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L^1-impulses method as an alternative method of harmonic components in the power theory of discrete time systems

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Abstract. The article presents the basic mathematical theory of the operational calculus of the $\boldsymbol{L}^1$-impulses in the discrete time domain. It presents the isomorphism between the rational function set of complex variable and the exponential $\boldsymbol{L}^1$ impulses set of positive and negative time domain. The paper shows how for any factorization of the rational function consisting of casual and noncasual parts can be directly obtained the N – periodic version of the original signal using for the individual components of the $\boldsymbol{L}^1$ impulses N – copy formula. It is done by the distribution of the convolution – the type admitance operator Y of electrical circuit to the two commutative convolution operators and on this basis is obtained the distribution of electrical circuit current to two components: the active current and the reactive current in the discrete time domain using the cyclic convolutions. The distribution of current in the time domain for signals significantly different from the sinusoidal is much more favorable than the distribution in the frequency domain.

Key words: $\boldsymbol{L}^1$ impulses, time-discrete $\boldsymbol{L}^1$ impulses operational method, time domain.

1. $\boldsymbol{L}^1$-impulses and operators

The time-continous $\boldsymbol{L}^1$-impulse is an absolutely summable signal (an element of $\boldsymbol{L}^1$ space)

$$x\colon \int_{-\infty}^{+\infty} |x(t)|\, dt < \infty \qquad t \in \boldsymbol{R},$$

and the time-discrete $\boldsymbol{L}^1$-impulse is a signal $\{x_n\}$, such that

$$\sum_{n=-\infty}^{\infty} |x_n| < \infty, \tag{1}$$

where $n \in \boldsymbol{Z}$ (Integers).

The inner product of the $\boldsymbol{L}^1$-impulses is defined as follows

$$(x,y) = \sum_{m=-\infty}^{\infty} h_{n-m}x_m, \qquad m \in \boldsymbol{Z} \tag{2}$$

and the convolution operator is defined as follows,

$$[h * n]_n = \sum_{m=-\infty}^{\infty} h_{n-m}x_m, \qquad m \in \boldsymbol{Z}, \tag{3}$$

maps the $\boldsymbol{L}^\infty$ space in itself (is the stable operator), only if $h \in \boldsymbol{L}^1$ ($\boldsymbol{L}^\infty$-space of bounded signals).

The sequence of two stable convolution operators acts as $\boldsymbol{L}^\infty$ into $\boldsymbol{L}^\infty$ mapping, which means at the same time that $h * g \in \boldsymbol{L}^1$ when $h \in \boldsymbol{L}^1$, $g \in \boldsymbol{L}^1$.

Thus the convolution of the $\boldsymbol{L}^1$-impulses also produces the $\boldsymbol{L}^1$-impulse.

The H^* operator is the adjoint operator for the linear operator H which meets the condition

$$(Hx, y) = (x, H^*y) \tag{4}$$

for any x, y belonging to the $\boldsymbol{L}^1$-impulses.

For the convolution operator (3) the adjoint ratio for its characterizing function (the impulse function) has the form [1]

$$h_n^* = h_{-n} \tag{5}$$

N – periodical extensions of the $\boldsymbol{L}^1$-impulse x is called N – periodic signal:

$$\widetilde{x}_n = \sum_{n=-\infty}^{\infty} x_{n+pN}, \tag{6}$$

where $N \in \{0,1,2,3,\ldots\} = Z_+$, $p \in Z$.

For purposes of the signal theory, the concept of one-sided $\boldsymbol{L}^1$-impulses is introduced. The right-hand $\boldsymbol{L}^1$-impulse (casual) is called the signal which meets the condition:

$$\widetilde{\boldsymbol{L}}^1_{\boldsymbol{N}} = \left\{ \widetilde{x}\colon x_n = \sum_{p=-\infty}^{\infty} x_{n+pN}, \quad x \in \boldsymbol{L}^1 \right\}$$

and the left-hand $\boldsymbol{L}^1$-impulse (noncasual) is called the signal which meets the condition:

$$x_n = 0 \qquad \text{for} \qquad n > 0.$$

The special case are the functions

$$\mathbf{1}_n = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad n \geq 0 \end{cases}$$

and

$$\mathbf{1}_{-n} = \begin{cases} 0 & \text{for} \quad n > 0 \\ 1 & \text{for} \quad n \leq 0 \end{cases}.$$

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2. The time-discrete L^1 impulses operational calculus. The $\widetilde{L}_N^1$ space

The very important issue is discussed in this section – the isomorphism between the rational function set of complex variable and the exponential $\boldsymbol{L}^1$ impulse set of positive and negative time domain, and their N – periodical extensions sets.

The following relation defines the isomorphism between right-hand (casual) exponential $\boldsymbol{L}^1$ impulse and the partial fraction of complex variable

$$\frac{1}{1-\sigma^{-1}z} \longleftrightarrow \sigma^{-n}\mathbf{1}_n \quad \text{for} \quad |\sigma|>1. \tag{7}$$

The relation (7) describes a simple causal recursive digital filter, whose impulse response $\{h_n\} \in \boldsymbol{L}^1$ meets the recursive equation:

$$\boldsymbol{h}_n = \begin{cases} 0 & \text{for} \quad n<0 \\ \delta_n + \sigma^{-1}h_{n-1} & \text{for} \quad n \geq 0 \end{cases} \tag{8}$$

where

$$\boldsymbol{\delta}_n = \begin{cases} 0 & \text{for} \quad n \neq 0 \\ 1 & \text{for} \quad n=0 \end{cases} \quad \text{– the Kronecker delta.}$$

For the partial fraction with the pole σ: $|\sigma|<1$ the isomorphism is given as follows

$$\frac{1}{1-\sigma z^{-1}} \longleftrightarrow \sigma^{-n}\mathbf{1}_{-n} \quad \text{for} \quad |\sigma|<1. \tag{9}$$

The relation (9) describes a simple noncausal recursive digital filter, whose impulse response $\{h_n\} \in \boldsymbol{L}^1$ meets the recursive equation:

$$\mathbf{h}_n = \begin{cases} \delta_n + \sigma h_{n+1} & \text{for} \quad n \leq 0 \\ 0 & \text{for} \quad n>0 \end{cases} \tag{10}$$

In the relations (7) and (9) the complex variable z represents the unit delay operator, i.e.

$$(z\,x)_n \doteqdot x_{n-1}.$$

For any factorization of a rational function, its original consisting of casual and noncasual parts is given as follows

$$\begin{aligned} H(z) &= \sum_{|\sigma|>1} \frac{a(\sigma)}{1-\sigma^{-1}\,z} - \sum_{|\sigma|<1} \frac{a(\sigma)}{1-\sigma\,z^{-1}} \leftrightarrow \\ h_n &= \sum_{|\sigma|>1} a(\sigma)\,\sigma^{-n}\mathbf{1}_n - \sum_{|\sigma|<1} a(\sigma)\,\sigma^{-n}\mathbf{1}_{-n}, \end{aligned} \tag{11}$$

where the residuals coefficients are given by the formula:

$$\begin{aligned} a(\sigma) &= \lfloor H(z)(1-\sigma^{-1}z) \rfloor_{z\to\sigma} \\ &= -\left[H)(z)(1-\sigma z^{-1})\right]_{z\to\sigma} \\ &= -\frac{L(\sigma)}{\sigma\left[\dfrac{dM}{dz}\right]_{z\to\sigma}}, \end{aligned} \tag{12}$$

where: $L(z)$, $M(z)$ – the polynomials of numerator and denominator rational function $H(z)$.

The factorization (11) can directly obtain the N – periodic version of the original signal using N – copy formula for the individual components of the $\boldsymbol{L}^1$ impulse

$$\begin{aligned} \frac{1}{1-\sigma^{-1}\,z} &\to \sigma^{-n}\mathbf{1}_n \to \sum_{p=-\infty}^{\infty} \sigma^{-(n+pN)}\mathbf{1}_{n+pN} \\ &= \sigma^{-n}\sum_{p=0}^{\infty}\left(\sigma^{-N}\right)^p = \frac{\sigma^{-n}}{1-\sigma^{-N}} \\ &\text{for} \quad n \in \{0,\ 1,\ 2,\ ...,\ N-1\}, \quad |\sigma|>1 \end{aligned} \tag{13}$$

and

$$\begin{aligned} \frac{1}{1-\sigma\,z^{-1}} &\to \sigma^{-n}\mathbf{1}_{-n} \to \sum_{p=-\infty}^{\infty} \sigma^{-(n+pN)}\mathbf{1}_{-n-pN} \\ &= \sigma^{-n}\sum_{p=1}^{\infty}\left(\sigma^{N}\right)^p = \frac{\sigma^{N-n}}{1-\sigma^{N}} \\ &\text{for} \quad n \in \{1,\ 2,\ ...,\ N-1\}, \quad |\sigma|<1 \\ \frac{1}{1-\sigma\,z^{-1}} &\to \sum_{p=0}^{\infty}\left(\sigma^{N}\right)^p = \frac{1}{1-\sigma^{N}} \\ &\text{for} \quad n=0, \quad |\sigma|<1 \end{aligned} \tag{14}$$

Thus N – periodic time – discrete original of the Z – transform of factorization (11) takes the following form:

$$\begin{aligned} \widetilde{h}_o &= \sum_{|\sigma|>1} a(\sigma)\frac{1}{1-\sigma^{-N}} - \sum_{|\sigma|<1} a(\sigma)\frac{1}{1-\sigma^{N}}, \\ \widetilde{h}_n &= \sum_{|\sigma|>1} a(\sigma)\frac{\sigma^{-n}}{1-\sigma^{-N}} - \sum_{|\sigma|<1} a(\sigma)\frac{\sigma^{N-n}}{1-\sigma^{N}} \\ &\text{for} \quad n \in \{1,\ 2,\ ...,\ N-1\}. \end{aligned} \tag{15}$$

Following the same pattern as when deriving the relation (13) and (14), a more general forms can be obtained

$$\frac{z^M}{1-\sigma^{-1}z} \xrightarrow{\boldsymbol{L}^1} \sigma^{-(n-M)}\mathbf{1}_{n-m} \xrightarrow{\widetilde{L}_N^1}$$

$$\begin{cases} \dfrac{\sigma^{-n+M-N}}{1-\sigma^{-N}}; \ \ n \in \{0,\ 1,\ \ldots,\ M-1\}\ |\sigma|>1 \\ \dfrac{\sigma^{-n+M}}{1-\sigma^{-N}}; \ \ n \in \{M,\ M+1,\ \ldots,\ N-1\}\ |\sigma|>1 \end{cases} \tag{16}$$

$$\frac{z^M}{1-\sigma\,z^{-1}} \xrightarrow{\boldsymbol{L}^1} \sigma^{-(n-M)}\mathbf{1}_{-(n-m)} \xrightarrow{\widetilde{L}_N^1}$$

$$\begin{cases} \dfrac{\sigma^{-n+M}}{1-\sigma^{N}}; \ n \in \{0,\ 1,\ \ldots,\ M\}\ |\sigma|<1 \\ \dfrac{\sigma^{-n+M+N}}{1-\sigma^{N}}; \ n \in \{M+1,\ M+2,\ ...,\ N-1\}\ |\sigma|<1 \end{cases} \tag{17}$$

The N – periodic extensions space of $\boldsymbol{L}^1$ impulses is indicated $\widetilde{\boldsymbol{L}}^1_{\boldsymbol{N}}$, i.e.

$$\widetilde{\boldsymbol{L}}^1_{\boldsymbol{N}} = \left\{ \widetilde{x} \colon x_n = \sum_{p=-\infty}^{\infty} x_{n+pN}, \quad x \in \boldsymbol{L}^1 \right\} \tag{18}$$

The convolution operator in the $\widetilde{\boldsymbol{L}}^1_{\boldsymbol{N}}$ – space is defined as follows

$$\left[\widetilde{h} * \widetilde{x}\right]_n = \sum_{m=0}^{N-1} \widetilde{h}_{n\ominus m} \widetilde{x}_m, \tag{19}$$

where $n \ominus m$ is subtraction modulo N of two indices, i.e.

$$n, m \in \{0, 1 \ldots, N-1\} \colon n \ominus m$$
$$= \begin{cases} n - m & \text{for} \quad n - m \in \{0, 1, \ldots, N-1\} \\ n - m + N & \text{for} \quad n - m \notin \{0, 1, \ldots, N-1\} \end{cases}.$$

The inner product in the $\widetilde{\boldsymbol{L}}^1_{\boldsymbol{N}}$ – space is defined as follows

$$\left(\widetilde{h}, \widetilde{x}\right) = \frac{1}{N} \sum_{n=0}^{N-1} \widetilde{x}_n \widetilde{y}_n.$$

3. The power theory in the time domain. The distribution of two terminal electrical receiver operator

The convolution – the type admitance operator Y of elecrical circuit is distributed to the two commutative convolution operators:

$$Y = G + B, \tag{20}$$

where

$$\begin{aligned} G^* &= G \quad \text{(self adjoint operator)}, \\ B^* &= -B \quad \text{(skew hermite operator)}. \end{aligned} \tag{21}$$

The distribution (20) induces distribution current of electrical circuit to two components: the active current i^G and the reactive i^B current

$$i = Gu + Bu = i^G + i^B. \tag{22}$$

The distribution (20) is the unique distribution because, from the following system of equations

$$G + B = Y,$$
$$G - B = Y^*,$$

the operators: the active G and the reactive B can be determined uniquely

$$\begin{aligned} G &= \frac{1}{2}\left(Y + Y^*\right), \\ B &= \frac{1}{2}\left(Y - Y^*\right). \end{aligned} \tag{23}$$

Energy (the average power) of circuit meets the condition:

$$(u, i) = (Gu, u) + (Bu, u)$$

but

$$(Bu, u) = (B^* u, u) = -\left(Bu, u\right).$$

Therefore, the reactive component i^B of the current does not transfer the energy i.e.

$$(Bu, u) = 0.$$

In addition, components i^G and i^B are orthogonal, which is due to:

$$(Gu, Bu) = (GBu, u)$$

and

$$(GB)^* = B^* G = -GB \quad \text{(skew hermite operator)}.$$

In the particular case of the convolution operator characterized by the impulse function $\{h_n\} \colon n \in \boldsymbol{I}$, or its Z – transform

$$H(z) = \sum_{n=-\infty}^{\infty} h_n z^n$$

the adjoint relation has the form (see also (5)):

$$\begin{aligned} h_n^* \to h_{-n} \to H^*(z) &= \sum_{n=-\infty}^{\infty} h_{-n} z^n \\ &= \sum_{n=-\infty}^{\infty} h_n z^{-n} = H(z^{-1}). \end{aligned} \tag{24}$$

Thus the distributions (23) take the form

$$\begin{aligned} G(z) &= \frac{1}{2}\left(Y(z) + Y\left(z^{-1}\right)\right), \\ B(z) &= \frac{1}{2}\left(Y(z) - Y\left(z^{-1}\right)\right), \end{aligned} \tag{25}$$

where $Y(z)$ is the "*digital model*" of the admittance operator function of two terminal receiver $Y_A(s)$. Particularly good results are obtained from using the bilinear transformation.

$$Y(z) = Y_A\left(\frac{2}{\tau}\,\frac{1-z}{1+z}\right),$$

which is true for the continuous time

$$G_A(s) = \frac{1}{2}\left[Y_A(s) + Y_A(-s)\right],$$

$$B_A(s) = \frac{1}{2}\left[Y_A(s) - Y_A(-s)\right],$$

because

$$\frac{2}{\tau}\left(\frac{1-z}{1+z}\right) \xrightarrow{z \to z^{-1}} \frac{2}{\tau}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = -\frac{2}{\tau}\left(\frac{1-z}{1+z}\right).$$

Thus, the distribution of current (22) in the Z transform domain has the form

$$I(z) = G(z)\,U(z) + B(z)\,U(z) \tag{26}$$

and in the discrete time domain is expressed through the cyclic convolutions

$$\widetilde{i}_n = \sum_{m=0}^{N-1} \widetilde{g}_{n\ominus m} \widetilde{u}_m + \sum_{m=0}^{N-1} \widetilde{b}_{n\ominus m} \widetilde{u}_m, \tag{27}$$

where $\{\widetilde{g}_n\}$ and $\left\{\widetilde{b}_n\right\}$, $n \in \{0,\ 1,\ 2,\ \ldots,\ N-1\} - N$ – periodic original complex function $G(z)$ and $B(z)$ determined by the operator ratio (11), (15) and (16) and (17). These functions can also be determined directly, acting in the time domain, from the impulse response $\{y_n\} \in \boldsymbol{L}^1$ of the two terminal electrical receiver

$$\{y_n\} \leftrightarrow Y(z); \quad y_n = 0 \quad \text{for} \quad n < 0.$$

Using N - copy formula, the following is obtained:

$$\widetilde{y}_n = \sum_{p=-\infty}^{\infty} y_{n+pN}$$

and for $n \in \{0, 1, 2, \ldots, N-1\}$ it is

$$\widetilde{y}_n = \sum_{p=0}^{\infty} y_{n+pN}. \tag{28}$$

For the adjoint signal (see (5))

$$y^* = y_{-n}$$

the following is obtained:

$$\widetilde{y}_n^* = \sum_{p=-\infty}^{\infty} y_{-(n+pN)} = \sum_{p=-\infty}^{\infty} y_{pN-n} = \sum_{p=-\infty}^{\infty} y_{pN+N-n}$$

and for $n \in \{0, 1, 2, \ldots, N\}$ (see (28)) it is

$$\widetilde{y}_n^* = \sum_{p=0}^{\infty} y_{N-n+pN} = \widetilde{y}_{N-n}.$$

Thus, for the distribution in the $\boldsymbol{L}^1$ – impulse domain

$$g_n = \frac{1}{2}(y_n + y_{-n}),$$

$$b_n = \frac{1}{2}(y_n - y_{-n}),$$

there is the distribution in the $\widetilde{\boldsymbol{L}}_{\boldsymbol{N}}^1$ space

$$\widetilde{g}_n = \frac{1}{2}(\widetilde{y}_n + \widetilde{y}_{N-n}),$$
$$\widetilde{b}_n = \frac{1}{2}(\widetilde{y}_n - \widetilde{y}_{N-n}) \tag{29}$$

for $n \in \{0, 1, 2, \ldots, N-1, N\}$.

The N – periodic distribution in the time domain (29) corresponds to the distribution in the complex domain (25). Equation (22) and the fact that $i^G \perp i^B$ results in:

$$\|i\|^2 = (i, i) = \left\|i^G\right\|^2 + \left\|i^B\right\|^2$$

but there is

$$(u, i) = (u, i_G). \tag{30}$$

Assuming that the inner product (30) is given the value of energy (the averaged power – the Active Power) consumed by the receiver, the third component of the current is introduced

$$i^A = \frac{(u, i)}{(u, u)} u = \frac{(u, i^G)}{(u, u)} u \tag{31}$$

and the following distribution of the current is reached

$$i = i^A + \left(i^G - i^A\right) + i^B. \tag{32}$$

In the distribution (32) all the components are mutually orthogonal:

$$\left(i^A, i^B\right) = \frac{(u, i^G)}{(u, u)} (u, Bu) = 0,$$

$$\left(i^A, i^G - i^A\right) = \frac{(u, i^G)}{(u, u)} \left(u, i^G\right) - \frac{(u, i^G)^2}{(u, u)^2} (u, u) = 0,$$

$$\left(i^G - i^A, i^B\right) = (GBu, u) - \frac{(u, i^G)}{(u, u)} (Bu, u) = 0 - 0 = 0.$$

On the other hand, it can be noted that the receiver does not affect the current component i^A (the energy value (u, i^G) is given the value), so from the equality

$$\|i\|^2 = \left\|i^A\right\|^2 + \left\|i^G - i^A\right\|^2 + \left\|i^B\right\|^2$$

the following is obtained:

$$MIN \ \|i\|^2 = \left\|i^A\right\|^2,$$

$$(u, i) = \left(u, i^G\right) = P$$

which means that i^A is the minimal rms current that transfers all the given energy P of the two terminal electrical receiver.

Example. The use of the $\boldsymbol{L}^1$ – impulse method and N-periodic extension and the cyclic convolution in the discrete time domain is shown here. The example presents RL circuit powered with the voltage signal. The diagram and the voltage waveform are shown in Fig. 1.

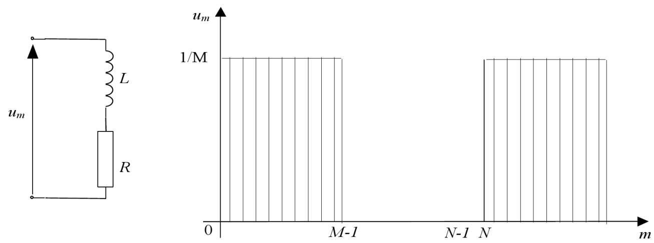

Fig. 1. Series RL circuit powered by a periodic unipolar square wave

The circuit admittance is

$$y(s) = \frac{1}{R + sL} = \frac{1}{L} \frac{1}{a + s},$$

where $a = \dfrac{R}{L}$.

Using the unchanging pulse function method, the time-continuous impulse response figures out to

$$y(t) = \frac{1}{L} e^{-at} \mathbf{1}(t)$$

and after sampling with an interval τ

$$y_n = \frac{1}{L} e^{-\beta n} \mathbf{1}_n,$$

where $\beta = a\tau$.

With applying the N – copy formula, the admittance $\widetilde{y}_n$ in the $\widetilde{L}^1_N$ – space is obtained

$$\widetilde{y}_n = \sum_{p=0}^{\infty} e^{-\beta(n+pN)} = \frac{1}{L} e^{-\beta n} \sum_{p=0}^{\infty} (e^{-\beta N})^p = \frac{1}{L} \frac{e^{-\beta n}}{1 - e^{-\beta N}} = \frac{1}{2L} \frac{e^{-\beta\left(n - \frac{N}{2}\right)}}{sh\beta \frac{N}{2}}$$

and for the adjoint signal is:

$$\widetilde{y}^*_n = \widetilde{y}_{N-n} = \frac{1}{2L} \frac{e^{\beta\left(n - \frac{N}{2}\right)}}{sh\beta \frac{N}{2}}.$$

The components of the admittance assumed in a shape of the conductance

$$\widetilde{g}_n = \begin{cases} y_0 & \text{for} \quad n = 0, \\ \frac{1}{2}(\widetilde{y}_n + \widetilde{y}_{N-n}) & \text{for} \quad n \in \{1,\ 2,\ ...,\ N-1\}, \end{cases}$$

the susceptance

$$\widetilde{b}_n = \begin{cases} 0 & \text{for} \quad n = 0, \\ \frac{1}{2}(\widetilde{y}_n - \widetilde{y}_{N-n}) & \text{for} \quad n \in \{1,\ 2,\ ...,\ N-1\}. \end{cases}$$

For the RL circuit the following outcome is obtained:

$$\widetilde{g}_n = \begin{cases} \frac{1}{L} \frac{1}{1 - e^{-\beta N}} = \frac{1}{2L} \frac{e^{\beta \frac{N}{2}}}{sh\beta \frac{N}{2}} & \text{for} \ \ n = 0, \\ \frac{1}{2L} \frac{ch\beta \left(n - \frac{N}{2}\right)}{sh\beta \frac{N}{2}} & \text{for} \ \ n \in \{1,\ 2,\ ...,\ N-1\}, \end{cases}$$

$$\widetilde{b}_n = \begin{cases} 0 & \text{for} \ \ n = 0, \\ -\frac{1}{2L} \frac{sh\beta \left(n - \frac{N}{2}\right)}{sh\beta \frac{N}{2}} & \text{for} \ \ n \in \{1,\ 2,\ ...,\ N-1\}. \end{cases}$$

The active and reactive current components can be determined by cyclic convolutions:

$$\begin{aligned} \widetilde{i}^g_n &= \sum_{m=0}^{n} \widetilde{g}_{n-m} u_m + \sum_{m=n+1}^{N-1} \widetilde{g}_{n-m+N} u_m, \\ \widetilde{i}^b_n &= \sum_{m=0}^{n} \widetilde{b}_{n-m} u_m + \sum_{m=n+1}^{N-1} \widetilde{b}_{n-m+N} u_m. \end{aligned} \tag{33}$$

The current values for the voltage signal u_m take the form:

$$\widetilde{i}^g_n = \frac{1}{M} \begin{cases} y_0 + \sum\limits_{m=1}^{M-1} \widetilde{g}_{N-m} & \text{for} \quad n = 0, \\ y_0 + \sum\limits_{m=0}^{n-1} \widetilde{g}_{n-m} + \sum\limits_{m=n+1}^{M-1} \widetilde{g}_{n-m+N} & \\ \text{for} \quad n \in \{1,\ 2,\ ...,\ M-2\} & \\ y_0 + \sum\limits_{m=0}^{M-2} \widetilde{g}_{M-1-m} & \text{for} \quad n = M-1, \\ \sum\limits_{m=0}^{M-1} \widetilde{g}_{n-m} & \text{for} \quad n \in \{M,\ ...,\ N-1\}, \end{cases}$$

$$\widetilde{i}^b_n = \frac{1}{M} \begin{cases} \sum\limits_{m=1}^{M-1} \widetilde{b}_{N-m} & \text{for} \quad n = 0, \\ \sum\limits_{m=0}^{n-1} \widetilde{b}_{n-m} + \sum\limits_{m=n+1}^{M-1} \widetilde{b}_{n-m+N} & \\ \text{for} \quad n \in \{1,\ 2,\ ...,\ M-2\}, & \\ \sum\limits_{m=0}^{M-2} \widetilde{b}_{M-1-m} & \text{for} \quad n = M-1, \\ \sum\limits_{m=0}^{M-1} \widetilde{b}_{n-m} & \text{for} \quad n \in \{M,\ ...,\ N-1\}. \end{cases}$$

Especially for the RL circuit they take on the particular form as follows:

$$\widetilde{i}^b_n = \frac{\left(sh\beta \frac{N}{2}\right)^{-1}}{-2L} \frac{1}{M} \begin{cases} \sum\limits_{m=1}^{M-1} sh\beta \left(\frac{N}{2} - m\right) \\ \text{for} \quad n = 0, \\ \sum\limits_{m=0}^{n-1} sh\beta \left(n - m - \frac{N}{2}\right) \\ + \sum\limits_{m=n+1}^{M-1} sh\beta \left(n - m + \frac{N}{2}\right) \\ \text{for} \quad n \in \{1,\ 2,\ ...,\ M-2\}, \\ \sum\limits_{m=0}^{M-2} sh\beta \left(M - 1 - m - \frac{N}{2}\right) \\ \text{for} \quad n = M-1, \\ \sum\limits_{m=0}^{M-1} sh\beta \left(n - m - \frac{N}{2}\right) \\ \text{for} \quad n \in \{M,\ ...,\ N-1\}. \end{cases}$$

For instance, if the reactive current for $M = 4$ is designated as

$$\widetilde{i}_0^b = \sum_{m=1}^{3} sh\beta\left(\frac{N}{2} - m\right),$$

$$\widetilde{i}_1^b = \sum_{m=0}^{0} sh\beta\left(1 - m - \frac{N}{2}\right) + \sum_{m=2}^{3} sh\beta\left(1 - m + \frac{N}{2}\right),$$

$$\widetilde{i}_2^b = \sum_{m=0}^{1} sh\beta\left(2 - m - \frac{N}{2}\right) + \sum_{m=3}^{3} sh\beta\left(2 - m + \frac{N}{2}\right)$$

$$\widetilde{i}_3^b = \sum_{m=0}^{2} sh\beta\left(3 - m - \frac{N}{2}\right)$$

it is demonstrable that the following subordination – $(u, i^b) = 0$ – is fulfilled. When calculating the additional components of the reactive current

$$\widetilde{i}_0^b = sh\beta\left[\left(\frac{N}{2} - 1\right) + \left(\frac{N}{2} - 2\right) + \left(\frac{N}{2} - 3\right)\right],$$

$$\widetilde{i}_1^b = sh\beta\left[\left(1 - \frac{N}{2}\right) + \left(-1 + \frac{N}{2}\right) + \left(-2 + \frac{N}{2}\right)\right],$$

$$\widetilde{i}_2^b = sh\beta\left[\left(2 - \frac{N}{2}\right) + \left(1 - \frac{N}{2}\right) + \left(-1 + \frac{N}{2}\right)\right],$$

$$\widetilde{i}_3^b = sh\beta\left[\left(3 - \frac{N}{2}\right) + \left(2 - \frac{N}{2}\right) + \left(1 - \frac{N}{2}\right)\right]$$

the following equation is acquired

$$\widetilde{i}^b = \sum_{n=0}^{3} i_n^b = 0.$$

Consequently, it appears to be evident that $(u, i^b) = 0$.

With the increase of M a matrix of the size $Mx(M-1)$ is augmented, whose words add up to zero.

The conclusions of this example can be easily extended to more complex circuits, which transmittance is easily to be decomposed into the partial fractions with real poles, such as RL circuits, RC circuits or RLC circuits with a predominance of attenuation.

The received set of expressions, which are calculated from the convoluted relationships (33), should be multiplied by a factor τ, which follows directly from the principle of sampling:

$$\int b\,(t - t')u(t')dt' \xrightarrow{t=n\tau} \sum \tau b_{n-m}u_m.$$

The factor τ appears automatically while other methods of digital modeling such as the rectangle method or the trapezoidal method are applied.

The rectangle method. In the expression defining the admittance of the receiver, the operator s is modeled by $s \to \frac{1}{\tau}(1 - z)$

$$Y(s) = \frac{1}{L}\frac{1}{a + s} \xrightarrow{s \to \frac{1}{\tau}(1-z)} \frac{1}{L}\frac{1}{a + \frac{1}{\tau}(1 - z)} = \frac{1}{L}\frac{\tau}{1 + a\tau}\frac{1}{1 - (1 + a\tau)^{-1} z}.$$

For the above expression the N – copy formula (13) can be employed

$$\widetilde{y}_n = \frac{1}{L}\frac{\tau}{1 + a\tau}\frac{(1 + a\tau)^{-n}}{1 - (1 + a\tau)^{-N}} = \frac{1}{L}\frac{\tau}{1 + a\tau}\frac{\left[(1 + a\tau)^{\frac{1}{a\tau}}\right]^{-a\tau n}}{1 - \left[(1 + a\tau)^{\frac{1}{a\tau}}\right]^{-a\tau N}}$$

$$n \in \{0,\ 1,\ ...,\ N - 1\}.$$

So that the above pulse function could be compared to the function obtained by the unchanging pulse function method mentioned below the expression substituted:

$$\left[(1 + a\tau)^{\frac{1}{a\tau}}\right]^{a\tau} = e^{\beta} \to$$

$$\beta = a\tau \ln(1 + a\tau)^{\frac{1}{a\tau}} = \ln(1 + a\tau).$$

It can be proven that for $x \to 0$

$$\ln(1 + x) = \left[\frac{d\ln(1 + x)}{dx}\right]_{x=0} x = \left[\frac{1}{1 + x}\right]_{x=0} x = x.$$

It results in

$$\beta \approx a\tau \quad \text{for} \quad \tau \to 0.$$

Finally, the form of the admittance $\widetilde{y}_n$ in the $\widetilde{L}_N^1$ – space can be written as:

$$\widetilde{y}_n = \frac{1}{L}\frac{\tau}{1 + a\tau}\frac{e^{-\beta n}}{1 - e^{-\beta N}} = \frac{\tau}{1 + a\tau}\frac{1}{2L}\frac{e^{-\beta\left(n - \frac{N}{2}\right)}}{sh\beta\frac{N}{2}}$$

and for the adjoint signal is

$$\widetilde{y}_n^* = \widetilde{y}_{N-n} = \frac{\tau}{1 + a\tau}\frac{1}{2L}\frac{e^{\beta\left(n - \frac{N}{2}\right)}}{sh\beta\frac{N}{2}}.$$

Because the expression

$$1 + a\tau \approx 1 \quad \text{for} \quad \tau \to 0$$

is the obtained from the rectangle method impulse function, it differs in τ – fold from the impulse function received from the unchanging impulse function method.

The trapezoidal method. In the expression defining the admittance of the receiver, the operator s is modeled by $s \to \frac{2}{\tau}\frac{1 - z}{1 + z}$

$$Y(s) = \frac{1}{L}\frac{1}{a + s} \xrightarrow{s \to \frac{2}{\tau}\frac{1-z}{1+z}} \frac{1}{L}\frac{1}{a + \frac{2}{\tau}\frac{1 - z}{1 + z}}$$

$$= \frac{1}{L}\frac{\tau}{2 + a\tau}\frac{1 + z}{1 - \left(\frac{2 + a\tau}{2 - a\tau}\right)^{-1} z}.$$

Assuming that $\sigma = \left(\dfrac{2+a\tau}{2-a\tau}\right)$, it can be written:

$$Y(z) = \frac{1}{L}\frac{\tau}{2+a\tau}\frac{1+z}{1-\sigma^{-1}z}.$$

The above expression should be converted to a form which allows to make use of the N – copy formula (13).

For this purpose, the impulse response of the filter described by transmittance $Y(z)$ needs to be determined (in the following expressions a constant factor $\dfrac{1}{L}\dfrac{\tau}{2+a\tau}$ is omitted).

The filter operational equation takes the form

$$\left(1-\sigma^{-1}z\right) y = (1+z)\, x$$

and its impulse response meets a recursive equation

$$h_n = \delta_n + \delta_{n-1} + \sigma^{-1}h_{n-1}.$$

The solution of the above equation is

$$\begin{aligned} h_0 &= 1, \\ h_1 &= 1+\sigma^{-1}, \\ h_2 &= \sigma^{-1}(1+\sigma^{-1}), \\ h_3 &= \sigma^{-2}(1+\sigma^{-1}), \ldots \end{aligned}$$

Generally, it can be written as

$$h_n = \sigma^{-(n-1)}(1+\sigma^{-1}).$$

Taking the constant factor $\dfrac{1}{L}\dfrac{\tau}{2+a\tau}$ into account, it is obtained as follows

$$\begin{aligned} h_n &= \frac{1}{L}\frac{\tau}{2+a\tau}\left(\frac{2+a\tau}{2-a\tau}\right)^{-n(n-1)}\left(1+\frac{2-a\tau}{2+a\tau}\right) \\ &= 4\frac{1}{L}\frac{\tau}{(2+a\tau)^2}\left(\frac{2+a\tau}{2-a\tau}\right)^{-(n-1)} \\ &= \frac{1}{L}\frac{4\tau}{4-(a\tau)^2}\left(\frac{2+a\tau}{2-a\tau}\right)^{-n} \approx \tau\frac{1}{L}\left(\frac{2+a\tau}{2-a\tau}\right)^{-n}. \end{aligned}$$

The above expression can be simplified for $\tau \to 0$ as it can be made fairly evident that

$$\begin{aligned} \left.\frac{2+x}{2-x}\right|_{x\to 0} &= \frac{(2+x)^2}{4-x^2} = \frac{4+4x+x^2}{4-x^2} \\ &\approx \frac{4+4x}{4} = 1+x. \end{aligned}$$

At last, it can be written that

$$y_n = \tau\frac{1}{L}\left(1+a\tau\right)^{-n},$$

where the form of the admittance $\widetilde{y}_n$ in the $\widetilde{L}^1_N$ – space comes from

$$\begin{aligned} \widetilde{y}_n &= \tau\frac{1}{L}\frac{(1+a\tau)^{-n}}{1-(1+a\tau)^{-N}} = \tau\frac{1}{L}\frac{e^{-\beta n}}{1-e^{-\beta N}} \\ &= \tau\frac{1}{2L}\frac{e^{-\beta\left(n-\frac{N}{2}\right)}}{sh\beta\dfrac{N}{2}} \end{aligned}$$

and for the adjoint signal the result is

$$\widetilde{y}^*_n = \widetilde{y}_{N-n} = \tau\frac{1}{2L}\frac{e^{\beta\left(n-\frac{N}{2}\right)}}{sh\beta\dfrac{N}{2}}.$$

Comparing the results to those received by means of the rectangle method, from the trapezoidal method obtained impulse functions differ in τ – fold from the impulse functions which were obtained from the unchanging impulse function method.

4. Summary

In the article the distribution directly in the discrete time domain to the active and reactive currents using the original method proposed by one of the authors – the $\boldsymbol{L}^1$-impulse method [1, 2] is presented. This method uses the isomorphism between the rational function set of complex variable and the exponential $\boldsymbol{L}^1$ impulses set of positive and negative time domain and their N – periodic extensions set. It is also shown that for periodic alternating receiver current, the active and the reactive current can be also separated in the discrete time domain using the distribution of impulse response of the receiver admittance to even and odd components (equivalent to active and reactive components) and N-periodic extension and then the cyclic convolution.

For signals significantly different from the sinusoidal the distribution in the time domain is much more favorable than the distribution in the frequency domain [3–5]. Additionally, taking into account the possibility of using digital signal processors for direct controlling of the compensators in real-time, the time domain seems to be much more effective than the frequency domain.

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