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Local controllability of nonlinear discrete-time fractional order systems

D. MOZYRSKA[1]* and E. PAWŁUSZEWICZ[2]

[1] Faculty of Computer Science, Bialystok University of Technology, 45A Wiejska St., 15-351 Białystok, Poland
[2] Faculty of Mechanical Engineering, Bialystok University of Technology, 45C Wiejska St., 15-351 Białystok, Poland

Abstract. The Riemann-Liouville, Caputo and Grünwald-Letnikov fractional order difference operators are discussed and used to state and solve the controllability problem of a nonlinear fractional order discrete-time system. It is shown that independently of the type of fractional order difference, such a system is locally controllable in q steps if its linear approximation is globally controllable in q steps.

Key words: fractional difference operator, fractional difference initial value problem, nonlinear fractional order system, linear approximation, controllability problem.

1. Introduction

Considering the increase in practical use of fractional integro-derivatives or fractional differences in systems modeling real behaviors (see for example [1–3]), there has been recently a growing interest in developing this topic, from both theoretical and practical points of view. The fractional calculus in continuous and discrete cases includes different notions of definitions of derivatives, e.g. Riemann-Liouville, Grünwald-Letnikov, Caputo and generalized function approach [2, 4–8].

The aim of this paper is to study the problem of local controllability of nonlinear discrete-time fractional order systems with different notions of fractional differences. We use three types of forward differences: the fractional Riemann-Liouville type difference defined in [5], fractional Caputo type difference defined in [9, 10] and fractional Grünwald-Letnikov type difference presented in [2, 11–13]. The obtained results are based on [14, 15].

In order to find a solution of the controllability problem stated for the considered fractional-order systems we use linear approximation of it. In this case the solution of the linear state-space equation is derived using a discrete version of Mittag-Leffler two parameters matrix function. In case of fractional Grünwald-Letnikov type difference, Kalman's type controllability condition is shown in [6]. Our main result indicates that the nonlinear fractional order control system, for any of the discussed fractional order difference operators, is locally controllable in a finite number of steps if its linear approximation is globally controllable in the same number of steps. For the proof we use results obtained by Graves in [16] and used by Walczak [17] to the controllability problem in a continuous-time case.

2. Preliminaries

Let

$$\mathbb{N}_a := \{a, a+1, \ldots\}$$

for any real number a. For $t \in \mathbb{R}\backslash\mathbb{Z}_-$ and $\alpha \in \mathbb{R}$ the factorial function is defined by

$$t^{(\alpha)} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \tag{1}$$

where $\mathbb{Z}_- = \{-1, -2, \ldots\}$, Γ is the Euler gamma function and we use the convention that division at a pole yields zero.

Let $\alpha > 0$. The *α-th fractional sum* for any function $\varphi : \mathbb{N}_a \to \mathbb{R}$, is defined by, (follow [5]),

$$\left({}_a\Delta^{-\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)}\varphi(s), \tag{2}$$

where ${}_a\Delta^{-\alpha}\varphi$ is defined for all $t \in \mathbb{N}_{a+\alpha}$. We assume that $\left({}_a\Delta^{0}\varphi\right)(t) := \varphi(t)$. Observe that according to (2) we have that $\left({}_a\Delta^{-\alpha}\varphi\right)(a+\alpha) = \varphi(a)$.

Moreover, for $\varphi(s) = (s-a+\mu)^{(\mu)}$ holds, see [5, 9, 18],

$$\left({}_a\Delta^{-\alpha}\varphi\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha)} \tag{3}$$

for any $t \in \mathbb{N}_{a+\mu+\alpha}$ and $\mu \notin \mathbb{Z}_-$ with the convention that division at a pole yields zero. In particular, if C is a constant, then (3) implies that $\left({}_a\Delta^{-\alpha}C\right)(t) = \dfrac{C}{\Gamma(\alpha+1)}(t-a)^{(\alpha)}$ for $t \in \mathbb{N}_{a+\alpha}$. The formula (3) can be also presented as, see [18, 19],

$${}_{a+\mu}\Delta^{-\alpha}(t-a)^{(\mu)} = \mu^{(-\alpha)}(t-a)^{(\mu+\alpha)}$$

for any $t \in \mathbb{N}_{a+\mu+\alpha}$.

Proposition 1 [5]. Let φ be a real-valued function defined on $\mathbb{N}_a$ and let $\alpha, \beta > 0$. Then,

$$({}_{a+\beta}\Delta^{-\alpha}({}_a\Delta^{-\beta}\varphi))(t) = ({}_a\Delta^{-(\alpha+\beta)}\varphi)(t) = \left({}_{a+\alpha}\Delta^{-\beta}\left({}_a\Delta^{-\alpha}\varphi\right)\right)(t)$$

for any $t \in \mathbb{N}_{a+\alpha+\beta}$.

We present three different approaches to defining fractional differences. In the next section we use them as operators

*e-mail: d.mozyrska@pb.edu.pl

acting in control systems and show that for each kind of system we have a similar controllability condition.

Definition 2 [9, 10]. Let $\alpha \in (0,1]$ and $\varphi : \mathbb{N}_a \to \mathbb{R}$. The *$\alpha$-th fractional Caputo type difference* of the function φ is defined as

$$\left({}_a\Delta_*^\alpha \varphi\right)(t) = \left({}_a\Delta^{-(1-\alpha)}\Delta\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{s=a}^{t-(1-\alpha)} (t-s-1)^{(-\alpha)}(\Delta\varphi)(s),$$

where $t \in \mathbb{N}_{a+(1-\alpha)}$ and $(\Delta\varphi)(s) = \varphi(s+1) - \varphi(s)$ is the classical forward difference.

Definition 3 [5]. Let $\alpha \in (0,1]$ and $\varphi : \mathbb{N}_a \to \mathbb{R}$. The *$\alpha$-th fractional Riemann-Liouville type difference* of the function φ is defined as

$$\left({}_a\Delta^\alpha \varphi\right)(t) = \Delta\left({}_a\Delta^{-(1-\alpha)}\varphi\right)(t), \tag{4}$$

where $t \in \mathbb{N}_{a+(1-\alpha)}$.

Particularly for $\alpha = 1$ we have that $\left({}_a\Delta_*^1 \varphi\right)(t) = \left({}_a\Delta^1\varphi\right)(t) = (\Delta\varphi)(t)$.

It can be shown (see [10]) that for any $\varphi : \mathbb{N}_a \to \mathbb{R}$ the following relation between Riemann-Liouville type difference operator and Caputo type difference operator is true.

$$\left({}_a\Delta^\alpha \varphi\right)(t) = \left({}_a\Delta_*^\alpha \varphi\right)(t) + \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)}\varphi(a).$$

The third definition of a fractional order difference can be presented as follows, see for example [2, 11, 12].

Definition 4. Let α be any real number. The *α-th fractional Grünwald-Letnikov type difference* is defined as

$$\left({}_a\Delta_\natural^\alpha \varphi\right)(t) = \sum_{s=0}^{t-a}(-1)^s\binom{\alpha}{s}\varphi(t-s),$$

where $t \in \mathbb{N}_a$ and $\binom{\alpha}{s} = \dfrac{\Gamma(\alpha+1)}{\Gamma(s+1)\Gamma(\alpha-s+1)}$ is the binomial coefficient.

In [4] versions of solutions of scalar fractional order difference equation are given. We adopt them to multi-variable fractional order linear case and define Mittag-Leffler matrix functions as well. Firstly let us introduce the Mittag-Leffler matrix function.

Definition 5. Let A be a square real matrix of degree n and let $\beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$. The *discrete Mittag-Leffler two-parameters matrix function* is defined as

$$E_{(\alpha,\beta)}(A,z) = \sum_{k=0}^{\infty} A^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k+\beta)}.$$

For $\beta = 1$ we write

$$E_{(\alpha)}(A,z) = E_{(\alpha,\beta=1)}(A,z) = \sum_{k=0}^{\infty} A^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k+1)}.$$

For our purpose we use the following form for the Mittag-Leffler two-parameters matrix function:

$$E_{(\alpha,\alpha-1)}(A,z) = \sum_{k=0}^{\infty} A^k \frac{(z+k(\alpha-1))^{(k\alpha+\alpha-1)}}{\Gamma((k+1)\alpha)} \tag{5}$$

and the following technical lemma.

Lemma 6 [4]. For $\alpha > 0$, $k \in \mathbb{N}$ and $f : \mathbb{N}_0 \to \mathbb{R}$ holds

$$\left({}_0\Delta^{-\alpha}\left({}_0\Delta^{-k\alpha}f\right)\right)(t+k\mu) = \left({}_0\Delta^{-(k+1)\alpha}f\right)(t+k\mu).$$

3. Fractional initial value problems

In this Section we discuss the problem of solvability of fractional order systems of difference equations, for each of the operators defined in Preliminaries. For the Caputo type fractional difference we use the following form of systems, similarly to scalar case in [4, 9].

Theorem 7. Let $f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\alpha \in (0,1]$ and $\mu = \alpha - 1$. The initial value problem (IVP)

$$\left({}_\mu\Delta_*^\alpha x\right)(t) = f(t, x(t+\mu)), \qquad t \in \mathbb{N}_0, \tag{6}$$

$$x(\mu) = x_0, \qquad x_0 \in \mathbb{R}^n, \tag{7}$$

has the unique solution given by the recurrence formula

$$x(t) = x_0 + \left({}_0\Delta^{-\alpha}\overline{f}\right)(t) = x_0 + \sum_{s=0}^{t-\alpha}\frac{(t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)} f(s, x(s+\mu)),$$

where $t \in \mathbb{N}_\alpha = \{\alpha, \alpha+1, \ldots\}$ and $\overline{f}(s) = f(s, x(s+\mu))$.

Proposition 8. In IVP problem (6)–(7), let the function f have values defined by $f(t,x) = Ax + Bu(t)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, $\alpha \in (0,1]$, $\mu = \alpha - 1$ and $u(t), t \in \mathbb{N}_0$, is a fixed control. The linear initial value problem

$$\left({}_\mu\Delta_*^\alpha x\right)(t) = Ax(t+\mu) + Bu(t), \qquad t \in \mathbb{N}_0,$$

$$x(\mu) = x_0, \qquad x_0 \in \mathbb{R}^n,$$

has the unique solution given by the formula

$$x(t) = E_{(\alpha)}(A,t)\,x_0 + \sum_{s=0}^{t-\alpha} E_{(\alpha,\alpha-1)}(A, t-\sigma(s))\,Bu(s), \tag{8}$$

where $t \in \mathbb{N}_\mu$ and $\sigma(s) := s+1$.

Proof. We only need to apply the method of successive approximations as it was done in [4] using the crucial power rule formula. Then, let us define the following sequence. Set $x_0(t) = x_0$ and

$$x_m(t) = x_0 + A\left({}_0\Delta^{-\alpha}\overline{x}_{m-1}\right)(t+\mu) + B\left({}_0\Delta^{-\alpha}u\right)(t),$$

where $m \in \mathbb{N}$ and $\overline{x}_{m-1}(s) = x_{m-1}(s+\mu) \in \mathbb{R}^n$. For $m = 1$ we have

$$x_1(t) = \left(I + A\frac{t^{(\alpha)}}{\Gamma(\alpha+1)}\right)x_0 + B\left({}_0\Delta^{-\alpha}u\right)(t).$$

Next for $m = 2$:

$$x_2(t) = \left(I + A \frac{t^{(\alpha)}}{\Gamma(\alpha+1)} + A^2 \frac{(t+\mu)^{(2\alpha)}}{\Gamma(2\alpha+1)} \right) x_0 + B \left({}_0\Delta^{-\alpha} u \right)(t) + AB \left({}_0\Delta^{-2\alpha} u \right)(t+\mu).$$

Using Lemma 6 and taking $m \to \infty$ we obtain the direct solution

$$x(t) = E_{(\alpha)}(A,t)\, x_0 + \sum_{k=1}^{\infty} A^{k-1} B \left({}_0\Delta^{-k\alpha} u \right)(t + (k-1)\mu).$$

Then

$$x(t) = E_{(\alpha)}(A,t)\, x_0 + \sum_{k=0}^{\infty} A^k \frac{1}{\Gamma(k\alpha+\alpha)} \sum_{s=0}^{t-\alpha} (t + k\mu - \sigma(s))^{(k\alpha+\mu)} Bu(s).$$

Finally we get the formula (8).

For the Riemann-Liouville type fractional difference we use the following form of systems, similarly to the scalar case in [18].

Theorem 9. Let $f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\alpha \in (0,1]$ and $\mu = \alpha - 1$. The initial value problem (IVP)

$$\left({}_\mu\Delta^{\alpha} x \right)(t) = f(t, x(t+\mu)), \qquad t \in \mathbb{N}_0, \tag{9}$$

$$x(\mu) = x_0, \qquad x_0 \in \mathbb{R}^n, \tag{10}$$

has the unique solution given by the recurrence formula

$$x(t) = \frac{t^{(\mu)}}{\Gamma(\alpha)} x_0 + \left({}_0\Delta^{-\alpha} \overline{f} \right)(t) = \frac{t^{(\mu)}}{\Gamma(\alpha)} x_0 + \sum_{s=0}^{t-\alpha} \frac{(t-s-1)^{(\alpha-1)}}{\Gamma(\alpha)} f(s, x(s+\mu)), \tag{11}$$

for any $t \in \mathbb{N}_\mu$ and $\overline{f}(s) = f(s, x(s+\mu))$.

Proposition 10. In IVP problem (9)–(10), let the function f have values defined by $f(t,x) = Ax + Bu(t)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$, $\alpha \in (0,1]$, $\mu = \alpha - 1$ and $u(t), t \in \mathbb{N}_0$, is a fixed control. The linear initial value problem

$$\left({}_\mu\Delta^{\alpha} x \right)(t) = Ax(t+\mu) + Bu(t), \qquad t \in \mathbb{N}_0,$$

$$x(\mu) = x_0, \qquad x_0 \in \mathbb{R}^n,$$

has the unique solution given by the formula

$$x(t) = E_{(\alpha,\alpha-1)}(A,t)\, x_0 + \sum_{s=0}^{t-\alpha} E_{(\alpha,\alpha-1)}(A, t - \sigma(s))\, Bu(s), \tag{12}$$

for any $t \in \mathbb{N}_\mu$.

Proof. We use a similar method as in the proof of Proposition 8. Starting with the recurrence formula (11) we define the following sequence. Set

$$x_0(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} x_0$$

and

$$x_m(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} x_0 + A \left({}_0\Delta^{-\alpha} \overline{x}_{m-1} \right)(t) + B \left({}_0\Delta^{-\alpha} u \right)(t),$$

where $m \in \mathbb{N}$ and $\overline{x}_{m-1}(s) = x_{m-1}(s+\mu) \in \mathbb{R}^n$. For $m = 1$ we have

$$x_1(t) = \left(\frac{t^{(\alpha-1)}}{\Gamma(\alpha)} I + A \frac{(t+\mu)^{(2\alpha-1)}}{\Gamma(2\alpha)} \right) x_0 + B \left({}_0\Delta^{-\alpha} u \right)(t).$$

Next for $m = 2$:

$$x_2(t) = \left(\frac{t^{(\alpha-1)}}{\Gamma(\alpha)} I + A \frac{(t+\mu)^{(2\alpha-1)}}{\Gamma(2\alpha)} + A^2 \frac{(t+2\mu)^{(3\alpha-1)}}{\Gamma(3\alpha)} \right) x_0 + B \left({}_0\Delta^{-\alpha} u \right)(t) + AB \left({}_0\Delta^{-2\alpha} u \right)(t+\mu).$$

Now using Lemma 6, formula (5) and taking $m \to \infty$ we obtain the direct solution

$$x(t) = E_{(\alpha,\alpha-1)}(A,t)\, x_0 + \sum_{k=1}^{\infty} A^{k-1} B \left({}_0\Delta^{-k\alpha} u \right)(t + (k-1)\mu).$$

Then

$$x(t) = E_{(\alpha,\alpha-1)}(A,t)\, x_0 + \sum_{k=0}^{\infty} A^k \frac{1}{\Gamma(k\alpha+\alpha)} \sum_{s=0}^{t-\alpha} (t + k\mu - \sigma(s))^{(k\alpha+\mu)} Bu(s).$$

Finally we get the formula (12).

For the Grünwald-Letnikov type fractional difference we use the following form of systems see [3, 11, 12, 20].

Theorem 11. Let $f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\alpha \in (0,1]$. The initial value problem (IVP)

$$\left({}_0\Delta^{\alpha}_{h} x \right)(t+1) = f(t, x(t)), \qquad t \in \mathbb{N}_0, \tag{13}$$

$$x(0) = x_0, \qquad x_0 \in \mathbb{R}^n, \tag{14}$$

has the unique solution given by the recurrence formula

$$x(t+1) = f(t, x(t)) - \sum_{s=1}^{t+1} (-1)^s \binom{\alpha}{s} x(t-s+1)$$

for any $t \in \mathbb{N}_0$.

Proposition 12 [15, 21]. In IVP problem (13)–(14) let the function f have values defined by $f(t,x) = Ax + Bu(t)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$, $\alpha \in (0,1]$ and $u(t), t \in \mathbb{N}_0$, is a fixed control. The linear initial value problem

$$\left({}_0\Delta^{\alpha}_{h} x \right)(t+1) = Ax(t) + Bu(t), \qquad t \in \mathbb{N}_0,$$

$$x(0) = x_0, \qquad x_0 \in \mathbb{R}^n,$$

has the unique solution given by the formula

$$x(t) = \Phi(t) x_0 + \sum_{s=0}^{t-1} \Phi(t-s-1) Bu(s)$$

for any $t \in \mathbb{N}_0$ and $n \times n$ dimensional state transition matrices $\Phi(t)$, are determined by the recurrence formula

$$\Phi(t+1) = (A + I_n \alpha)\, \Phi(t) + \sum_{i=2}^{t+1} (-1)^{i+1} \binom{\alpha}{i} \Phi(t-i+1)$$

with $\Phi(0) = I_n$, where I_n is $n \times n$ dimensional identity matrix.

4. Controllability conditions

Let us consider nonlinear fractional order discrete-time control systems

$$\begin{aligned}({}_{\mu}\Delta_*^{\alpha}x)(t) &= f(x(t+\mu), u(t)),\\ x(\mu) &= x_0, \quad t \in \mathbb{N}_0,\end{aligned} \tag{15}$$

$$\begin{aligned}({}_{\mu}\Delta^{\alpha}x)(t) &= f(x(t+\mu), u(t)),\\ x(\mu) &= x_0, \quad t \in \mathbb{N}_0,\end{aligned} \tag{16}$$

$$\begin{aligned}({}_{0}\Delta_{\natural}^{\alpha}x)(t+1) &= f(x(t), u(t)),\\ x(0) &= x_0, \quad t \in \mathbb{N}_0.\end{aligned} \tag{17}$$

Since some definitions and facts that we discuss are the same for each type system, difference operators in the left hand side in equations (15)–(17) are denoted by the common symbol defined by its values

$$\left({}_{a}\Upsilon^{\alpha}x\right)(t) = \begin{cases} ({}_{\mu}\Delta_*^{\alpha}x)(t) \text{ or } ({}_{\mu}\Delta^{\alpha}x)(t) & \text{for } a = \mu \\ ({}_{0}\Delta_{\natural}^{\alpha}x)(t+1) & \text{for } a = 0 \end{cases}.$$

Hence we consider the following common form for control systems (15)–(17):

$$\begin{aligned}\left({}_{a}\Upsilon^{\alpha}x\right)(t) &= f(x(t+a), u(t)),\\ x(a) &= x_0, \quad t \in \mathbb{N}_0,\end{aligned} \tag{18}$$

where $x(\cdot) \in \mathbb{R}^n$ denotes the state vector, the values $u(t)$ of control u are elements of an arbitrary set $\Omega \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$. We assume that $0 \in \operatorname{int}\Omega$ and

A1. f is a (classically) differentiable function at $(0, 0)$.

A2. $f(0, 0) = 0$.

The set Ω is usually called the control space and satisfies the following property: $\Omega \subseteq \mathbb{R}^m$ is such that $\Omega \subseteq \overline{\operatorname{int}\Omega}$ and any two points in the same connected component of Ω can be jointed by a smooth curve lying in $\operatorname{int}\Omega$, except for end points.

Let $J_0(m)$ denotes the set of all sequences $U = (u_0, u_1, \ldots)$ where $u_t := u(t) \in \Omega$, $t \in \mathbb{N}_0$. Let $\gamma(\cdot, x_0, U)$ be defined by its values $\gamma(t, x_0, U) = x(t)$, $t \in \mathbb{N}_a$, and denote the state forward trajectory of system (18), i.e. a solution which is uniquely defined by initial state x_0 and control sequence $U \in J_0(m)$. From Theorems 7, 9, 11, it is clear that for a given initial condition and for a given control sequence $U \in J_0(m)$ there exists the unique solution of nonlinear fractional difference equations, respectively (15)–(17).

The reachable set for the given initial state x_0 at q steps, denoted as $R^q(x_0)$ is defined as the set of all states to which the given system can be steered from the prescribed initial state at q steps by control $U \in J_0(m)$, i.e.

$$\begin{aligned}R^q(x_0) &:= \{x \in \mathbb{R}^n \colon x = \gamma(q + a, x_0, U),\ U \in J_0(m)\},\\ R^0(x_0) &:= \{x_0\}.\end{aligned}$$

Note that a set $R(x_0) := \bigcup_{q \in \mathbb{N}_0} R^q(x_0)$ is the set of all states reachable from x_0. This set is nonempty.

The following Definition extends the definition of local controllability of semi-linear discrete-time systems given in [15].

Definition 13. The fractional order discrete-time control system given by (18) is *locally controllable in q steps* if there exists a neighborhood $V \subset \mathbb{R}^n$ of the point $x_0 = 0$ such that $V \subset R^q(x_0)$.

Definition 14. The fractional order discrete-time control system given by (18) is *controllable in q steps* if for zero initial condition $x_0 = 0$ we have that $R^q(0) = \mathbb{R}^n$.

Considering the assumption A1 let us define matrices

$$A := \frac{\partial f}{\partial x}(0, 0), \qquad B := \frac{\partial f}{\partial u}(0, 0)$$

and consider a linear fractional order discrete-time system

$$\left({}_{a}\Upsilon^{\alpha}x\right)(t) = Ax(t+a) + Bu(t), \quad x(a) = 0,\ t \in \mathbb{N}_0. \tag{19}$$

This system is called a *linear approximation* of the nonlinear one given by (18). Again, Propositions 8, 10, 12, imply that for a given initial condition and for an arbitrary sequence of controls $U \in J_0(m)$, there exists the unique solution of linear approximation (19).

Lemma 15. The linear fractional order discrete-time control systems

$$\begin{aligned}\left({}_{\mu}\Delta_*^{\alpha}x\right)(t) &= Ax(t+\mu) + Bu(t),\\ x(\mu) &= x_0 = 0, \quad t \in \mathbb{N}_0\end{aligned} \tag{20}$$

and

$$\begin{aligned}\left({}_{\mu}\Delta^{\alpha}x\right)(t) &= Ax(t+\mu) + Bu(t),\\ x(\mu) &= x_0 = 0, \quad t \in \mathbb{N}_0\end{aligned} \tag{21}$$

are controllable in q steps if and only if

$$\operatorname{rank}[B,\ AB,\ \ldots,\ A^{q-1}B] = n\,.$$

Proof. The idea of the proof comes from [6]. Let us consider the control systems described by equations (20) or (21). From the proof of Proposition 8 for system (20), from Proposition 10 for system (21) it follows that under zero initial conditions

$$x_f = \gamma(q + \mu, 0, U) = \sum_{s=1}^{q} A^{s-1}B({}_{0}\Delta^{-s\alpha}(t + (s-1)\mu)$$

$$= [B,\ A,\ \ldots,\ A^{q-1}B]\begin{bmatrix} ({}_{0}\Delta^{-\alpha}u)(t) \\ ({}_{0}\Delta^{-2\alpha}u)(t+\mu) \\ \vdots \\ ({}_{0}\Delta^{-q\alpha}u)(t + (q-1)\mu) \end{bmatrix}.$$

Putting

$$u_0 = ({}_{0}\Delta^{-\alpha}u)(t),$$

$$u_1 = ({}_{0}\Delta^{-2\alpha}u)(t+\mu), \ldots,$$

$$u_{q-1} = ({}_{0}\Delta^{-q\alpha}u)(t + (q-1)\mu)$$

and using classical arguments (as for example Kronecker-Capelli's Theorem) we can see that the above system has the solution for every x_f if and only if $\mathrm{rank}[B,\ AB,\ \ldots,\ A^{q-1}B] = n$.

Lemma 16 [6]. The linear fractional order discrete-time control system

$$\begin{aligned}\left({}_0\Delta_{\natural}^{\alpha} x\right)(t+1) &= Ax(t) + Bu(t), \\ x(\mu) = x_0 = 0, &\quad t \in \mathbb{N}_0\end{aligned} \tag{22}$$

is controllable in q steps if and only if

$$\mathrm{rank}[B,\ \Phi_1 B,\ \ldots,\ \Phi_{q-1}B] = n\,.$$

Corollary 17. Observe that using properties of rank of matrices the following conditions are equivalent:

(i) The linear fractional order discrete time control system (22) is controllable in q steps.
(ii) $\mathrm{rank}[B,\ \Phi_1 B,\ \ldots,\ \Phi_{q-1}B] = n$.
(iii) $\mathrm{rank}[B,\ A_\alpha B,\ \ldots,\ A_\alpha^{q-1}B] = n$, where $A_\alpha = A + I_n\alpha$.
(iv) $\mathrm{rank}[B,\ AB,\ \ldots,\ A^{q-1}B] = n$.

From Lemma 16 and Corollary 17 we can present the general result.

Proposition 18. The linear fractional order discrete time control system

$$\left({}_a\Upsilon^{\alpha} x\right)(t) = Ax(t+a) + Bu(t), \qquad x(a) = x_0 = 0, \quad t \in \mathbb{N}_a$$

is controllable in q steps if and only if

$$\mathrm{rank}[B,\ AB,\ \ldots,\ A^{q-1}B] = n\,.$$

Before stating the main results, let us recall some facts from functional analysis that we shall use in proofs (based on [14, 15]).

Lemma 19 [14, 16]. Let $F : Z \to Y$ be a nonlinear operator from Banach space Z into a Banach space Y and such that $F(0) = 0$, F has the Fréchet derivative $\mathrm{d}F(0) : Z \to Y$ whose image coincides with the whole space Y. Then the image of the operator F contains a neighborhood of the point $F(0) \in Y$.

Theorem 20. The nonlinear fractional order control system (18) with zero initial condition is locally controllable in q steps if its linear approximation (19) is globally controllable in q steps.

Proof. Since the proof is similar to the one in the classical nonlinear discrete-time case and in the semilinear fractional order control system case given in [14, 15] we present only sketch of it.

Let us consider a projection

$$\pi : J_0(m) \to \underbrace{\mathbb{R}^m \times \mathbb{R}^m \times \ldots \times \mathbb{R}^m}_{q\ times} = (\mathbb{R}^m)^q,$$

$$U = (u_0, u_1, \ldots) \mapsto (u_0, u_1, \ldots, u_{q-1}).$$

Let the operator $F : (\mathbb{R}^m)^q \to \mathbb{R}^n = Y$ transform a finite sequence of controls $\pi(U)$ into the space of solutions in q steps to system (18). More precisely operator F is defined as

$$(F \circ \pi)(U) = \gamma(q + a, 0, U), \tag{23}$$

where $\gamma(q + a, 0, U)$ is the solution in q steps to system (18) corresponding to a sequence of controls $U \in J_0(m)$ and zero initial condition. Then

$$\mathrm{d}F(0)(\pi(U)) = \overline{\gamma}(q + a, 0, U),$$

where $\mathrm{d}F(0)$ is the Fréchet derivative of map F at 0 and $\overline{\gamma}(q + a, 0, U)$ is the solution in q steps to the linear system (19) corresponding to the same sequence of controls $U \in J_0(m)$ as in (23) and with respect to zero initial condition. Assumption A2 implies that $F(0) = 0$. If the linear approximation (19) of the system (18) is controllable in q steps, then $\mathrm{Im}(\mathrm{d}F(0)) = \mathbb{R}^n$. Hence, by Lemma 19, the map F covers a neighborhood of $0 \in \mathbb{R}^n$ and, by Definition 13, the system (18) is locally controllable in q steps.

Example 21. Let us consider the system

$$\left({}_a\Upsilon^{\alpha} x\right)(t) = \begin{bmatrix} x_2(t+a) \\ -0.1\sin x_1(t+a) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\,. \tag{24}$$

Then its linearization has the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is easily seen that the system $\left({}_a\Upsilon^{\alpha} x\right)(t) = Ax(t + a) + Bu(t)$ is controllable in $q = 2$ step. As we take the initial condition $x_0 = 0$, we have two different approaches. We compare two difference models by taking constant control $u(t) \equiv -0.1$. Moreover we take the order $\alpha = 0.8$. Solutions are presented on Figs. 1 and 2.

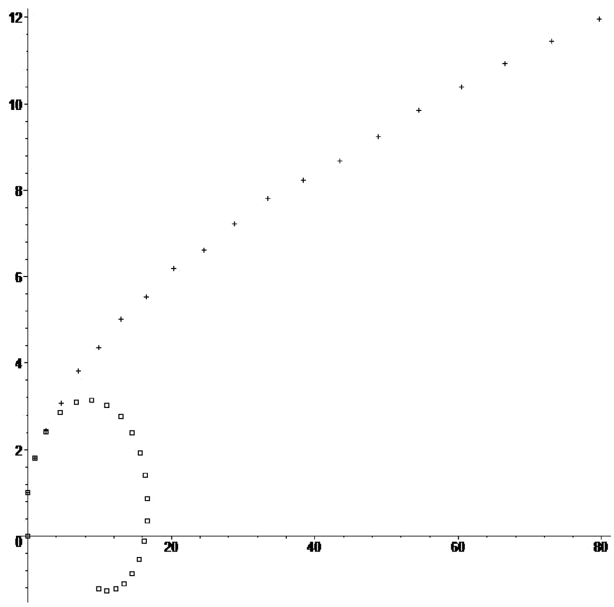

Fig. 1. Comparison of two trajectories of the nonlinear system from Example 21 and for Caputo type operator ${}_a\Upsilon^{0.8}x = {}_{(-0.2)}\Delta^{0.8}x$. For ten steps and nonlinear case – "crosses" and its linearization – "boxes", both for $u \equiv -0.1$.

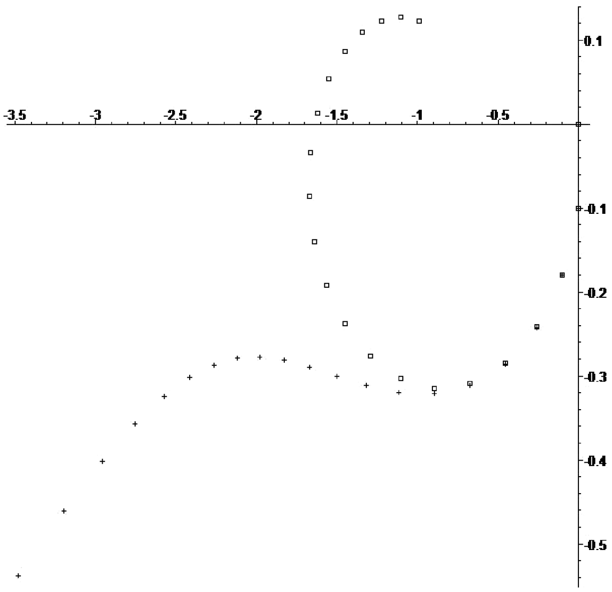

Fig. 2. Comparison of two trajectories of the nonlinear system from Example 21 for Grünwald-Letnikov type operator ${}_a\Upsilon^{0.8}x = {}_0\Delta_h^{0.8}x$. For ten steps and nonlinear case – "crosses" and its linearization – "boxes", both for $u \equiv -0.1$.

5. Conclusions

In the paper there have been presented three approaches to a definition of forward fractional order difference, namely the Riemann-Liouville, Caputo and Grünwald-Letnikov fractional order operators. The problem of existence of solutions to initial value problems for linear and nonlinear fractional order systems with left hand side given by each one of these operators have been shown. Obtained results have been used in the problem of controllability of linear approximation of nonlinear discrete-time fractional order systems. It occurs that for each case of a difference fractional operator the similar Kalman condition can be proved. Moreover, the result stated in [14] for continuous-time case and for semi-linear fractional order control systems stated in [15] can be extended on the whole class of nonlinear discrete-time fractional order systems.

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