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On Rayleigh wave in generalized magneto-thermoelastic media with hydrostatic initial stress

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Abstract. The governing equations of generalized magneto-thermoelasticity with hydrostatic initial stress are solved for surface wave solutions. The particular solutions in the half-space are applied to the boundary conditions at the free surface of the half-space to obtain the frequency equation of Rayleigh wave. The frequency equation is approximated for small thermal coupling and small reduced frequency. The velocity of propagation and amplitude-attenuation factor of Rayleigh wave are computed numerically for a particular material. Effects of magnetic field and hydrostatic initial stress on the velocity of the propagation and amplitude-attenuation factor are shown graphically.

Key words: generalized thermoelasticity, hydrostatic initial stress, magnetic field, Rayleigh wave, frequency equation.

1. Introduction

The classical dynamical coupled theory of thermoelasticity was extended to generalized thermoelasticity theories by Lord and Shulman [1] and Green and Lindsay [2]. These theories consider heat propagation as a wave phenomenon rather than a diffusion phenomenon and predict a finite speed of heat propagation. Ignaczak and Ostoja-Starzewski [3] presented the analysis of above two theories in their book on "Thermoelasticity with Finite Wave Speeds". The representative theories in the range of generalized thermoelasticity are reviewed by Hetnarski and Ignaczak [4].

Surface waves in elastic solids were first studied by Lord Rayleigh [5] for an isotropic elastic solid. Thermoelastic Rayleigh waves in semi-infinite isotropic solids are studied by Lockett [6], Deresiewicz [7], Nayfeh and Nemat-Nasser [8], Carroll [9], Agarwal [10], Dawn and Chakraborty [11], and many others with various additional parameters.

Initial stresses in solids have significant influence on the mechanical response of the material from an initially-stressed configuration and have applications in geophysics, engineering structures and in the behaviour of soft biological tissues. Initial stress arises from processes, such as manufacturing or growth, and is present in the absence of applied loads. Montanaro [12] formulated the isotropic thermoelasticity with hydrostatic initial stress. Singh et al. [13], Othman et al. [14], Singh [15], and many others have applied Montanro [12] theory to study the plane harmonic waves in context of generalized thermoelasticity.

In the present paper, the governing equations given by Montanaro [12] are modified in context of Lord and Shulman and Green and Lindsay theories with uniform magnetic field. These equations are solved for the surface wave solutions, which satisfy the required boundary conditions at the free surface and we obtain the frequency equation for the Rayleigh wave in the half-space. The frequency equation is approximated and analyzed numerically to observe the effects of hydrostatic initial stress and magnetic field on the velocity of propagation and amplitude-attenuation factor.

2. Basic equations

We consider an isotropic thermoelastic solid with hydrostatic initial stress under constant primary magnetic field $\mathbf{H_0}$ acting on y-axis. Following Lord and Shulman [1], Green and Lindsay [2] and Montanaro [12], the governing equations of linear, isotropic and homogenous magneto-thermoelastic solid with hydrostatic initial stress are

(i) The stress-strain-temperature relation:

$$\begin{aligned}\sigma_{ij} = -p(\delta_{ij}+\omega_{ij}) + \overline{\lambda} e_{pp}\delta_{ij} + 2\overline{\mu} e_{ij} \\ -\frac{\alpha}{\kappa_T}(T + a\dot{T})\delta_{ij},\end{aligned} \tag{1}$$

(ii) The displacement-strain relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

(iii) The small rotation-displacement relation:

$$\omega_{ij} = \frac{1}{2}(u_{j,i} - u_{i,j}), \tag{3}$$

(iv) The modified Fourier's law:

$$h_i + a^* \dot{h_i} = K\frac{\partial T}{\partial x_i}, \tag{4}$$

(v) The equation of motion:

$$\begin{aligned}\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \left(\overline{\mu} - \frac{p}{2}\right)\frac{\partial^2 u_i}{\partial x_p \partial x_p} \\ + \left(\overline{\lambda} + \overline{\mu} + \frac{p}{2}\right)\frac{\partial^2 u_p}{\partial x_i \partial x_p} - \frac{\alpha}{\kappa_T}\frac{\partial T}{\partial x_i} + \overline{\sigma}_{ip,p},\end{aligned} \tag{5}$$

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(vi) The equation of heat conduction:

$$\rho_0 c_v \left(1 + a^* \frac{\partial}{\partial t}\right) \frac{\partial T}{\partial t} + T_0 \frac{\alpha}{\kappa_T} \left(1 + \Delta a^* \frac{\partial}{\partial t}\right) \frac{\partial^2 u_p}{\partial t \partial x_i} = K \frac{\partial^2 T}{\partial x_p \partial x_p}, \tag{6}$$

(vii) Maxwell equations governing the electromagnetic field:

$$\nabla \times \mathbf{h} = \mathbf{j}, \qquad \nabla \times \mathbf{E} = -\mu_e \frac{\partial \mathbf{h}}{\partial t}, \qquad \nabla.\mathbf{h} = 0, \qquad \nabla.\mathbf{E} = 0, \tag{7}$$

(viii) Maxwell stresses:

$$\overline{\sigma}_{ij} = \mu_e [H_i h_j + H_j h_i - (\mathbf{H}.\mathbf{h})\delta_{ij}], \tag{8}$$

where $T = \Theta - T_0$ is small temperature increment, Θ is the absolute temperature of the medium, T_0 is the reference uniform temperature of the body chosen such that $\left|\frac{T}{T_0}\right| \ll 1$, ρ_0 is the mass density, q_i is the heat conduction vector, K is the thermal conductivity, c_v is the specific heat at constant strain, $\overline{\lambda}, \overline{\mu}$ are the counterparts of Lame parameters, α is the volume coefficient of thermal expansion, κ_T is the isothermal compressibility, σ_{ij} are the components of the stress tensor, u_i are the components of the displacement vector, e_{ij} are the components of the strain tensor, ω_{ij} are the components of the small rotation tensor, δ_{ij} is the Kronecker delta, $a, a^* \geq 0$ are the thermal relaxation times, p is the initial pressure, $\mathbf{h}$ is the perturbed magnetic field over $\mathbf{H_0}$, $\mathbf{j}$ is the electric current density, μ_e is the magnetic permeability, $\mathbf{h} = \nabla \times (\mathbf{u} \times \mathbf{H_0})$ and $\mathbf{H} = \mathbf{H_0} + \mathbf{h}$. The above governing equations reduce for L-S (Lord-Shulman) theory when $a = 0$, $\Delta = 1$ and for G-L(Green-Lindsay) theory, when $\Delta = 0$.

3. Formulation of the problem

For Rayleigh type waves in the half space $z \geq 0$, using the representation of displacement components

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \qquad u_2 = 0, \qquad u_3 = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \tag{9}$$

where ϕ and ψ are functions of x, z and t, Eqs. (5) and (6) are satisfied if

$$\frac{\partial^2 \phi}{\partial t^2} = {c_1}^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) - \frac{\gamma}{\rho_0}(T + a\dot{T}), \tag{10}$$

$$\frac{\partial^2 \psi}{\partial t^2} = {c_2}^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}\right), \tag{11}$$

$$\rho_0 c_v (\dot{T} + a^* \ddot{T}) + \gamma T_0 \left(1 + \Delta a^* \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right] = K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right), \tag{12}$$

where

$${c_1}^2 = \frac{\overline{\lambda} + 2\overline{\mu} + \mu_e {H_0}^2}{\rho_0}, \qquad {c_2}^2 = \frac{\overline{\mu} - \frac{p}{2}}{\rho_0}, \qquad \gamma = \frac{\alpha}{\kappa_T}.$$

Introducing the following non-dimensional quantities,

$$x' = \frac{x}{(c_1/\omega^*)}, \qquad z' = \frac{z}{(c_1/\omega^*)}, \qquad t' = t\omega^*,$$
$${u_1}' = \frac{u_1}{(c_1/\omega^*)}, \qquad u_3' = \frac{u_3}{(c_1/\omega^*)}, \qquad T' = \frac{\gamma T}{\rho_0 {c_1}^2},$$
$$\phi' = \frac{\phi}{(c_1/\omega^*)^2}, \qquad \psi' = \frac{\psi}{(c_1/\omega^*)^2},$$
$$a' = a\omega^*, \qquad {a^*}' = a^*\omega^*, \qquad \omega^* = \frac{\rho_0 c_v {c_1}^2}{\kappa},$$

in the Eqs. (9)–(12) and suppressing the primes, we obtain the equations in dimensionless form as

$$u_1 = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \qquad u_2 = 0, \qquad u_3 = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}, \tag{13}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) - (T + a\dot{T}), \tag{14}$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{v^2} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2}\right), \tag{15}$$

$$(\dot{T} + a^* \ddot{T}) + \epsilon \left(1 + \Delta a^* \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right] = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right), \tag{16}$$

where, the thermoelastic coupling is given by

$$\epsilon = \frac{\gamma^2 T_0}{{\rho_0}^2 c_v {c_1}^2}, \qquad v^2 = \frac{c_1^2}{c_2^2}. \tag{17}$$

The mechanical and thermal conditions at the boundary $z = 0$ are

$$\sigma_{13} + \overline{\sigma}_{13} = 0, \qquad \sigma_{33} + \overline{\sigma}_{33} = -p, \qquad \frac{\partial T}{\partial z} + hT = 0, \tag{18}$$

where $h \to 0$ corresponds the thermally insulated surface and $h \to \infty$ corresponds the isothermal surface.

4. Solutions and the frequency equation

For thermoelastic surface waves in the half-space propagating in x-direction, the functions (T, ϕ, ψ) may be taken in the form

$$\{T, \phi, \psi\} = \{\hat{T}(z), \hat{\phi}(z), \hat{\psi}(z)\} \exp\ \{i(\eta x - \chi t)\}. \tag{19}$$

Substituting (19) in equations (14)-(16) and noting that $\hat{T}, \hat{\phi}, \hat{\psi} \to 0$ as $z \to \infty$ for surface waves, the solution is obtained as

$$\phi = [A \exp(-\eta\beta_1 z) + B \exp(-\eta\beta_2 z)] \exp i(\eta x - \chi t), \tag{20}$$

$$\psi = C \exp\{(-\eta\beta_3 z) + i(\eta x - \chi t)\}, \tag{21}$$

$$T = \frac{1}{1 - ia\chi}[A\{\chi^2 + \eta^2({\beta_1}^2 - 1)\} \exp(-\eta\beta_1 z) + B\{\chi^2 + \eta^2({\beta_2}^2 - 1)\} \exp(-\eta\beta_2 z)] \exp\{i(\eta x - \chi t)\}, \tag{22}$$

where

$$\beta_3^2 = (1 - c^2 v^2), \qquad c^2 = \frac{\chi^2}{\eta^2},$$

and β_1, β_2 are the roots of the following equation with $Re(\beta) \gg 0$

$$\begin{aligned} &\beta^4 + \beta^2[-2 + c^2(\overline{\overline{a}} + 1 - i\chi\overline{a}\overline{\overline{a}}^{\Delta}\epsilon)] \\ &+1 - c^2(\overline{a} + 1 + i\chi\overline{a}\overline{\overline{a}}^{\Delta}\epsilon) + c^4\overline{\overline{a}} = 0, \end{aligned} \tag{23}$$

where

$$\overline{a} = a + \frac{i}{\chi}, \qquad \overline{\overline{a}} = a^* + \frac{i}{\chi}, \qquad \overline{\overline{a}}^{\Delta} = a^* + \Delta\frac{i}{\chi}.$$

The solutions (20)–(22) satisfy the boundary conditions (18), and we obtain the following frequency equation

$$\begin{aligned} &[2 - c^2v^2 + p_1(1 - c^2v^2)][\beta_1^2 + \beta_2^2 + \beta_1\beta_2 - 1 + c^2] \\ &-[4 + 2p_1 + (2p_2 + p_1p_2)v^2]\beta_1\beta_2\beta_3(\beta_1 + \beta_2) \\ &= -\frac{h}{\eta}[(\beta_1^2 + \beta_2^2)\{2 + (p_2 - c^2)v^2\} \\ &\cdot\{2 - c^2v^2 + p_1(1 - c^2v^2)\} \\ &-\{4 + 2p_1 + (2p_2 + p_1p_2)v^2\}\beta_3(\beta_1^2\beta_2^2 + 1 - c^2)], \end{aligned} \tag{24}$$

where

$$p_1 = \frac{p}{\rho_0 c_2^2}, \qquad p_2 = \frac{p}{\rho_0 c_1^2}, \qquad \overline{\gamma} = \frac{\gamma}{\rho_0}.$$

5. Limiting cases

(a) Small thermal coupling

For most of the materials, ϵ is small at normal temperature. Therefore, we approximated the frequency equation by assuming $\epsilon \ll 1$.

For $\epsilon \ll 1$, we obtain from equation (23) the approximated expressions for β_1 and β_2 as

$$\beta_1 \simeq (1 - c^2)^{1/2}\left[1 + \frac{\epsilon}{2}\frac{c^2\chi\overline{a}\overline{\overline{a}}^{\Delta}}{(\overline{\overline{a}} - 1)(1 - c^2)}\right], \tag{25}$$

$$\beta_2 \simeq (1 - \overline{\overline{a}}c^2)^{1/2}\left[1 - \frac{\epsilon}{2}\frac{c^2\chi\overline{a}\overline{\overline{a}}\overline{\overline{a}}^{\Delta}}{(\overline{\overline{a}} - 1)(1 - \overline{\overline{a}}c^2)}\right]. \tag{26}$$

These approximated expressions for β_1 and β_2 are inserted in equation (24) to obtain the approximated frequency equation.

(b) Small reduced frequency $\chi \ll 1$

For small reduced frequency $\chi \ll 1$, we obtain from equation (23), the following approximated expressions

$${\beta_1}^2 + {\beta_2}^2 \simeq 2 - i\frac{c^2}{\chi}(1 + \epsilon) - c^2[1 + a^* + \epsilon(a + \Delta a^*)], \tag{27}$$

$$\begin{aligned} &\beta_1\beta_2 \simeq \frac{1 - i}{\sqrt{2\chi}}(1 - c^2 + \epsilon)^{1/2} \\ &+\frac{1 + i}{2\sqrt{2}}\frac{[1 + a^* + \epsilon(a + \Delta aa^*)]}{(1 - c^2 + \epsilon)^{1/2}}\chi^{1/2}. \end{aligned} \tag{28}$$

6. Numerical analysis of the frequency equation

If we put $c^2 = {c^*}^2 + \epsilon(\xi_1 + i\xi_2)$, where c^* is the classical Rayleigh wave velocity and ξ_1 and ξ_2 are two reals depending on the reduced frequency χ and a, a^*, then

$$\eta = \frac{\chi}{c^*}\left(1 - \frac{\epsilon\xi_1}{2{c^*}^2} - i\frac{\epsilon\xi_2}{2{c^*}^2}\right). \tag{29}$$

The velocity of propagation is equal to $\left(c^* + \frac{\epsilon\xi_1}{2c^*}\right)$ and the amplitude-attenuation factor is equal to $\exp\left[\frac{\epsilon\chi\xi_2 x}{2{c^*}^3}\right]$ with $\xi_2 < 0$. The velocity of propagation and amplitude-attenuation factor are computed for the following material parameters
$E = 6.9 \times 10^{10}$ N. m^{-2}, $\sigma = 0.33$, $\rho_0 = 2700$ Kg.m^{-3},
$c_v = 987.9$ J. Kg^{-1}. K^{-1}, $K = 205.85$ J.m^{-1}. s^{-1}. K^{-1},
$\alpha = 0.01$, $\kappa_T = 0.05$, $\omega = 2\ s^{-1}$, $T_0 = 293$ K,
$x = .01$ m, $a = 0.05$ s, $a^* = 0.2$ s, $\epsilon = 0.05$,
$\mu_e = 1$, $c^* = 0.9554$, $\chi = 0.1$.

The generalized Lame's constants $\overline{\lambda}$ and $\overline{\mu}$ are related as

$$\overline{\lambda} = \frac{E\sigma}{\zeta(1 + \sigma)(1 - 2\sigma)}, \qquad \overline{\mu} = \frac{E}{2\zeta(1 + \sigma)} \tag{30}$$

where ζ is initial stress parameter, E is Young's modulus and σ is Poisson ratio. $\zeta = 1$ corresponds to the isotropic elastic medium.

The velocity of propagation and amplitude-attenuation factor are plotted against magnetic field parameter H in Figs. 1 and 2, respectively, when $p = -2, 0$ and 2. For $p = -2$, the velocity is 0.4774×10^3 near $H = 0$ and it decreases slowly with the increase in the magnetic field. It attains its value 0.4751×10^3 at $H = 20 \times 10^5$ oe. The variation of the velocity for $p = -2$ is shown by solid line in Fig. 1. For $p = 2$, the velocity is 0.73964×10^3 near $H = 0$ and it decreases sharply with the increase in the magnetic field. It attains its value 0.48064×10^3 at $H = 20 \times 10^5$ oe. The variation of the velocity for $p = 2$ is shown by solid line with circles as center symbols in Fig. 1. In absence of initial stress, the variations for $p = -2$ and 2 reduce to that shown by solid line with triangles as center symbols in Fig. 1. The amplitude-attenuation factors ($\times 10^3$) for $p = -2, 0$ and 2 are shown graphically against magnetic field in Fig. 2. From Figs. 1 and 2, it is observed that the velocity of propagation and amplitude-attenuation factors are significantly affected by initial stress parameter for lower range of the magnetic field parameter.

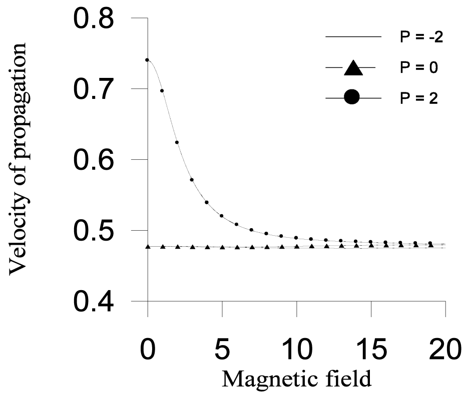

Fig. 1. Variations of the velocities of propagation of Rayleigh wave versus the magnetic field parameter, when $p = -2, 0$ and 2

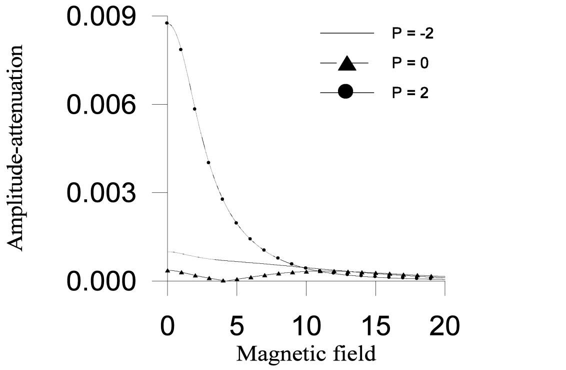

Fig. 2. Variations of the amplitude-attenuation factors of Rayleigh wave versus the magnetic field parameter, when $p = -2, 0$ and 2

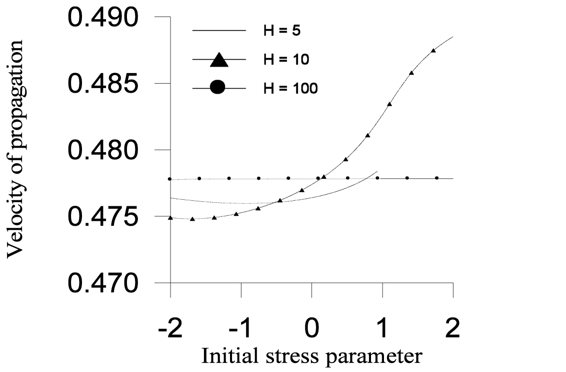

Fig. 3. Variations of the velocities of propagation of Rayleigh wave versus the hydrostatic initial stress parameter, when $H = 0, 10 \times 10^5$ oe and 100×10^5 oe.

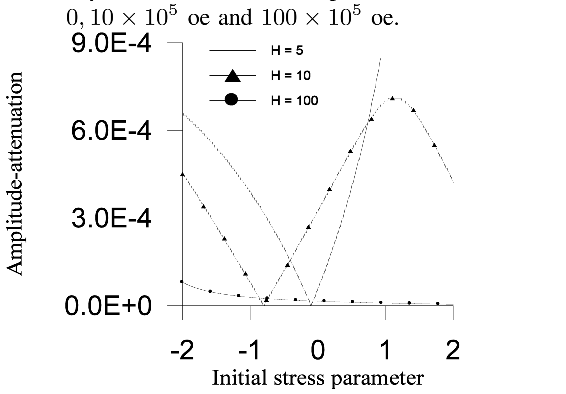

Fig. 4. Variations of the amplitude-attenuation factors of Rayleigh wave versus the hydrostatic initial stress parameter, when $H = 0, 10 \times 10^5$ oe and 100×10^5 oe.

The velocity of propagation and amplitude-attenuation factor are also plotted against initial stress parameter p in Figs. 3 and 4, respectively, when $H = 0, 10 \times 10^5$ oe and 100×10^5 oe. For $H = 10 \times 10^5$ oe, the velocity is 0.4749×10^3 at p = -2 and it decreases slowly to its minimum value 0.47481×10^3 at p = -1.7. Thereafter, it attains its maximum value 0.48851×10^3 at p = 2. The variation of the velocity for H = 10 is shown by solid line with triangles as center symbols in Fig. 3. For $H = 100 \times 10^5$ oe, the velocity is 0.47775×10^3 at $p = -2$ and it increases slowly for the given range of p and attains its maximum value 0.47783×10^3 at $p = 2$. The variation of the velocity for $H = 100$ is shown by solid line with circles as center symbols in Fig. 3. In absence of magnetic field, the variations for $H = 10 \times 10^5$ oe and 100×10^5 oe reduce to that shown by solid line without center symbols in Fig. 3. The amplitude-attenuation factors $(\times 10^3)$for $H = 0, 10 \times 10^5$ oe and 100×10^5 oe are shown graphically against initial stress in Fig. 4. From Figs. 3 and 4, it is observed that the velocity of propagation and amplitude-attenuation factors are significantly affected by magnetic field at each value of initial stress parameter.

7. Conclusions

The frequency equation of Rayleigh wave in a magneto-thermoelastic half-space with hydrostatic initial stress is obtained. The frequency equation is approximated for small thermal coupling and small reduced frequency and the expressions for the velocity of propagation and amplitude-attenuation factors are obtained and computed numerically for a particular material. The velocity and the amplitude-attenuation factor are significantly influenced by hydrostatic initial stress and magnetic field parameters.

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