis x and both move along the axis y).

In another publication [30], for a specific CMM solution (Fig. 2) an analogous model (this time instead of the vector of the point coordinates encumbered with a measurement error, the output quantity is vector $\boldsymbol{E}$, that is, the point measurement error) is presented in a simplified form:

$$\boldsymbol{E} = \boldsymbol{P} + \boldsymbol{A} \cdot \boldsymbol{X} + \boldsymbol{A}_P \cdot \boldsymbol{X}_P \tag{21}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & -ywz - xrz & zwx + xry + yry \\ 0 & 0 & -zwy - xrx - yrx \\ 0 & xrx & 0 \end{bmatrix}, \tag{22}$$

$$\boldsymbol{A}_P = \begin{bmatrix} 0 & -xrz - yrz - zrz & xry + yry + zry \\ xrz + yrz + zrz & 0 & -xrx - yrc - zrx \\ -xry - yry - zry & xrx + yrx + zrx & 0 \end{bmatrix}, \tag{23}$$

$$\boldsymbol{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \boldsymbol{X}_P = \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} xtx + ytx + ztx \\ yty + xty + zty \\ ztz + ztx + zty \end{bmatrix}. \tag{24}$$

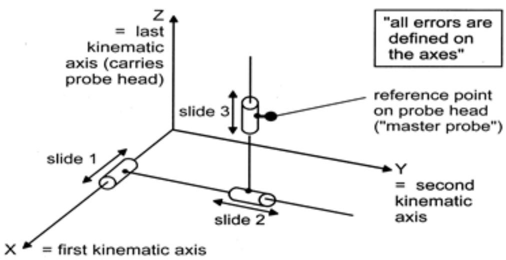

Fig. 2. CMM kinematic diagram (source: [30], original description).

The above models constitute a basis (they are first stage models) for evaluation of individual characteristics measurement uncertainty (dimensions, geometrical deviations) measured with the coordinate technique. In the second stage, geometrical features (*e.g.* planes, cylinders) are matched to the gathered points, that is, parameters of these features (a plane – 6 parameters: a point and

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a standard unit vector, a cylinder – 7 parameters: a point, a standard unit vector of an axis and a diameter or a radius) are the output quantities. Assuming that the output quantities from the first stage model are n coordinates of points x_i, y_i, z_i, a general second stage model for a plane has the form:

$$\begin{bmatrix} x_P \\ y_P \\ x_P \\ u \\ v \\ w \end{bmatrix} = f\left(x_i, y_i, z_i\right), i = 1, \ldots, n. \tag{25}$$

In the case of dimensions and deviations of form in the second and in other cases in the third stage, relevant characteristics are calculated. Here, the output quantity is a scalar (value of the dimension or the geometrical deviation).

For example, the value of the flatness deviation *flt* is calculated from the same input quantities as the plane parameters:

$$flt = f\left(x_i, y_i, z_i\right), i = 1, \ldots, n, \tag{26}$$

but one of a few options to calculate the value of the parallelism of two planes *prl* is to use the parameters u, v, w of both planes and also the size of the one which is a tolerated element (information contained in the coordinates of the points can be used to approximately evaluate the size r of the feature):

$$prl = f\left(u_1, v_1, w_1, u_2, v_2, w_2, r\right). \tag{27}$$

Information on the geometrical errors of CMM (data for the first stage model) can be obtained on the basis of a rather labour-intensive experiment. Only the producers of CMM software have full knowledge concerning the applied second and third stage models. Knowledge of these models shall not be necessary if the Monte Carlo method is used at the propagation stage and there is a possibility to use CMM software.

In several European metrological institutes, simulation software developed at the Physikalisch-Technische Bundesanstalt (PTB) known as VCMM is used. Information on the probability distributions of errors occurring in the machine model is obtained based on several days of CMM research under good environmental conditions of the given laboratory. This information must be periodically updated, which limits the area of application to calibration laboratories. Software for use in industry must allow the simulation of measurements under variable environmental conditions *e.g.*, [31].

In ISO/TS 15530-1 [19], Ch. 6.2, one can find the following statement: "*Since CMMs are complex measuring instruments, directly implementing this technique may only be possible for a limited number of measuring tasks*". That statement is now outdated as in many publications, and, in particular in [32, 33], the opposite has been shown. With appropriate assumptions, the measurement model (of a not too complex form) can include the essence of the coordinate measurements. The sensitivity coefficients present in the uncertainty budgets obtained on the basis of these models allow for an unambiguous indication of the weight of individual components.

The mentioned method/technique mentioned above (called "sensitivity analysis" in [32, 33]) allows determining the uncertainty for all geometrical characteristics, both for linear and angular dimensions, as well as for all geometrical deviations (form, orientation, location and runout) [11]. These models are based on known formulae [34], Table B.7 (symbols according to [34]):

– point – straight line distance

$$d\left(PT_1, SL_2\right) = \left|\left(\boldsymbol{A}_2 - \boldsymbol{PT}_1\right) \times \boldsymbol{u}_2\right| \tag{28}$$

– point – plane distance

$$d\,(PT_1, PL_2) = |(\boldsymbol{A}_2 - \boldsymbol{PT}_1) \cdot \boldsymbol{u}_2| \tag{29}$$

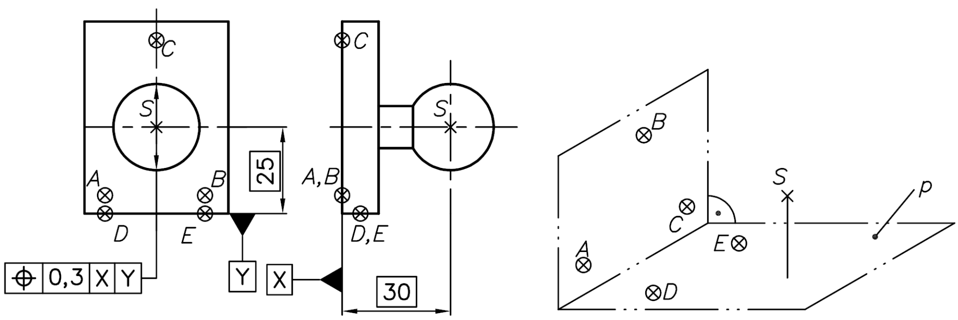
– point – point distance

$$d\,(PT_1, PT_2) = |\boldsymbol{PT}_1 - \boldsymbol{PT}_2| \tag{30}$$

– straight line – straight line distance

$$d\,(SL_1, SL_2) = \left|(\boldsymbol{A}_2 - \boldsymbol{A}_1) \cdot \frac{(\boldsymbol{u}_1 \times \boldsymbol{u}_1)}{|\boldsymbol{u}_1 \times \boldsymbol{u}_1|}\right| \tag{31}$$

As an example, a measurement model of the position of a point (centre of the sphere) relative to the secondary datum in the form of two perpendicular planes was provided (Fig. 3).



(a) (b)

Fig. 3. Measurement model of the position of a point from the secondary datum in the form of two perpendicular planes: a) a technical drawing, b) characteristic points for the measurement model.

To define a measurement model, a mathematically minimal number of points necessary to calculate the distance of the point *S* (centre of the sphere) from the plane *p* constituting the secondary datum is selected. The position is equal to the doubled value of the distance difference (observed and theoretically exact dimension, in the example it is equal to 25 mm) and it can be expressed as a function of differences of coordinates of 4 pairs of points (*AB*, *AC*, *DE* and *DS*), that is, a function of 12 input quantities:

$$l\,(\boldsymbol{AB}, \boldsymbol{AC}, \boldsymbol{DE}, \boldsymbol{DS}) = 2\left(\boldsymbol{DS} \cdot \frac{(\boldsymbol{AB} \times \boldsymbol{AC}) \times \boldsymbol{DE}}{|(\boldsymbol{AB} \times \boldsymbol{AC}) \times \boldsymbol{DE}|} - 25\right). \tag{32}$$

So far, more than 20 measurement models have been developed and programmed (Python, off-line version), enabling the evaluation of measurement uncertainty for most geometric measurements. Currently, research is underway on the online version.

7. Assignment of probability distributions

From the provisions of JCGM 104 [12] and JCGM 101 [14] it follows that probability distributions which are relevant for them must be assigned to all input quantities. It is not an accurate wording, since in some cases knowledge of standard deviation is enough. It is very often assumed that the input quantities have a normal distribution, less frequently t-distribution. It is often assumed that input quantities have a symmetrical distribution (in the case of error distribution – symmetrical in relation to zero). There are, however, cases where it is known, for example, that a given quantity or a measurement error of this quantity assumes only non-negative values. Examples of such quantities in mechanical engineering are all geometrical deviations (deviations of form, orientation, location, and runout). Identification of the form of probability distribution and/or estimation of distribution parameters of any random variable can be performed based on research/statistical analyses or based on other information. Unfortunately, practically no documents on uncertainty address this subject. GUM allows "*using available knowledge"* [13], Ch. 3.3.5, "*for insight based on experience and general knowledge*" [13], Ch. 4.3.2, "*an a priori distribution*" [13], Ch. 4.1.6 or "*available information*" [14], Table 1 to assume that a given input quantity has a specific distribution (*e.g.* uniform, triangular, normal, arcsine (U)). It also allows for an approximate assumption of extreme values which a given random variable can assume [18], Table B.1). In the document JCGM 101 [14] the following probability distributions (except for those mentioned previously) are described in some detail (random number generators included): trapezoidal, curvilinear trapezoidal, exponential, gamma, and multivariate normal (Gaussian) distribution. Sometimes a histogram, also known as frequency distribution, is enough to describe a random variable [14]. The problem of assigning probability distributions to input quantities is related to the terms "type A uncertainty" and "type B uncertainty": "*Type A evaluations of standard uncertainty components are founded on frequency distributions, while Type B evaluations are founded on a priori distributions*" used in GUM. It is worth mentioning here as a side note that type A and type B evaluations are methods of determining uncertainty components and not generally understood methods of uncertainty determination. Both types of evaluation are based on probability distributions, and the uncertainty components resulting from either type are quantified by variances or standard deviations [13], Ch. 3.3.4.

8. Propagation of distributions

Distribution propagation is the method used to determine the probability distribution for an output quantity from the probability distributions assigned to the input quantities on which the output quantity depends [14], Ch. 3.17. In a general case, propagation can be implemented with three methods: analytically, using the central limit theorem (in accordance with GUF) or with the Monte Carlo method.

9. Analytical propagation

We deal with the analytical approach to uncertainty propagation only in extremely rare cases. This approach is based on the theorem that the distribution of the sum of two random variables is a convolution of these distributions (if f and g are the probability density function (PDF) of the independent random variables X and Y, then $f * g$ is the PDF of the random variable $X + Y$) The formula for the convolution of functions f and g has the following form:

$$h(x) = f * g = \int_{-\infty}^{\infty} f\,(x - t)\, g(t)\mathrm{d}t. \tag{33}$$

Practically, the only examples of application of measurement uncertainty in the analysis are as follows:

- determination of the distribution of the sum (also the difference) of any number of random variables with normal distributions; the sum (the difference) has a normal distribution of the expected value equal to the sum (the difference) of the expected values and a standard deviation equal to the geometric sum of the standard deviations,
- determination of the distribution of the sum of two random variables with uniform distributions; the sum of two random variables with uniform distributions has a trapezoidal distribution (see, *e.g.* [17], Ch. S10), and in a specific case (the sum of two random variables with the same uniform distributions) has a triangular distribution.

10. GUM uncertainty framework. Law of propagation of uncertainty

This is the case when the conditions of the central limit theorem are met: a sum of a large number of random variables (regardless of their distributions) has a normal distribution with expectation equal to the sum of the expectations, and a standard deviation equal to the geometric sum of the standard deviations. The requirement of a "large" number of elements can be alleviated to the requirement that at least 2 or 3 largest elements have standard deviations of a similar order of magnitude.

It should be noted explicitly that this is about the sum of random variables. In the case of linear models, the conditions of the central limit theorem need to be referred to products of function coefficients (sensitivity coefficients) and standard deviations. In the case of nonlinear functions their earlier linearization is necessary, and the conditions of the central limit theorem remain the same as for a linear function. Expansion of the function into a Taylor series is most frequently used for linearization, which is frequently not even mentioned – values of the partial derivatives of the measurement function are simply assumed as sensitivity coefficients:

$$u_c^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \cdot u_c\,(x_i) \right)^2 . \tag{34}$$

The partial derivatives present in the formula are called sensitivity coefficients and are marked as c_i. The described method of conduct (multiplication of the uncertainties of the input quantities by relevant sensitivity coefficients and calculation of the uncertainty of the output quantity as a geometric sum of these products) is GUF. JCGM 101 [14] defines "GUF" as "*application of the law of propagation of uncertainty and the characterization of the output quantity by a Gaussian distribution or a scaled and shifted t-distribution in order to provide a coverage interval*".

The term "*law of propagation of uncertainty*" is described in GUM [13], Ch. 5.1, 5.2. The simplest form of the formula for the propagation of uncertainty is (34) [13], formula (10), which applies to uncorrelated input quantities. Yet, in the case of significant non-linearity of the measurement function, a second component should be added in the form of: [13], Ch. 5.1.2

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left[\frac{1}{2} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 + \frac{\partial f}{\partial x_i} \frac{\partial^3 f}{\partial x_i \partial x_j^2} \right] u^2\,(x_i)\, u^2\,(x_j) . \tag{35}$$

For the case of correlated input quantities, the formula for the propagation of uncertainty has the form [13], formula (16):

$$u_c^2\,(y) = \sum_{i=1}^{N} c_i^2 u^2\,(x_i) + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j u\,(x_i)\, u\,(x_j)\, r\,(x_i, x_j) . \tag{36}$$

From this formula it follows that in the case of negative correlation the value of the uncertainty calculated according to the formula can be lower than that calculated according to formula (34). In an extreme case, when all input quantities are correlated and the correlation coefficients are equal to +1, the above formula is simplified to the following form (instead of the geometric sum present in formula (34), there is an algebraic sum):

$$u_c^2\left(y\right)=\left(\sum_{i=1}^{N}c_i u\left(x_i\right)\right)^2=\left(\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}u\left(x_i\right)\right)^2. \tag{37}$$

In ISO 14253-2, there is the following formula for uncertainty [17], combined formulae (18) and (19):

$$u_c=\sqrt{\left(\sum_{i=1}^{r}u_i\right)^2+\sum_{i=1}^{p}u_i^2}, \tag{38}$$

which means that some component uncertainties are summed arithmetically (there are r of them, and they are correlated components), and others (uncorrelated, there are p of them) geometrically. This is due to the fact that the standard suggests (for reasons of simplification) using only three values of the correlation coefficient: 0, +1 or -1 [18], Ch. 5, although lack of correlation was assumed in all the examples provided in this standard.

According to ISO 14253-2 [18] for transparent box models, the measurement uncertainty is calculated according to the formula:

$$u_c=\sqrt{\left(\sum_{i=1}^{r}\frac{\partial Y}{\partial X_i}u_{Xi}\right)^2+\sum_{i=1}^{p}\left(\frac{\partial Y}{\partial X_i}u_{Xi}\right)^2}. \tag{39}$$

Compared to the formula for the "black box", partial derivatives were added, which were previously equal to one.

11. Summary

The main objective of the stage called summarising is calculation of expanded uncertainty in order to obtain the possibility to record the measurement result in the form of $y \pm U$. If the measurement result is used as an input quantity for another measurement task, knowledge of standard uncertainty u or expansion coefficient k (in calibration certificates these are U and k) is also needed. The expansion probability p, which most frequently amounts to 0.95, is important additional information.

The most frequently applied approach to propagation and summarising is GUF or, in other words, the law of uncertainty propagation. Then the whole analysis is most often presented in the form of an uncertainty budget. The uncertainty budget contains names and values of all uncertainty components, as well as other information, such as sensitivity coefficients or the name of the applied method of determination of particular components (type A or type B evaluation). As a side note: a clear distinction should be made between the Monte Carlo method used for distribution propagation and simulation (perhaps also with the Monte Carlo method) used to determine the probability distribution of the uncertainty component (determination of distribution of roundness deviation of the measured workpiece described in [26] is a good example). Examples of uncertainty budgets can be found in, *e.g.* [17, 18], but there are no examples in GUM [13].

In the case of using the Monte Carlo method, the direct result of propagation is constituted by the empirical distribution of the output quantity Y, from which all necessary information can be obtained. The entire document JCGM 101 [14] is devoted to the issue of uncertainty propagation using the Monte Carlo method. However, it should be noted that it lacks information on methods for identifying probability distributions and estimating their parameters. Without going into further details: the type of probability distribution is most frequently recognised visually on the basis of a histogram, and distribution parameters are calculated with the method of moments or the method of the maximum likelihood estimation. However, a considerable number of observations is necessary to plot a histogram. Another possibility is to use probability nets available for normal distributions, but also for other distributions, *e.g.* for the Weibull distribution. Appropriate tools are available in software containing statistical functions (Minitab, Python). The term "maximum likelihood" related to one of the methods to estimate the parameters of the probability distributions appears in [35]. An example of the development of the results of analytical propagation of distributions can be found in [17], Ch. S10.

The above statements relate to the most frequently occurring cases when the coverage interval is symmetrical. There are numerous situations for which the coverage interval is not symmetrical to the measurement result, however, the standards lack appropriate examples.

From the perspective of uncertainty propagation, the Monte Carlo method is a universal approach. In cases where we encounter asymmetrical distributions, the Monte Carlo method becomes the only feasible solution. The publication [14], which describes random number generators for various probability distributions, enables the resolution of more complex tasks related to the determination of measurement uncertainty.

12. Conclusions

In the machinery industry, it is required to specify and document the method of determining the uncertainty of all measurements that affect the decision on whether a product is compliant with the requirements or should be rejected as non-compliant. To achieve this, it is necessary to systematize and standardize documents of the status of international standards. The GUM guide, along with its supplements, largely organizes the subject of determining measurement uncertainty. From the perspective of industry professionals, documents that facilitate the creation of procedures for determining uncertainty for a wide range of measurement tasks and equipment are needed.

In coordinate measurements, a troublesome issue is the correlation between the input quantities, which are the coordinates of probed points resulting from the sampling of a large cloud of points. This problem can be solved by building measurement models based on the minimal mathematically required number of points. Such models also allow for consideration of specifics of individual geometric characteristics of the measured objects. The analyses performed on the developed models indicate a considerable range of measurement uncertainty for different characteristics measured by the same CMM. In the briefly described new method for estimating the uncertainty of coordinate measurements, conclusions from the analysis were used.

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