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DEDICATED PAPERS

Dedicated to Professor Tadeusz Kaczorek in honour of his 80th birthday

Stability analysis of continuous-time linear systems consisting of *n* subsystems with different fractional orders

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Abstract. The stability problem of continuous-time linear systems described by the state equation consisting of n subsystems with different fractional orders of derivatives of the state variables has been considered. The methods for asymptotic stability checking have been given. The method proposed in the general case is based on the Argument Principle and it is similar to the modified Mikhailov stability criterion known from the stability theory of natural order systems. The considerations are illustrated by numerical examples.

Key words: linear system, continuous-time, fractional, stability, Mikhailov criterion.

1. Introduction

A dynamical system represented by differential (or difference) equations with not necessarily integer orders of derivatives (or differences) can be considered as a fractional order system. The real objects are generally fractional, however, for many of them the fractionality is very low. Therefore, the fractional order representation is more adequate to describe real world systems than the integer order models.

In the last decades, the problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations has been considered in many papers and monographs, see [1–6], for example.

The problems of stability of linear continuous-time and discrete-time fractional order systems have been investigated in [7–17] and [18–21], respectively.

The new class of linear fractional order systems, namely the positive systems of fractional order has been considered in [21–26].

The aim of the paper is to give the methods for asymptotic stability analysis of fractional continuous-time linear systems described by the state-space model consisting of n subsystems with different fractional orders of derivatives of state variables. Such models have been considered in [13, 17, 24].

In the paper the following notations are used: $\Re^{n\times m}$ – the set of $n\times m$ real matrices and $\Re^n=\Re^{n\times 1}$; Z_+ – the set of non-negative integers; I_n – the identity $n\times n$ matrix.

2. Preliminaries and problem formulation

Consider a continuous-time linear system of fractional orders described by the homogeneous state equation

$$ {}_0D_t^{\overline{\alpha}}x(t)=Ax(t), \tag{1}$$

where

$$x(t)=\begin{bmatrix} x_1(t)\\ \vdots \\ x_n(t)\end{bmatrix},\quad {}_0D_t^{\overline{\alpha}}x(t)=\begin{bmatrix} {}_0D_t^{\alpha_1}x_1(t)\\ \vdots \\ {}_0D_t^{\alpha_n}x_n(t)\end{bmatrix}, \tag{2}$$

with $x_k(t)\in\Re^{n_k}$, $k=1,...,n$, $x(t)\in\Re^N$, $N=n_1+\cdots+n_n$, and

$$A=\begin{bmatrix} A_{11} & ... & A_{1n}\\ \vdots & ... & \vdots \\ A_{n1} & ... & A_{nn}\end{bmatrix}, \tag{3}$$
$$A_{kr}\in\Re^{n_k\times n_r}\quad (k,r=1,...,n).$$

Initial conditions for (1) have the form

$$x_k^{(r)}(0)=x_{k0}^{(r)}\in\Re^{n_k}, \tag{4}$$

where $x_k^{(r)}(0)=(d^r/dt^r)x_k(t)_{|t=0}$ for $k=1,...,n$; $r=0,1,...,p_k-1$.

In (2) the following Caputo definition of the fractional α_k-order derivative has been used

$${}_0D_t^{\alpha_k}x_i(t)=\frac{1}{\Gamma(p_k-\alpha_k)}\int_0^t\frac{x_i^{(p_k)}(\tau)d\tau}{(t-\tau)^{\alpha_k+1-p_k}}, \tag{5}$$

where

$$x_i^{(p_k)}(t)=\frac{d^{p_k}x_i(t)}{dt^{p_k}},\qquad p_k-1\le\alpha_k\le p_k, \tag{6}$$

p_k is a positive integer and

$$\Gamma(\alpha_k)=\int_0^\infty e^{-t}t^{\alpha_k-1}dt,\qquad \mathrm{Re}\alpha_k>0, \tag{7}$$

is the Euler gamma function.

The Laplace transform of the fractional derivative of the state vector $x(t)$ with zero initial conditions has the form

$$L\{{}_0D_t^{\overline{\alpha}}x(t)\}=\begin{bmatrix} s^{\alpha_1}X_1(s)\\ \vdots \\ s^{\alpha_n}X_n(s)\end{bmatrix}, \tag{8}$$

where $X_k(s)=L\{x_k(t)\}$, $k=1,...,n$.

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The characteristic matrix of the fractional system (1)

$$H(s) = \begin{bmatrix} I_{n_1}s^{\alpha_1} - A_{11} & \cdots & -A_{1n} \\ -A_{21} & \cdots & -A_{2n} \\ \vdots & \ddots & \vdots \\ -A_{n1} & \cdots & I_{n_n}s^{\alpha_n} - A_{nn} \end{bmatrix} \tag{9}$$

can be computed from the formula

$$H(s) = I(s) - A, \tag{10}$$

where

$$I(s) = \begin{bmatrix} I_{n_1}s^{\alpha_1} & 0 & \cdots & 0 \\ 0 & I_{n_2}s^{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_n}s^{\alpha_n} \end{bmatrix}. \tag{11}$$

From (10) and (11) it follows that the characteristic function of the system (1)

$$w(s) = \det(I(s) - A), \tag{12}$$

is a polynomial of fractional degree

$$\delta = n_1\alpha_1 + n_2\alpha_2 + ... + n_n\alpha_n. \tag{13}$$

We consider the following three cases:

Case 1. The fractional order system (1) is of a commensurate order. In this case there exists a real number $\alpha > 0$ such that

$$\alpha_i = k_i\alpha, \qquad i = 1, 2, ..., n, \qquad k_i \in Z_+. \tag{14}$$

Case 2. The fractional order system (1) is of a rational order. In this case the following conditions hold

$$\alpha_i = v_i/u_i, \quad v_i,\ u_i \in Z_+ \ \ (i = 1, ..., n), \tag{15}$$

where v_i and u_i are coprime.

Case 3. The fractional order system (1) is of a non-commensurate order. In this case the conditions (14) and (15) do not hold.

From the theory of stability of linear fractional order systems (see [7, 8, 12], for example) we have the following theorem.

Theorem 1. The fractional order system (1) is asymptotically stable if and only if

$$w(s) = \det H(s) \neq 0 \quad \text{for} \quad \text{Re}\, s \geq 0. \tag{16}$$

In [15] it was shown that if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ and the condition (16) holds then components of the vector $x(t)$ decay to 0 not exponentially but following to the function $t^{-\mu}$, $t > 0$, $\mu > 0$. Therefore, the condition (16) is necessary and sufficient for asymptotic stability (but not for asymptotic exponential stability) of the system (1).

The aim of the paper is to give the methods for checking the condition (16) for fractional system (1) in three above mentioned cases.

3. Problem solution

3.1. Stability of the system of a commensurate order. If the condition (14) holds then substitution of

$$\lambda = s^\alpha \tag{17}$$

in (10), (11) gives the natural degree characteristic matrix

$$\widetilde{H}(\lambda) = I(\lambda) - A \tag{18}$$

associated with the fractional degree characteristic matrix (9), where

$$I(\lambda) = \begin{bmatrix} I_{n_1}\lambda^{k_1} & 0 & \cdots & 0 \\ 0 & I_{n_2}\lambda^{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_n}\lambda^{k_n} \end{bmatrix}. \tag{19}$$

Hence, the natural degree polynomial associated with the fractional degree polynomial (12) has the form

$$\widetilde{w}(\lambda) = \det \widetilde{H}(\lambda) = \lambda^p + a_{p-1}\lambda^{p-1} + ... + a_0, \tag{20}$$

where a_k $(k = 0,\ 1, ..., p-1)$ are constant coefficients,

$$p = \sum_{i=1}^{n} n_i k_i \tag{21}$$

and natural numbers k_i $(i = 1, 2, ..., n)$ are defined in (14).

From the theory of stability of linear fractional order systems ([7, 8, 12], for example) we have that in Case 1 the condition (16) holds for the fractional polynomial (12) if and only if the condition

$$|\arg \lambda_i| > \alpha\frac{\pi}{2}, \qquad i = 1, 2, ..., p, \tag{22}$$

is satisfied for all roots λ_i $(i = 1,\ 2, ..., p)$ of the associated natural degree polynomial (20), where α is defined in (14).

From the above we have the following theorem.

Theorem 2. The fractional order system (1) of a commensurate order ((14) holds) is asymptotically stable if and only if $\gamma > \alpha\pi/2$, where

$$\gamma = \min_i |\arg \lambda_i|, \quad i = 1, 2, ..., p. \tag{23}$$

From [8] it follows that the fractional system with the characteristic polynomial

$$w(s) = s^{p\alpha} + a_{p-1}s^{(p-1)\alpha} + ... + a_0 \tag{24}$$

is unstable for all $\alpha > 2$. Therefore, in this paper we consider the fractional order systems (1) in Case 1 with $0 < \alpha < 2$.

The asymptotic stability regions of the system (1), described by (22), are shown in Figs. 1 and 2 for $0 < \alpha < 1$ and for $1 < \alpha < 2$, respectively.

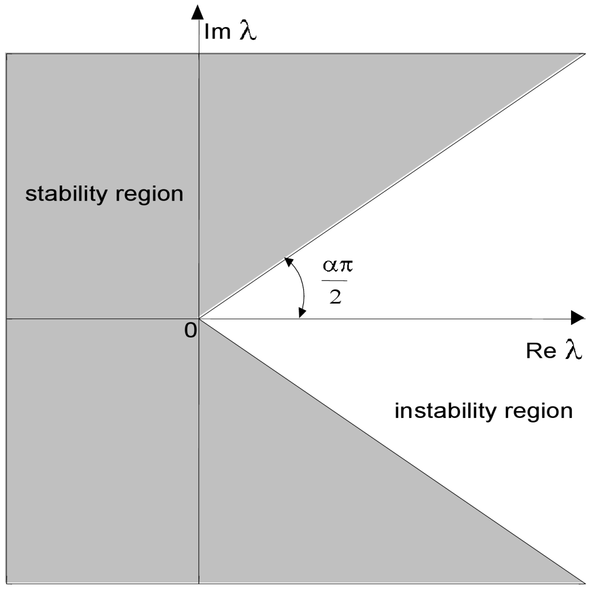

Fig. 1. Asymptotic stability region of the system (1) in Case 1 with $0 < \alpha < 1$

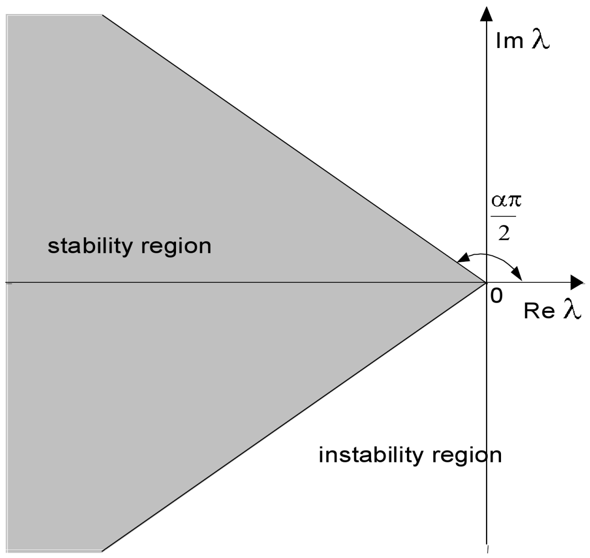

Fig. 2. Asymptotic stability region of the system (1) in Case 1 with $1 < \alpha < 2$

Parametric description of the boundary of the asymptotic stability region has the form

$$(j\omega)^\alpha = |\omega|^\alpha e^{j\pi\alpha/2}, \qquad \omega \in (-\infty,\ \infty). \tag{25}$$

From Theorem 2 and Figs. 1 and 2 we have the following lemma and remarks for the system (1) in the Case 1.

Lemma 1. If the associated natural degree polynomial (20) has no real non-negative roots then the fractional order system (1) is asymptotically stable if and only if $\alpha \in (0,\ \alpha_0)$, where

$$\alpha_0 = 2\gamma/\pi \tag{26}$$

and γ is defined by (23).

Remark 1. If $0 < \alpha < 1$ then the fractional order system (1) may be asymptotically stable when not all roots of the polynomial (20) lie in open left half-plane. Moreover, the system may be asymptotically stable when all roots of (20) are complex conjugate with positive real parts.

Remark 2. If $1 < \alpha < 2$ then the fractional order system (1) may be unstable when all roots of the polynomial (20) lie in open left half-plane.

Remark 3. If $0 < \alpha < 2$ then the fractional order system (1) is unstable if the polynomial (20) has at least one non-negative real root. In particular, this holds if

$$\widetilde{w}(0) = a_0 = \det(-A) = 0. \tag{27}$$

Example 1. Consider the fractional commensurate order system (1) with $n_1 = 2$, $n_2 = 1$ and matrix A of the form (3) with $n = 2$, where

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} -1 & -2 \end{bmatrix}, \qquad A_{22} = -3. \tag{28}$$

For the system of a fractional commensurate order the condition (14) holds. We check stability of the system in two cases: a) $k_1 = 1$, $k_2 = 2$, b) $k_1 = k_2 = 1$.

In the case a) the characteristic polynomial has the form

$$w(s) = \det \begin{bmatrix} s^\alpha & -1 & 0 \\ 0 & s^\alpha & -1 \\ 1 & 2 & s^{2\alpha}+3 \end{bmatrix} = s^{4\alpha} + 3s^{2\alpha} + 2s^\alpha + 1. \tag{29}$$

Substitution $\lambda = s^\alpha$ in (29) gives the associated polynomial of natural degree

$$\widetilde{w}(\lambda) = \lambda^4 + 3\lambda^2 + 2\lambda + 1. \tag{30}$$

The polynomial (30) has the following roots: $\lambda_{1,2} = -0.3497 \pm j0.4390$ and $\lambda_{3,4} = 0.3497 \pm j1.7470$. From (23) and (26) we have $\gamma = 1.3732$ and $\alpha_0 = 2\gamma/\pi = 0.8742$.

From Lemma 1 it follows that the system with $k_1 = 1$ and $k_2 = 2$ in (14) is asymptotically stable if and only if $\alpha \in (0,\ 0.8742)$.

It easy to check that in the case b) the polynomial of natural degree, associated with the characteristic polynomial of the system has the form

$$\widetilde{w}(\lambda) = \lambda^3 + 3\lambda^2 + 2\lambda + 1. \tag{30a}$$

This polynomial has the following roots: $\lambda_1 = -2.3247$; $\lambda_{2,3} = -0.3376 \pm j0.5623$ and from (23) and (26) it follows that $\gamma = 2.1116$ and $\alpha_0 = 2\gamma/\pi = 1.3443$. From Lemma 1 we have that the system with $k_1 = k_2 = 1$ in (14) is asymptotically stable if and only if $\alpha \in (0,\ 1.3443)$.

From (18)–(20) and Example 1 it follows that if the condition $k_1 = k_2 = \cdots = k_n = 1$ does not hold then the associated natural degree polynomial (20) has at least one coefficient a_k $(k = 0,\ 1, ..., p-1)$ equal to zero. In this case, according to the Hurwitz stability criterion, there exists at least one root of polynomial (20) with non-negative real parts. Hence, we have the following remark.

Remark 4. If for the system (1) of fractional commensurate order ((14) holds) the condition $k_1 = k_2 = \cdots = k_n = 1$

is not satisfied then this system may be stable only for $\alpha \in (0,\ \alpha_0)$, where $\alpha_0 < 1$.

3.2. Stability of the system of a rational order. For the system (1) of a fractional rational order the condition (15) holds.

Denote by m the lowest common multiple of all u_i $(i = 1, ..., n)$, defined in (15).

In this case we can write

$$\alpha_i = k_i \alpha, \qquad i = 1, 2, ..., n, \qquad k_i \in Z_+, \tag{31}$$

where

$$\alpha = 1/m, \qquad k_i = m\alpha_i. \tag{32}$$

From the above it follows that if the condition (15) holds then the system (1) is of a rational commensurate order. This means that in this case we can use the methods described in Subsec. 3.1 to asymptotic stability analysis. Because $\alpha = 1/m < 1$, the fractional order system (1) in Case 2 is asymptotically stable if and only if all root of the associated polynomial (20) lie in the stability region shown in Fig. 1.

Example 2. Consider the fractional rational order system (1) with $n_1 = 2$, $n_2 = 1$ and the matrix A of the form (3), (28). Check asymptotic stability of the system in two cases: a) $\alpha_1 = 2/3$, $\alpha_2 = 3/4$ and b) $\alpha_1 = 1/3$, $\alpha_2 = 6/5$.

In case a) according to (15) and (31), (32) we have $u_1 = 3$, $u_2 = 4$ and $m = 12$, $\alpha = 1/12$, $k_1 = m\alpha_1 = 8$, $k_2 = m\alpha_2 = 9$.

From (18)–(20) one obtains

$$\widetilde{H}(\lambda) = \begin{bmatrix} \lambda^8 & -1 & 0 \\ 0 & \lambda^8 & -1 \\ 1 & -2 & \lambda^9 + 3 \end{bmatrix} \tag{33}$$

and

$$\widetilde{w}(\lambda) = \det \widetilde{H}(\lambda) = \lambda^{25} + 3\lambda^{16} - 2\lambda^8 + 1. \tag{34a}$$

Computing roots λ_i $(i = 1, 2, ..., 25)$ of the polynomial (34a) and using (23) we obtain $\gamma = 0.1154$. It easy to see that

$$\gamma = 0.1154 < \alpha\pi/2 = \pi/24 = 0.1309.$$

This means that the condition of Theorem 2 does not hold and the system in case a) is unstable.

In case b) we have $u_1 = 3$, $u_2 = 5$, $m = 15$, $\alpha = 1/15$, $k_1 = 5$, $k_2 = 18$ and

$$\widetilde{w}(\lambda) = \lambda^{28} + 3\lambda^{10} - 2\lambda^5 + 1. \tag{34b}$$

From (23) for roots of (34b) we have $\gamma = 0.1888$. Because $\alpha\pi/2 = \pi/30 = 0.1017$ the condition $\gamma > \alpha\pi/2$ of Theorem 2 is satisfied and the system in case b) is asymptotically stable.

The method of Theorem 2 requires computation of roots of the associated polynomial (20). These roots are different from eigenvalues of the state matrix A. Moreover, the degree of polynomial (20) depends on α defined in (14). It is easy to see that investigation of asymptotic stability of the fractional order system (1) by checking the condition (22) (or (23)) can be inconvenient with regard on high degree of the associated polynomial (20).

To asymptotic stability analysis of the fractional order system (1) of commensurate order we can apply the frequency domain method described in the next section. This method is a general method which can be applied to asymptotic stability checking of the fractional order system (1) with commensurate or non-commensurate fractional orders of derivatives.

3.3. Stability of the system of a non-commensurate order. The methods described in the above sections can not be applied to asymptotic stability analysis of the fractional order system (1) in Case 3, i.e. in the case of non-commensurate orders of fractional derivatives. In this case we apply the frequency domain method.

The frequency domain methods have been proposed, respectively, in [9–11, 18], (see also [26], Chapter 9) for asymptotic stability investigation of fractional order continuous-time and discrete-time linear systems described by the transfer function. These methods have been applied in [12] to asymptotic stability analysis of continuous-time linear systems described by state space models with the same fractional order of derivatives of all state variables and in [13] with different fractional commensurate orders.

Denote by $w_r(s)$ the reference asymptotically stable fractional polynomial of degree δ (see (13)), that is of the same fractional degree as the characteristic polynomial (12) of the fractional order system (1).

Let us consider the rational function

$$\psi(s) = \frac{w(s)}{w_r(s)} = \frac{\det(I(s) - A)}{w_r(s)}. \tag{35}$$

The reference asymptotically stable fractional degree polynomial can be chosen in the form

$$w_r(s) = (s + c)^{\delta}, \qquad c > 0. \tag{36}$$

Theorem 3. The fractional order system (1) (with non-commensurate or commensurate fractional orders of derivatives) is asymptotically stable if and only if

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} \psi(j\omega) = 0, \tag{37}$$

where $\psi(j\omega) = \psi(s)$ for $s = j\omega$ and $\psi(s)$ is defined by (35).

Proof. From (35) it follows that

$$\Delta \arg \psi(j\omega) = \Delta \arg w(j\omega) - \Delta \arg w_r(j\omega). \tag{38}$$

From the Argument Principle it follows that the fractional degree characteristic polynomial (12) is asymptotically stable if and only if

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} w(j\omega) = \underset{\omega \in (-\infty,\infty)}{\Delta \arg} w_r(j\omega). \tag{39}$$

From (38) it follows that (39) holds if and only if (37) is satisfied.

Satisfaction of (37) means that plot of the function $\psi(j\omega)$ does not encircle or cross the origin of the complex plane as ω runs from $-\infty$ to ∞.

From (10)–(12), (35) and (36) we have

$$\psi(\infty) = \lim_{\omega\to\pm\infty} \psi(j\omega) = 1 \tag{40}$$

and

$$\psi(0) = \frac{\det(-A)}{c^\delta}. \tag{41}$$

From (41) it follows that $\psi(0) \leq 0$ if $\det(-A) \leq 0$. Hence, from Theorem 3 we have the following lemma.

Lemma 2. If $\det(-A) \leq 0$, then the fractional order system (1) is unstable.

Example 3. Consider the fractional order system (1) with $n_1 = n_2 = 2$, $\alpha_1 = 1.1$, $\alpha_2 = \sqrt{2}$ and the matrix A of the form (3) with $n = 2$, where

$$A_{11} = \begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 1 & 2 \\ -7 & 3 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} -2 & 2 \\ 6 & -3 \end{bmatrix}. \tag{42}$$

In this case the characteristic matrix has the form

$$H(s) = \begin{bmatrix} I_2 s^{1.1} - A_{11} & -A_{12} \\ -A_{21} & I_2 s^{\sqrt{2}} - A_{22} \end{bmatrix}. \tag{43}$$

From (43) and (13) it follows that the characteristic polynomial of the system has the fractional degree

$$\delta = n_1\alpha_1 + n_2\alpha_2 = 2.2 + 2\sqrt{2}. \tag{44}$$

Plot of the function

$$\psi(j\omega) = \frac{\det H(j\omega)}{(j\omega+3)^\delta}, \qquad \omega \in (-\infty,\ \infty), \tag{45}$$

is shown in Fig. 3.

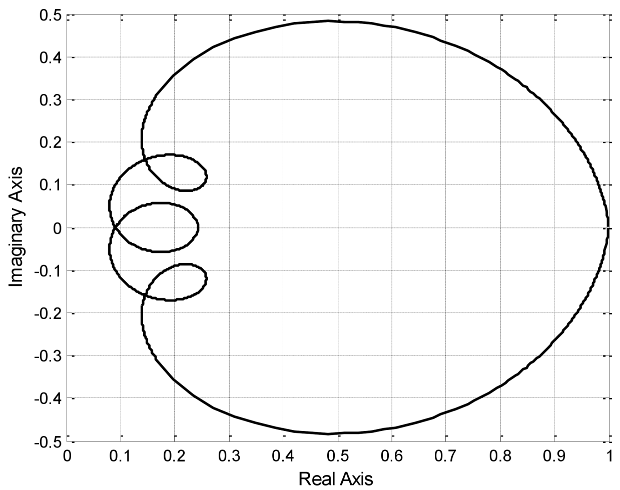

Fig. 3. Plot of the function (45)

According to (40) and (41) we have

$$\psi(\infty) = 1, \qquad \psi(0) = \det(-A)/3^{(2.2+\sqrt{2})} = 0.2433.$$

From Fig. 3 it follows that plot of (45) does not encircle the origin of the complex plane. This means, according to Theorem 3, that the fractional order system is asymptotically stable.

4. Concluding remarks

The asymptotic stability problem of continuous-time linear system (1) consisting of n subsystems with different fractional orders of derivatives of state variables has been considered. It has been shown that in the case of commensurate or rational orders of derivatives, asymptotic stability of the system is equivalent to satisfaction of the condition of Theorem 2 for all roots of the associated natural degree polynomial (20).

In the general case of non-commensurate orders of fractional derivatives, the frequency domain method has been proposed in Theorem 3. This method is based on the Argument Principle and it is a generalisation of the classical modified Mikhailov asymptotic stability criterion to the class of fractional order systems (1).

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