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Unified summation equations and their applications in tribology wear process

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Abstract. This paper presents some applications of summation equations with regard to the calculation prognosis of micro-bearing wear parameters. Summation equations are presented in a new form of difference and recurrence equations where the unknown micro-bearing wear function occurs as the argument of the reciprocal unified operator of summation (UOS). In this paper the properties of the UOS and reciprocal UOS as well as the unitary translation operator (UTO) are defined and applied to a micro-bearing wear determination. The approach of both dual and interaction between summation and recurrence equations and micro-bearing wear determination, makes this paper unique.

Key words: unified operator of summation, summation equations, micro-bearing wear.

1. Preliminaries

Information on summation equations is very scarce. Many more papers have been published on difference equations [1–6].

The result of an increased interest in recurrence equations is among others the fact that differential equations can be simulated by means of summation and recurrence equations, and some linear summation and recurrence equations can be reduced to those differential equations that are equivalent with respect to solutions to recurrence equations. It is to be noted that those phenomena are described with summation equations whose results or values change in a discrete manner. A completely different description of phenomena with the aid of differential equations concerns dependencies and values with continuous changes, which as we know are a large approximation of the surrounding reality [7–9].

In the presented paper the summation or recurrence equations are applied for solving the problems connected with the wear prognosis during the HDD-microbearing Seagate Barakuda exploitation [10–13].

Owing to a constant growth of the potential of computers, a development of those methods that are approximate in the area of partial differential equations is being observed. In this research area, recurrence equations are becoming more and more widely applied. Initially, differential methods were developed. Further, the methods by Runge-Kutta, W. Ritz and B.G. Galerkin were created together with many other analytical and numerical methods, where the solutions of recurrence equations were required [9, 14].

Another Finite Elements Method is to be noted [15–20]. The advantage of this method is obtaining solutions of differential or recurrence equations in areas with very complex shapes. A disadvantage of this method is obtaining approximate solutions which do not strictly satisfy the equation which governs the process, or its satisfaction with a very low degree of the shape function. In order to obtain a small error, the area needs to be divided into many thousands of elements. The result of this division, however, is a large propagation of computational errors and lengthening of time required for calculations. The initial and boundary conditions in this method are satisfied in nodes only. This drawback is usually eliminated by means of a condensation of the network of numerical calculations. The recent achievements [15–20] have been done in Finite Elements computation modeling involving both validation of mathematical models and verification of numerical schemes with an error propagation.

The result of an improvement of the Finite Elements Method with the use of recurrence equations is an application of E. Trefftz functions as base functions in the Finite Elements Method [5]. The Trefftz function method cannot be omitted when discussing the solutions of recurrence equations [5]. The idea of E. Trefftz method consists in the determination of the system of the functions of polynomials which satisfy strictly the differential equation in question [5].

The presented paper is devoted to both mathematical theory of solutions of summation or recurrence equations and applications to HDD micro-bearing wear process determination and its convergences in succeeding time units of operation.

2. Some properties of the unified operator

Let S be a general operator of summation. We define general operator S and its properties in the following form [6, 11]:

$$S^1_{\varepsilon\rho}(...) \equiv S^1_0(....) + \varepsilon\rho(...), \tag{1}$$

where

$$(...) \equiv f_n, \qquad S^1_0(f_n) \equiv f_{n+1}. \tag{2}$$

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We denote: $\varepsilon = +1$ or $\varepsilon = -1$ and f_n – complex functions defined on the natural numbers, $\varepsilon\rho$ – basis of the unified operator of summation S, ρ – complex number of the complex variable, $m = 1$ – the first rank as an upper index of the operator S^m of unified summation.

Operation S on the function f_n is known as the unified summation. The particular case of operation S presented by the relation (1) for $\rho = 0$, has the form S_0^1 and can be denoted in the form S_0 and is known as unitary translation operator (UTO). The operation of the UTO on the function f_n was univocal defined by the Eq. $(2)_2$.

It is easy to see that the unified operation of summation (UOS) defined by the Eq. (1) in particular case for $\varepsilon = -1$, $\rho = z$ leads to the classical difference and the reciprocal difference operator, namely [6]:

$$S_{-z}^1 \equiv \Delta_z, \qquad S_{-z}^{-1} \equiv \Delta_z^{-1}. \tag{3}$$

Unified operation (1) for $\varepsilon = +1$, $\rho = z$ describes a new form of summation. Both forms are as follows [6, 11]:

$$S_z^1, \qquad S_z^{-1}. \tag{4}$$

The properties of UOS (the unified operator of summation) will be now presented. At first we show some particular operator properties for $\rho = 0$ namely:

$$S_0^1(...). \tag{5}$$

It is easy to see that the following operation is true [1, 6]:

$$S_0^1(S_0^1 f_n) = S_0^1(f_{n+1}) = f_{n+2}. \tag{6}$$

If the operator S has rank k, defined in the following recurrence form [1, 6]:

$$\begin{gathered} S_0^0(f_n) = f_n, \\ S_0^k(f_n) = S_0^1\left[S_0^{k-1}(f_n)\right] \quad \text{for} \quad k = 1, 2, 3, ... \end{gathered} \tag{7}$$

then is easy to prove the following dependences [1, 6]:

$$\begin{gathered} S_0^k(f_n) = f_{n+k}, \\ S_0^k(f_n g_n) = f_{n+k} g_{n+k}, \end{gathered} \tag{8}$$

From equations: (8) follows the multiplicative property of the UTO. Let be $k = 0$. In this case the unitary translation operator has order zero. For arbitrary natural number k the operator has order k. The UTO of the order k, is linear i.e. is simultaneously additive and homo-geneous, hence we can written as the following equation:

$$S_0^k(\alpha f_n + \beta f_n) = \alpha S_0^k(f_n) + \beta S_0^k(f_n), \tag{9}$$

where α, β are two arbitrary constants independent of n.

Moreover the UTO for two arbitrary natural orders s and k satisfies the following iterative summation law [1, 6]:

$$S_0^k S_0^s(f_n) = S_0^{k+s}(f_n). \tag{10}$$

The properties from (6) to (10) refer to the UTO of the order k where $k = 0, 1, 2, 3, \ldots$. Now we go to define the recurrent form of UOS of order k [1, 16, 17]:

$$\begin{gathered} S_{\varepsilon\rho}^0(f_n) \equiv f_n, \qquad S_{\varepsilon\rho}^k(f_n) \equiv S_{\varepsilon\rho}^1\left[S_{\varepsilon\rho}^{k-1}(f_n)\right] \\ \text{for} \quad k = 1, 2, 3, ... \end{gathered} \tag{11}$$

From the definition (11) implies the following iterative equations are implied [1, 6]:

$$\begin{gathered} S_{\varepsilon\rho}^k(f_n) \equiv \left(S_0^1 + \varepsilon\rho\right)^k f_n \\ \text{for} \quad k = 1, 2, 3, ... \end{gathered} \tag{12}$$

which is easy to prove in a mathematical induction way.

3. Reciprocal unified operator of summation

The reciprocal UOS regarding the unified operator of summation defined in Eq. (1) is denoted by the following description [1, 2, 6, 11]:

$$S_{\varepsilon\rho}^{-1}(...). \tag{13}$$

Reciprocal unified operator of summation will be defined in the following form [1, 2, 6, 11]:

$$S_{\varepsilon\rho}^{-1}(f_n) \equiv F_n, \quad \text{because} \quad S_{\varepsilon\rho}^{+1}(F_n) \equiv f_n, \tag{14}$$

where F_n, f_n are the functions determined for the natural numbers $n = 1, 2, 3, \ldots$.

Reciprocal unified operator of summation is denoted by the upper index (-1) and is not always univocal. To explain such a property we define the following.

Lemma 1

If function F_n presents a result of operation of reciprocal unified operator of summation $S_{\varepsilon\rho}^{-1}$ on the function f_n, then each function [6, 11]:

$$F_n + C \cdot (-\varepsilon\rho)^n \tag{15}$$

is also a consequence of the operation of reciprocal UOS on the function f_n where C is the arbitrary constant (independent of n).

Proof of Lemma 1

Let the UOS operate on the expression (15) and we need its linear and multiplicative properties defined by Eqs. (8), (9). After operation we obtain [1, 2, 6]:

$$\begin{gathered} S_{\varepsilon\rho}^{+1}\left[F_n + C \cdot (-\varepsilon\rho)^n\right] \\ = f_n + C \cdot (-\varepsilon\rho)^{n+1} + C \cdot (\varepsilon\rho)(-\varepsilon\rho) = f_n. \end{gathered} \tag{16}$$

Calculation (16) completes the proof of Lemma 1.

Now we show some selected characteristic properties of reciprocal UOS. At first we assume that the following operations [1, 6]:

$$S_{\varepsilon\rho}^1(....), \qquad S_{\varepsilon\rho}^{-1}(....), \tag{17}$$

with the same basis $\varepsilon\rho$ are for the two functions f_n, F_n reciprocal in following sense [1, 2, 6]:

$$\begin{gathered} S_{\varepsilon\rho}^{+1}\left[S_{\varepsilon\rho}^{-1}(f_n)\right] = f_n, \\ S_{\varepsilon\rho}^{-1}\left[S_{\varepsilon\rho}^{+1}(F_n)\right] = F_n + C \cdot (-\varepsilon\rho)^n. \end{gathered} \tag{18}$$

Equations (18) follow from the Definition (14) and are presented as the law of rank reduction.

4. Characteristic examples of a reciprocal operator of summation

In Table 1 we show some characteristic reciprocal transformations occurring in wear values summation [6, 11]. Now the UOS and the reciprocal UOS operator will be needed for the calculations of the sums of finite series and the sums of infinite quantity of terms in convergent series. Before the beginning the calculation process of the finite and infinite terms of the convergent series in a classical sense, first will be formulated the following lemma:

Lemma 2

If the following operation is valid:

$$S_{-1}^{+1}(F_n) = f_n, \tag{19}$$

then the sums of series with finite m terms and infinite $m \to \infty$ terms for natural numbers k, m where $k \leq m$, have the following form of the limits of subtracts [1, 6, 11]:

$$\begin{aligned}\sum_{n=k}^{m} f_n &= F_{m+1} - F_k \\ &= \lim_{n\to m+1}\left[S_{-1}^{-1}(f_n)\right] - \lim_{n\to k}\left[S_{-1}^{-1}(f_n)\right],\end{aligned} \tag{20a}$$

or

$$\begin{aligned}\sum_{n=1}^{\infty} f_n &= F_\infty - F_1 \\ &= \lim_{n\to \infty}\left[S_{-1}^{-1}(f_n)\right] - \lim_{n\to 1}\left[S_{-1}^{-1}(f_n)\right].\end{aligned} \tag{20b}$$

Proof of Lemma 2

On the grounds of the assumption (19), the following assumption is true [2, 7]:

$$\sum_{n=k}^{m}\left[S_{-1}^{+1}(F_n)\right] = \sum_{n=k}^{m} f_n. \tag{21}$$

By virtue of UOS definitions (1), (2) we can write:

$$\begin{gathered}\left[S_{-1}^{1}(F_n) = F_{n+1} - F_n\right] \Rightarrow \\ \left\{\sum_{n=k}^{m}\left[S_{-1}^{+1}(F_n)\right] = \sum_{n=k}^{m}(F_{n+1} - F_n)\right\}.\end{gathered} \tag{22}$$

From simple calculations we have:

$$\begin{aligned}\sum_{n=k}^{m}(F_{n+1} - F_n) &= \sum_{n=k}^{m}(F_{n+1}) - \sum_{n=k}^{m}(F_n) \\ &= F_{m+1} - F_k = (F_n)\big|_{n=k}^{n=m+1}.\end{aligned} \tag{23}$$

Table 1

Characteristics reciprocal UOS transformations with arbitrary summation constants C for arbitrary constants K and a and natural numbers k, s, n

	f_n	$S_{\varepsilon\rho}^{m}(f_n)$	$\varepsilon\rho$	m
1	K	$\frac{K}{1+\varepsilon\rho} + C\cdot(-\varepsilon\rho)^n$	$\varepsilon\rho \neq -1$	-1
2	K	$Kn + C$	$\varepsilon\rho = -1$	-1
3	a^n	$\frac{a^n}{a+\varepsilon\rho} + C\cdot(-\varepsilon\rho)^n$	$\varepsilon\rho \neq -a$	-1
4	a^n	$na^{n-1} + C\cdot(a)^n$	$\varepsilon\rho = -a$	-1
5	na^n	$\frac{an-n-a}{(a-1)^2}a^n + C$ for $a \neq 1$	$\varepsilon\rho = -1$	-1
6	na^n	$\frac{an+n-a}{(a+1)^2}a^n + C(-1)^n$ for $a \neq -1$	$\varepsilon\rho = +1$	-1
7	$arctg n$	$arctg\left(\frac{1}{n^2+n+1}\right)$	$\varepsilon\rho = -1$	-1
8	$\frac{1}{n}$	$\Psi(n) + C$ Ψ – Psi-Euler Function	$\varepsilon\rho = -1$	-1
9	n	$nJ - J^2 + C\cdot(-\varepsilon\rho)^n$ for $\frac{1}{1+\varepsilon\rho} \equiv J$	$\varepsilon\rho = -1$	-1
10	n	$\frac{n(n-1)}{2} + C$	$\varepsilon\rho = -1$	-1
11	1	$2^{-k} + (-1)^n C$	$\varepsilon\rho = +1$	$-k$
12	$v_n \times S_{\varepsilon\rho}^{1}(u_n)$	$u_n v_n - S_{\varepsilon\rho}^{-1}\left[u_{n+1} S_{-1}^{+1}(v_n)\right]$	Arbitrary $\varepsilon\rho$	-1
13	$n^{(r)} = \frac{n!}{(n-k)!}$	$r^{(s)} n^{(r-s)}$ $r^{(s)} n^{(r-s)} = 0 \; for \; s > r$	$\varepsilon\rho = -1$	s

If we combine equation (21), the result of implication (22) with the first and last hands side of Eq. (23), then we obtain:

$$\sum_{n=k}^{m} f_n = (F_n)|_{n=k}^{n=m+1} = \lim_{n\to m+1}[F_n] - \lim_{n\to k}[F_n]. \quad (24)$$

Imposing the reciprocal UOS operations on the assumption (19) leads to the following formula:

$$F_n = S_{-1}^{-1}(f_n), \qquad n = 1, 2, ... \quad (25)$$

If we put discrete function F_n obtained in formula (25) into the last r.h.s. of equation (24), then Lemma 2 is completed described by formula (20a) for finite series. If m tends to infinity for infinite series, then formula (20a) tends to formula (20b) and is completed Lemma 2 described by formula (20).

5. Linear summation equations

A difference equation, where unknown function f_n with real or complex discrete values determined on natural numbers n, occurs as the argument of a reciprocal unified m-order operator of summation $S_{\varepsilon\rho}^{-m}$, will be referred to as the m-order summation equation. In the general case, the linear summation equation in technical applications has the following form [11, 13]:

$$\begin{gathered} a_m S_{\varepsilon\rho}^{-m}(f_n) + a_{m-1} S_{\varepsilon\rho}^{-m+1}(f_n) \\ + a_{m-2} S_{\varepsilon\rho}^{-m+2}(f_n) + ... + a_1 S_{\varepsilon\rho}^{-1}(f_n) + a_0 f_n = A, \end{gathered} \quad (26)$$

where a_m – coefficients of linear summation equations, A – free term of a linear summation equation, f_n – unknown function of a linear summation equation.

Equation (26) describes the micro-bearing wear values sequence f_n in succeeding time units $n = 1, 2, 3,$ during the operating process [10, 11]. At first we consider the following.

Lemma 3

A non-homogeneous second order summation equation of the second and first kind with constant coefficients:

$$a_2 S_1^{-2}(f_n) + a_1 S_1^{-1}(f_n) + a_0 f_n = A, \quad (27a)$$

$$a_2 S_{-1}^{-2}(f_n) + a_1 S_{-1}^{-1}(f_n) + a_0 f_n = A, \quad (27b)$$

where a_1, a_2, a_0, A are real numbers that are independent from $n = 1, 2.3, ...$, can always be transformed to equivalent linear second order recurrent equations with constant coefficients. Function f_n describes micro-bearing wear changes in operating time units $n = 1, 2, \ldots$.

Proof of Lemma 3

A summation equation is equivalent to the recurrence equation if both equations have the same particular solutions. At first we assume the summation equation of the first kind (27a). On both sides of Eq. (27a), we operate with a unified operator of summation S_1^1 with basis one, hence we obtain [1, 6]:

$$a_2 S_1^{-1}(f_n) + a_1 f_n + a_0(f_{n+1} + f_n) = A + A. \quad (28)$$

On both sides of Eq. (28), we again operate with a unified operator of summation S_1^1 with basis one, thus we obtain:

$$\begin{gathered} a_2 f_n + a_1(f_{n+1} + f_n) + a_0(f_{n+2} + 2f_{n+1} + f_n) \\ = 2A + 2A. \end{gathered} \quad (29)$$

Equation (29) after ordering has the form:

$$a_0 f_{n+2} + (2a_0 + a_1) f_{n+1} + (a_0 + a_1 + a_2) f_n = 4A. \quad (30)$$

Obtained Eq. (30) presents a linear, non-homogeneous recurrent equation of the second order with constant coefficients. Symbol f_n denotes the discrete unknown function. Now we assume the summation equation of the second kind (27b). On both sides of Eq. (27b), we operate with the unified operator of summation S_{-1}^1 with basis minus one, hence we obtain:

$$a_2 S_{-1}^{-1}(f_n) + a_1 f_n + a_0(f_{n+1} - f_n) = A - A. \quad (31)$$

On both sides of Eq. (31), we again operate with the unified operator of summation S_{-1}^1 thus we obtain:

$$a_2 f_n + a_1(f_{n+1} - f_n) + a_0(f_{n+2} - 2f_{n+1} + f_n) = 0. \quad (32)$$

After ordering, Eq. (32) gives:

$$a_0 f_{n+2} + (-2a_0 + a_1) f_{n+1} + (a_0 - a_1 + a_2) f_n = 0. \quad (33)$$

Equation (33) presents a linear homogeneous second order recurrent equation with constant coefficients. Symbol f_n describes the discrete unknown function.

Expressions presented in Eqs. (30), (33) completes the proof of Lemma 3.

Theorem 1

Linear non-homogeneous summation equations of the second order and the second kind with variable coefficients in the form of a first degree polynomial presented in the following form [6, 10, 11]:

$$\begin{gathered} (a_{02} + a_{12} n) S_1^{-2}(f_n) + (a_{01} + a_{11} n) S_1^{-1}(f_n) \\ + (a_{00} + a_{10} n) f_n = A_0 + A_1 n, \end{gathered} \quad (34)$$

we can always transform the following to an equivalent linear recurrent equation:

1° of the third order if $a_{00} \neq 0$ or $a_{10} \neq 0$,

2° of the second order if $a_{00} = 0$, $a_{10} = 0$,

and if simultaneously $a_{01} \neq 0$, $a_{11} \neq 0$,

3° of the apparent third order transformable to the second order if $a_{12} = a_{11} = a_{10} = 0$, i.e. coefficients of the summation equations are constant.

Proof of Theorem 1

On both sides of Eq. (34), we operate with a unified operator of summation S_1^1 with basis one, hence we obtain [6]:

$$\begin{gathered} a_{02} S_1^1 \left[S_1^{-2}(f_n)\right] + a_{12} S_1^1 \left[n \cdot S_1^{-2}(f_n)\right] \\ + a_{01} S_1^1 \left[S_1^{-1}(f_n)\right] + a_{11} S_1^1 \left[n \cdot S_1^{-1}(f_n)\right] \\ + a_{00} S_1^1(f_n) + a_{10} S_1^1(n \cdot f_n) \\ = S_1^1(A_0) + S_1^1(n \cdot A_1). \end{gathered} \quad (35)$$

After operations and applying iteration low (10), Equation (35) obtains:

$$\begin{gathered} a_{02} S_1^{-1}(f_n) \\ + a_{12}\left[(n+1) \cdot S_1^{-2}(f_{n+1}) + n \cdot S_1^{-2}(f_n)\right] + a_{01} f_n \\ + a_{11} S_1^1 \left[(n+1) \cdot S_1^{-1}(f_{n+1}) + n \cdot S_1^{-1}(f_n)\right] + \\ + a_{00}(f_{n+1} + f_n) + a_{10}\left[(n+1) f_{n+1} + n f_n\right] \\ = A_0 + A_0 + A_1(n+1) + n \cdot A_1. \end{gathered} \quad (36)$$

After ordering the terms in Eq. (36), we obtain:

$$\begin{aligned} &a_{02} S_1^{-1}(f_n) \\ &+a_{12}(n+1) S_1^{-2}(f_{n+1}+f_n) - a_{12} S_1^{-2}(f_n) \\ &+a_{11}(n+1) S_1^{-1}(f_{n+1}+f_n) - a_{11} S_1^{-1}(f_n) \\ &+(a_{01}+a_{00}+n\cdot a_{10}) f_n \\ &+(a_{00}+a_{10}+n\cdot a_{10}) f_{n+1} = 2A_0 + A_1(2n+1). \end{aligned} \tag{37}$$

Now we require the following evident transformations[6]:

$$S_1^{-2}(f_{n+1}+f_n) = S_1^{-2}\left[S_1^1(f_n)\right] = S_1^{-1}(f_n), \tag{38}$$

$$S_1^{-1}(f_{n+1}+f_n) = S_1^{-1}\left[S_1^1(f_n)\right] = f_n. \tag{39}$$

If we put relations (38), (39) into Eq. (37), we obtain:

$$\begin{aligned} &a_{02} S_1^{-1}(f_n) + a_{12}(n+1) S_1^{-1}(f_n) - a_{12} S_1^{-2}(f_n) \\ &+a_{11}(n+1) f_n - a_{11} S_1^{-1}(f_n) \\ &+(a_{01}+a_{00}+n\cdot a_{10}) f_n \\ &+(a_{00}+a_{10}+n\cdot a_{10}) f_{n+1} \\ &= 2A_0 + A_1(2n+1). \end{aligned} \tag{40}$$

After ordering the coefficients by the operators and functions S_1^{-2}, S_1^{-1}, f_{n+1}, f_n, Eq. (40) tends to the form:

$$\begin{aligned} &-a_{12} S_1^{-2}(f_n) \\ &+\left[a_{02}+a_{12}(n+1)-a_{11}\right] S_1^{-1}(f_n) \\ &+\left[a_{00}+(n+1)a_{10}\right] f_{n+1} \\ &+\left[a_{01}+a_{00}+n\cdot a_{10}+a_{11}(n+1)\right] f_n \\ &= 2A_0 + A_1(2n+1). \end{aligned} \tag{41}$$

On both sides of Eq. (41), we operate two times with a unified operator of summation S_1^1, and in following steps we utilize the relations (38), (39). After ordering we attain the recurrence equation following form:

$$\begin{aligned} &\left[a_{00}+(n+3)a_{10}\right] f_{n+3} \\ &+\left[a_{01}+3a_{00}+3(n+2)\cdot a_{10}+(n+3)a_{11}\right] f_{n+2} \\ &+\left[2a_{01}+3a_{00}+3(n+1)\cdot a_{10}+a_{02}\right. \\ &\left.+(2n+3)a_{11}+a_{12}(n+3)\right] f_{n+1} \\ &+\left[a_{01}+a_{00}+n\cdot a_{10}+a_{02}+na_{12}+na_{11}\right] f_n \\ &= 8A_0 + 4(2n+3)A_1. \end{aligned} \tag{42}$$

Expression (42) presents in general a non-homogeneous, third order recurrent equation with variable coefficients in the form of polynomials of first degree. Symbol f_n denotes the unknown function. It can be easily seen that the coefficient by discrete function f_{n+3} is different from zero if $a_{00} \neq 0$ or $a_{10} \neq 0$ and if both coefficients a_{00}, a_{10} are different from zero. This fact completes the proof of Theorem 1 in Point 1°.

The coefficient by discrete function f_{n+3} is equal to zero if $a_{00} = 0$ and $a_{10} = 0$. In this case, recurrent Eq. (42) attains the second order but only in the case if $a_{01} \neq 0$ or $a_{11} \neq 0$ and if both are different from zero. Such conditions give in recurrent Eq. (42) the coefficient by function f_{n+2} which is different from zero. This fact completes the proof of Theorem 1 in Point 2°.

Now we take into consideration the case where $a_{10} = a_{11} = a_{12} = A_1 = 0$. In this case, the second order summation equation (34) leads to the second order summation equation of the type (26) with constant coefficients. Recurrent Eq. (42), transformed from the summation Eq. (34) now takes on the form:

$$\begin{aligned} &a_{00}\cdot f_{n+3} + \left[a_{01}+3a_{00}\right] f_{n+2} \\ &+\left[2a_{01}+3a_{00}+a_{02}\right] f_{n+1} \\ &+\left[a_{01}+a_{00}+a_{02}\right] f_n = 8A_0. \end{aligned} \tag{43}$$

In principle, we obtain a third order recurrent equation. It can be easily seen that the third order is apparent because the recurrent equation (43) can be written in the following form:

$$\begin{aligned} &a_{00}\cdot(f_{n+3}+f_{n+2}) \\ &+(a_{01}+2a_{00})(f_{n+2}+f_{n+1}) \\ &+(a_{01}+a_{00}+a_{02})(f_{n+1}+f_n) = 8A_0. \end{aligned} \tag{44}$$

Using the UOS operator, we can write:

$$\begin{aligned} f_{n+3}+f_{n+2} &= S_1^1(f_{n+2}), \\ f_{n+2}+f_{n+1} &= S_1^1(f_{n+1}), \\ f_{n+1}+f_n &= S_1^1(f_n). \end{aligned} \tag{45}$$

If we put Expressions (45) into Eq. (44), we obtain:

$$\begin{aligned} &a_{00}\cdot S_1^1(f_{n+2}) + (a_{01}+2a_{00})\cdot S_1^1(f_{n+1}) \\ &+(a_{01}+a_{00}+a_{02})\cdot S_1^1(f_n) = 8A_0. \end{aligned} \tag{46}$$

On both sides of Eq. (46), we operate with the reciprocal unified operator of summation. Hence, we obtain the following second order recurrent equation with constant coefficients:

$$\begin{aligned} &a_{00}\cdot f_{n+2} + (a_{01}+2a_{00})\cdot f_{n+1} \\ &+(a_{01}+a_{00}+a_{02})\cdot f_n = 4A_0. \end{aligned} \tag{47}$$

Assuming that $a_{00} \equiv a_0$, $a_{01} \equiv a_1$, $a_{02} \equiv a_2$, Eq. (47) is identical with Eq. (30). If coefficient a_{00} is different from zero, then the second order of Eq. (47) will be kept. The second order recurrence equation (47) is equivalent to the third order recurrence equation (43). This fact completes the proof of Theorem 1 in Point 3°.

Corollary 1

If in summation equation (34) we assume that [6, 11]:

$$\begin{aligned} &a_{02} \equiv 0, \quad a_{12} \equiv 0, \quad a_{01} \equiv 0, \quad a_{11} \equiv 0, \\ &a_{00} \neq 0, \quad a_{10} \neq 0, \quad A_0 \equiv 0, \quad A_1 \equiv 0, \end{aligned} \tag{48}$$

then we obtain the following equation:

$$(a_{00}+a_{10}n) f_n = 0. \tag{49}$$

In the case of assumption (48), recurrent Eq. (42) being equivalent with summation Eq. (34) leads to the following third order recurrent equation [6, 11]:

$$\begin{aligned} &\left[a_{00}+(n+3)a_{10}\right] f_{n+3} \\ &+\left[3a_{00}+3(n+2)a_{10}\right] f_{n+2} \\ &+\left[3a_{00}+3(n+1)a_{10}\right] f_{n+1} \\ &+\left[a_{00}+na_{10}\right] f_n + 0. \end{aligned} \tag{50}$$

Equation (50) is equivalent with Eq. (49).

Proof of corollary 1

To prove that equation (50) leads to Eq. (49), we show that recurrent equation (50) can be written in the following form [1, 6]:

$$\begin{aligned} &S_1^1\left\{[a_{00}+(n+2)a_{10}]\,f_{n+2}\right\}\\ &+S_1^1\left\{[2a_{00}+2(n+1)a_{10}]\,f_{n+1}\right\}\\ &+S_1^1\left\{[a_{00}+na_{10}]\,f_n\right\}=0. \end{aligned} \tag{51}$$

Imposing operator S_1^{-1} on both sides of (51), we obtain:

$$\begin{aligned} &[a_{00}+(n+2)a_{10}]\,f_{n+2}\\ &+[2a_{00}+2(n+1)a_{10}]\,f_{n+1}\\ &+[a_{00}+na_{10}]\,f_n=0. \end{aligned} \tag{52}$$

Equation (52) can be written in the following form:

$$\begin{aligned} &S_1^1\left\{[a_{00}+(n+1)a_{10}]\,f_{n+1}\right\}\\ &+S_1^1\left\{[a_{00}+na_{10}]\,f_n\right\}=0. \end{aligned} \tag{53}$$

Imposing operator S_1^{-1} on both sides of (53), we obtain:

$$[a_{00}+(n+1)a_{10}]\,f_{n+1}+[a_{00}+na_{10}]\,f_n=0. \tag{54}$$

Equation (54) can be presented in the following form:

$$S_1^1\left\{[a_{00}+na_{10}]\,f_n\right\}=0. \tag{55}$$

Imposing operator S_1^{-1} on both sides of Eq. (55), we obtain Eq.(49). Hence, the proof of Corollary 1 is completed.

6. Uniform mega- algorithm of solutions for ordinary recurrence equations

The results of applied mathematical achievements are presented in the form of the Uniform Mega- Algorithm elaboration for linear independent particular solutions $u_n^{[1]},\ldots,u_n^{[n]},u_n^{[b]}$ of the determination of an ordinary, linear n-order homogeneous and non-homogeneous following difference or recurrent equation occurring in wear process determination [6, 11]:

$$\begin{aligned} &p_k(n)u_{n+k}+p_{k-1}(n)u_{n+k-1}+\ldots.+p_2(n)u_{n+2}\\ &+p_1(n)u_{n+1}+p_0(n)u_n=b(n), \end{aligned} \tag{56}$$

where coefficients p_j, b for $j=0,1,\ldots,k$ depend on variable n in neighborhood of regular or non-regular points. Equation (56) describes the micro-bearing wear values sequence f_n in succeeding time units $n=1,2,3$, during the operating process [1, 6, 11].The linear independent particular solutions of the recurrent equation are presented in the sequence form and are satisfying the imposed boundary conditions. If coefficients p_j and b are independent of n, then equation (56) has the following general solution [2, 6]:

$$\begin{aligned} u_n^* &= C_1u_n^{[1]}+\ldots+C_ku_n^{[k]}+u_n^{[b]}\\ &=\sum_{s=1}^{r}\chi_s^n\left(\sum_{m=0}^{\nu_s-1}C_{sm}\cdot n^m\right)+\frac{b\cdot n^q}{q!\ \cdot\ Q\left(\chi=1\right)}, \end{aligned} \tag{57}$$

where C_{sm}, C_k are constants; $n=1,2,3,\ldots$. Associated polynomial Q and symbol χ_s for $s=1,2,3,\ldots,r$; $r\leq k$ denotes the successive different, roots of characteristic algebraic equations [1, 2, 6, 11]:

$$\begin{aligned} &Q(\chi)=\frac{p(\chi)}{(\chi-1)^q},\\ &p(\chi)=p_k\chi^k+p_{k-1}\chi^{k-1}+\ldots\ldots\ldots+p_1\chi+p_0=0, \end{aligned} \tag{58}$$

with multiple ν_s attributed to the roots χ_s whereas the sum of manifolds of roots is equal to the order of the recurrence equation, namely [11]:

$$\nu_1+\nu_2+\ldots+\nu_{r-1}+\nu_r=k. \tag{59}$$

Number q is the multiplicity of the root $\chi=1$ in characteristic equation $p(\chi)$.

Conclusion 1

Linear, non-homogeneous, first order recurrent equations with variable coefficient a_n and variable free term b_n [1, 2, 6]:

$$u_{n+1}+a_nu_n=b_n, \tag{60}$$

have the following general solution [1]:

$$u_n=(-1)^{n-1}\cdot\prod_{j=1}^{n-1}a_j\left\{C+\sum_{k=1}^{n-1}\left[\frac{b_k}{(-1)^k\prod\limits_{s=1}^{k}(a_s)}\right]\right\} \tag{61}$$

$$\text{for}\quad n=2,3,4,\ldots$$
$$u_1=C,$$

where u_n – an unknown discrete function, C-an arbitrary constant, indexes j and k belong to set $1,2,\ldots,n-1$ whereas $s=1,2,\ldots,k$.

Analytical proof of conclusion 1

If we divide both sides of Eq. (60) by the product $a_1a_2a_3\cdots a_n$, we obtain [1, 2, 6, 11]:

$$\frac{u_{n+1}}{\prod\limits_{j=1}^{n}a_j}+\frac{a_nu_n}{\prod\limits_{j=1}^{n}a_j}=\frac{b_n}{\prod\limits_{j=1}^{n}a_j}. \tag{62}$$

By reducing a second fraction by a_n, we obtain:

$$\frac{u_{n+1}}{\prod\limits_{j=1}^{n}a_j}+\frac{u_n}{\prod\limits_{j=1}^{n-1}a_j}=\frac{b_n}{\prod\limits_{j=1}^{n}a_j}. \tag{63}$$

Now we take into account the following assumption [1]:

$$\begin{aligned} &g_n\equiv\frac{u_n}{\prod\limits_{j=1}^{n-1}a_j},\\ &n>1\Rightarrow g_{n+1}\equiv\frac{u_{n+1}}{\prod\limits_{j=1}^{n}a_j}, \end{aligned} \tag{64}$$

$$g_1\equiv u_1, \tag{65}$$

$$B_n\equiv\frac{b_n}{\prod\limits_{j=1}^{n}a_j}\Rightarrow B_k\equiv\frac{b_k}{\prod\limits_{j=1}^{k}a_j}. \tag{66}$$

By using the formulae (64), (65), (66) we can write the recurrence equation (64) in following form [1, 2]:

$$g_{n+1}+g_n=B_n. \tag{67}$$

By virtue of the particular solutions of Eq. (60) for $a_n = 1$, the general solution of the recurrence Eq. (67) has the following form [1, 2, 6]:

$$g_n = C \cdot (-1)^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k-1} \cdot B_k, \tag{68}$$

where C denotes the arbitrary constant.

If we replace B_k in the r.h.s. of Eq. (68) with the expression (66), we obtain [1, 2, 6]:

$$\begin{gathered} g_n = C \cdot (-1)^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k-1} \cdot \frac{b_k}{\prod\limits_{j=1}^{k} a_j}, \\ \text{for} \quad n = 2, 3, 4, ... \\ g_1 = C. \end{gathered} \tag{69}$$

If we replace g_n in the r.h.s. of Eq. (69) with expressions (64), (65), we have [1, 2, 6]:

$$\begin{gathered} \frac{u_n}{\prod\limits_{j=1}^{n-1} a_j} = C \cdot (-1)^{n-1} + \sum_{k=1}^{n-1} (-1)^{n-k-1} \cdot \frac{b_k}{\prod\limits_{j=1}^{k} a_j}, \\ \text{for} \quad n = 2, 3, 4, ... \\ u_1 = C. \end{gathered} \tag{70}$$

It is easy to see that after conformal transformations the obtained formula (70) is identical with the general solution presented in formula (61) which completes Conclusion 1.

Conclusion 2

The linear, non-homogeneous, k-order recurrence equation with constant coefficients p_k and variable exponential free term [1, 6, 11]

$$\sum_{j=0}^{k} p_j u^*_{n+j} = b \cdot a^n, \quad a, b \neq 0, \tag{71}$$

has the following general solution:

$$\begin{gathered} u^*_n = u_n + u^b_n, \\ \text{for} \quad u^b_n \equiv \frac{a^{n-q} \cdot b \cdot n^q}{q! \cdot Q\,(\chi = a)}, \end{gathered} \tag{72}$$

where function u_n describes wear process values and denotes a general solution of the homogeneous, k-order recurrence equation created from the non-homogeneous Eq. (71) for $b = 0$, and moreover u^b_n presents a certain particular solution of the non-homogeneous equation, whereas number q is the multiplicity of the root $\chi = a$ of characteristic equation $p(\chi) = 0$. The associated polynomial is defined in following form [1, 2]:

$$Q(\chi) \equiv \frac{p(\chi)}{(\chi - a)^q}. \tag{73}$$

Proof of conclusion 2

If we put solution (72) into l.h.s. of Eq. (71), then we obtain [1, 2]:

$$p(S^1_0) \left[u_n + \frac{a^{n-q} \cdot b \cdot n^q}{q! \cdot Q\,(\chi = a)} \right] \stackrel{?}{=} b \cdot a^n. \tag{74}$$

Taking into account the polynomial translation operator and the properties of (UTO) unitary translation operator (9) for example additive properties, we can write the proved equality (74) in following form [1, 2, 6]:

$$p(S^1_0) u_n + p(S^1_0) \left[\frac{a^{n-q} \cdot b \cdot n^q}{q! \cdot Q\,(\chi = a)} \right] \stackrel{?}{=} b \cdot a^n. \tag{75}$$

The first term on the l.h.s. of the proved equation (75) equals zero, because u_n is the solution of the homogeneous recurrence equation. Taking into account the property of the homogeneity of the polynomial operator of translation, we can exclude outside the operator all the constant terms independent of n occurring in the second term on the l.h.s. of Eq. (75), and then we obtain [1, 2, 6]:

$$\frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} p(S^1_0) a^n n^q \stackrel{?}{=} b \cdot a^n. \tag{76}$$

If in Eq. (73) we insert in the place χ the unified operator of summation, then we have [1, 2]:

$$p(\chi) = (\chi - a)^q Q(\chi) \Rightarrow p(S^1_0) = (S^1_0 - a)^q \cdot Q(S^1_0). \tag{77}$$

If the result of implication (77) is put into l.h.s. of Eq. (76),then we obtain [1, 2]:

$$\frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} (S^1_0 - a)^q \cdot Q(S^1_0) a^n \cdot n^q \stackrel{?}{=} b \cdot a^n. \tag{78}$$

The operators on the l.h.s. of Eq. (78) can be transposed on the grounds of multiplicative property of the unified operator of translation (8). Hence the expression (78) attains the form:

$$\frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} Q(S^1_0)(S^1_0 - a)^q \cdot a^n \cdot n^q \stackrel{?}{=} b \cdot a^n. \tag{79}$$

By virtue of the formula (12), we have [1, 2, 6]:

$$\left(S^1_0 - a\right)^q = S^q_{-a}. \tag{80}$$

By using the (80) in Eq. (79), we obtain:

$$\frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} Q(S^1_0) \cdot S^q_{-a} \left(a^n \cdot n^q\right) \stackrel{?}{=} b \cdot a^n. \tag{81}$$

By using the formula [1, 2, 4]:

$$n^q = d^*_{q,1} n^{(1)} + d^*_{q,2} n^{(2)} + ... + d^*_{q,q-1} n^{(q-1)} + n^{(q)}, \tag{82}$$

and Sterling numbers $d^*_{q,j}$ of the second kind we can write the equation (81) in the following form [1, 2, 4]:

$$\begin{gathered} \frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} Q(S^1_0) \cdot \left\{ d^*_{q,1} S^q_{-a} \left[a^n \cdot n^{(1)}\right] + ... \right. \\ \left. ... + d^*_{q,q-1} S^q_{-a} \left[a^n \cdot n^{(q-1)}\right] + S^q_{-a} \left[a^n \cdot n^{(q)}\right] \right\} . \stackrel{?}{=} b \cdot a^n. \end{gathered} \tag{83}$$

By virtue of the transformation indicated in row 13 of Table 1, we can determine the values of the following q-repeated summation operator [1, 2, 4, 6]:

$$S^q_{-a} \left[a^n \cdot n^{(r)}\right] \quad \text{for} \quad r = 1, 2, 3, ..., q-1, q. \tag{84}$$

Therefore we can obtain the values of the terms in the braces occurring in Eq. (83) for indexes r from 1 to q. It is easy to see, that only for $r = q$ the value mentioned is different from zero. In the case for $a = 1$ and $q = r$ expression (84) is equal to $q!$. After calculations in the braces occurring in Eq. (83), we finally obtain [1, 2, 6]:

$$\frac{b}{a^q \cdot q! \cdot Q\,(\chi = a)} Q(S_0^1) \cdot (q! \cdot a^q \cdot a^n) \stackrel{?}{=} b \cdot a^n. \tag{85}$$

Because factor $q!$ and a^q are independent of n, hence these factors we can exclude outsider the operator, and taking into account the properties of polynomial operator $p(S_0^1)$, we get [1, 2, 6]:

$$Q(S_0^1) \cdot a^n = a^n Q(a). \tag{86}$$

By using Eq. (86) in equation (84), we obtain:

$$\begin{aligned}&\frac{b \cdot a^q \cdot q!}{a^q \cdot q! \cdot Q\,(\chi = a)} Q(S_0^1) \cdot (a^n)\\ &= \frac{b}{Q\,(\chi = a)} a^n Q(a) = b \cdot a^n.\end{aligned} \tag{87}$$

The result (87) completes the proof of Conclusion 2.

7. Wear prognosis for HDD bearing system

Problem formulation

Differences of the surface wear of some HDD micro-bearing system between the next and foregoing year of the operating time during the succeeding years numbered $n = 1, 2, 3, \ldots$ are as follows [6, 11, 13]:

$$F_n = f_{n+1} - f_n, \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{88}$$

Determine the differences of wear F_n if we know, that the wear in the succeeding years is described by the sequence $\{f_n\}$ for $n = 1, 2, \ldots$. It follows from experimental measurements [13] that wear in each year f_n increased by the double difference of wear F_n, in two successive years is equal to a journal diameter decrease by 3000 pm multiplied by number 2^n. For this problem boundary conditions are imposed describing the fact, that the difference of wear $F_1 = f_2 - f_1$ between the second and first years of the exploitation equals to 10 000 pm.

Solution of the problem

From Eq. (88) it follows, that wear surface can be presented in the form:

$$S_{-1}^{-1} F_n = S_{-1}^{-1}(f_{n+1} - f_n) = S_{-1}^{-1}[S_{-1}^{+1}(f_n)] = f_n. \tag{89}$$

The solved problem can be described by the following, non-homogeneous, first order summation equation with a variable free term {compare Eq. (60)}:

$$S_{-1}^{-1} F_n + 2F_n = 3000 \cdot 2^n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{90}$$

with the following boundary condition:

$$F_1 = 10\,000. \tag{91}$$

The unknown of summation equation (90) is the sequence with the general term F_n. Imposing the UOS operator S_{-1}^{+1} on the both sides of Eq. (90), we obtain:

$$\begin{gathered}F_n + 2S_{-1}^{+1}(F_n) = 3000 \cdot S_{-1}^{+1}(2^n)\\ \text{for} \quad n = 1, 2, 3, \ldots.\end{gathered} \tag{92}$$

After simple and known operations performed in Eq. (92) we attain the following, non-homogeneous, first order recurrence equation with a constant coefficient and a variable free term:

$$F_{n+1} - \frac{1}{2} F_n = 1500 \cdot 2^n \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{93}$$

It is easy to see, that the recurrence equation (93) has the form (60) for $a_n = a = -0,5$, $b_n = 1500 \cdot 2^n$. Utilizing the solution (61), we can write general solution of the equation (93) in the following form:

$$\begin{gathered}F_n = C \cdot (-1)^{n-1} \cdot \left(-\frac{1}{2}\right)^{n-1} +\\ +1500 \sum_{k=1}^{n-1} (-1)^{n-k-1} \left(-\frac{1}{2}\right)^{n-k-1} \cdot 2^k,\\ \text{for} \quad n = 2, 3, \ldots\\ F_1 = C,\end{gathered} \tag{94}$$

where C denotes the arbitrary summation constant.

The general solution (94) after simple terms ordering, attain the following form:

$$\begin{gathered}F_n = \frac{C}{2^{n-1}} + \frac{3000}{2^n} \cdot \sum_{k=1}^{n-1} 2^{2k}, \quad \text{for} \quad n = 2, 3, \ldots.\\ F_1 = C.\end{gathered} \tag{95}$$

Taking into account the sum of $n - 1$ terms of geometrical sequence on the r.h.s. of Eq. (95), we obtain:

$$\begin{aligned}F_n &= \frac{C}{2^{n-1}} + \frac{3000}{2^n} \cdot 4\frac{4^{n-1} - 1}{4 - 1}\\ &= \frac{C}{2^{n-1}} + \frac{1000}{2^n} \cdot (4^n - 4)\\ &= \frac{1}{2^n}(2C - 4000) + 1000 \cdot 2^n, \quad \text{for} \quad n = 1, 2, \ldots\end{aligned} \tag{96}$$

Imposing the boundary condition (91) for $n = 1$ on the general solution (96b), we obtain the following unknown constant value:

$$C = 10\,000. \tag{97}$$

Hence, the particular solution i.e. the general term of the sequence of the differences of the wear surface in succeeding years, has the following form:

$$F_n = \frac{16000}{2^n} + 1000 \cdot 2^n, \quad \text{for} \quad n = 1, 2, \ldots \tag{98}$$

From the formula (98) it follows, that in the succeeding years the differences of the wear surfaces for the considered HDD micro-bearing system, between the next and the foregoing year, are as follows [11, 13]:

$$\begin{gathered}\{F_n\}_{n=1}^{\infty} = 10 \text{ nm}, \quad 8 \text{ nm}, \quad 10 \text{ nm},\\ 17 \text{ nm}, \quad 32,5 \text{ nm} \quad 64,25 \text{ nm}, \ldots\end{gathered} \tag{99}$$

8. Convergences of the wear prognosis

8.1. Problem formulation. The sequence of wear values $\{f_n\}$ i.e. the values of a decrease in nm of the HDD micro-bearing journal diameter equal to sum of wear in two foregoing successive months multiplied by dimensionless average coefficient $0 < a < 1,0$ plus some exponent dimensional function bD^n where $b \geq 0$, D-real number. Coefficients D, a, b obtained in experiment depend on the micro-bearing material, the journal angular velocity and standard deviation. The wear in two first time units attains dimensional values $f_1 = W_1$, $f_2 = W_2$ in nm. Determine the unknown analytical formula $\{f_n\}$ for a sequence of wear values numbered for $n = 1, 2, 3, \ldots$ operation time units if we know the dimensionless values $D[1]$, $a[1]$ and dimensional values b [nm], W_1 [nm], W_2 [nm].

8.2. Principles of solution. The problem is defined by the following difference equation [10, 11, 13]:

$$f_{n+2} = a(f_{n+1} + f_n) + bD^n \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{100}$$

From formulae (57), (72) it follows that the general solution of the recurrence equation (100) for two arbitrary constants C_1, C_2 has the following form:

$$f_n = C_1\chi_1^n + C_2\chi_2^n + f_n^b. \tag{101}$$

The characteristic equation (58) for the recurrence equation (100) has the form:

$$\chi^2 - a\chi - a = 0 \tag{102}$$

and has always the following two real roots:

$$\begin{aligned} &\chi_{1,2} = \frac{a}{2} \pm \sqrt{a + \frac{a^2}{4}}, \quad \text{for} \quad 0 < a < 1, \\ &D_2 \leq \chi_{1,2} < D_1, \qquad D_{1,2} = \frac{1 \pm \sqrt{5}}{2}. \end{aligned} \tag{103}$$

Imposing the boundary conditions on the general solution (100) we obtain the following system of equations:

$$\begin{aligned} C_1\chi_1 + C_2\chi_2 + f_1^b = W_1, \\ C_1\chi_1^2 + C_2\chi_2^2 + f_2^b = W_2. \end{aligned} \tag{104}$$

The roots of the system (104) are as follows C_1, C_2:

$$\begin{aligned} C_1 &= -\frac{W_2 - \chi_2 W_1}{\chi_1(\chi_2 - \chi_1)} + \frac{f_2^b - \chi_2 f_1^b}{\chi_1(\chi_2 - \chi_1)}, \\ C_2 &= \frac{W_2 - \chi_1 W_1}{\chi_2(\chi_2 - \chi_1)} - \frac{f_2^b - \chi_1 f_1^b}{\chi_2(\chi_2 - \chi_1)}. \end{aligned} \tag{105}$$

The region of considered parameters of considered wear process has the form: $[0 < D < D_1 \equiv 1,618] \times [0 < a < 1]$ and is presented in Fig. 1.

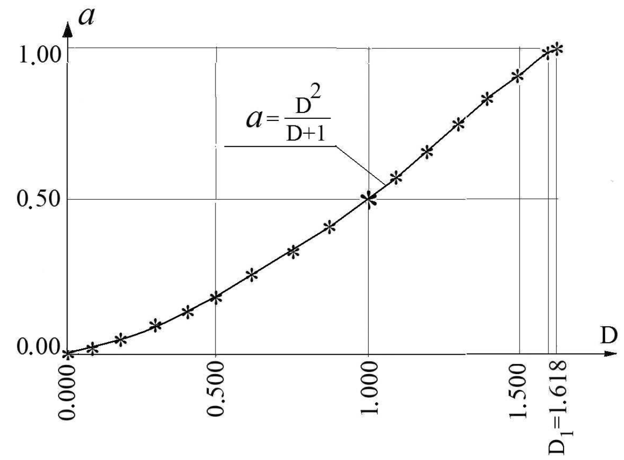


Fig. 1. Region of the wear consideration process for $0 < a < 1$, $0 < D < D_1$

8.3. Particular non-homogeneous solution for wear process for two dimensional sub-region. For the following assumptions:

$$\begin{gathered} 0 < a < \frac{1}{2}, \qquad \frac{1}{2} < a < 1, 0 \leq D \leq D_1, \\ a \neq \frac{D^2}{D+1}, \qquad \text{i.e.} \quad \chi_1, \chi_2 \neq D, \\ \chi_{1,2} = \frac{a}{2} \pm \sqrt{a + \frac{a^2}{4}}, \end{gathered} \tag{106}$$

and by virtue of Eq. (72) a particular solution of non-homogeneous equation (100)has the form presented below:

$$\begin{aligned} &f_n^b = \frac{bD^n}{D^2 - aD - a} \quad \text{for} \quad \chi_1, \chi_2 \neq D, \\ &0 < a < 1, \quad \text{i.e.} \quad a \neq \frac{D^2}{D+1}, \quad D_2 \leq D \leq D_1. \end{aligned} \tag{107}$$

The sum of non-homogeneous solution (101) i.e. sum of wear values in the N time units has the following form:

$$\begin{aligned} \sum_{n=1}^{N} f_n &= \frac{1}{\Delta\chi}\left(\chi_2\frac{1-\chi_1^N}{1-\chi_1} - \chi_1\frac{1-\chi_2^N}{1-\chi_2}\right)W_1 \\ &+, \frac{1}{\Delta\chi}\left(\frac{1-\chi_2^N}{1-\chi_2} - \frac{1-\chi_1^N}{1-\chi_1}\right)W_2 + \\ &-\frac{1}{\Delta\chi}\cdot\frac{bD}{D^2 - aD - a} \\ \times &\left[(\chi_2 - D)\frac{1-\chi_1^N}{1-\chi_1} - (\chi_1 - D)\frac{1-\chi_2^N}{1-\chi_2} - \frac{1-D^N}{1-D}\Delta\chi\right], \\ &\Delta\chi \equiv \chi_2 - \chi_1; \quad n = 1, 2, 3, ..., N. \end{aligned} \tag{108}$$

8.4. Two dimensional sub-region of convergent micro-bearing wear process. For following assumptions:

$$\begin{gathered} a \neq \frac{D^2}{D+1}, \qquad 0 < a < \frac{1}{2}, \\ 0 < D < 1, \quad \text{i.e.} \quad |\chi_{1,2}| < 1, \end{gathered} \tag{109}$$

we consider the following two dimensional sub-region of micro-bearing wear process presented in Fig. 2a.

For infinitely many time units, sum of wear (108) for each point including in presented sub-region, by virtue of (20b) tends to the following limit value presented in (110):

$$\sum_{n=1}^{\infty} f_n = \frac{1-\chi_1-\chi_2}{(1-\chi_1)\cdot(1-\chi_2)}W_1 + \frac{1}{(1-\chi_1)\ \cdot(1-\chi_2)}W_2 - \frac{bD}{D^2-aD-a}\left(\frac{1+D-\chi_1-\chi_2}{(1-\chi_1)\ \cdot(1-\chi_2)} - \frac{1}{D-1}\right),$$
$$\chi_1,\chi_2 \neq D, \qquad a \neq \frac{D^2}{D+1},$$
$$0<a<\frac{1}{2}, \qquad |\chi_{1,2}|<1,$$
$$0<D<1. \tag{110}$$

Calculation example. In two first successive time units HDD micro-bearing journal attains diameter decreases W_1 and W_2 nm. Determine the wear after infinite time units using measurements and stochastic data presented by $a[1]=1/6$, $D[1]=1/3$ for arbitrary b [nm].

Solution. From formulae (106), (109) we obtain following micro-bearing exploitation parameters:

$$a \neq \frac{D^2}{D+1} = \frac{1}{12}, \qquad \chi_1 = \frac{1}{2}, \qquad \chi_2 = -\frac{1}{3}. \tag{111}$$

Putting data (111) in formula (110) the convergent sequence of sums of wear after successive time units attain the form presented by the formula (112):

$$\text{for} \quad N=1, \;\; f_1 = W_1,$$
$$\text{for} \quad N=2, \;\; f_1+f_2 = W_1+W_2,$$
$$\text{for} \quad N=3, \; \sum_{n=1}^{3} f_n = 1.16(6)W_1 + 1.16(6)W_2 + 0.3(3)b,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\text{for} \quad N=12, \; \sum_{n=1}^{10} f_n = 1.2492W_1 + 1.4976W_2 + 0.7453b,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\text{for} \quad N=\infty, \; \sum_{n=1}^{\infty} f_n = 1.25W_1 + 1.50W_2 + 0.75b. \tag{112}$$

The graphical illustrations of results obtained in sequence (112) are presented in Fig. 2b.

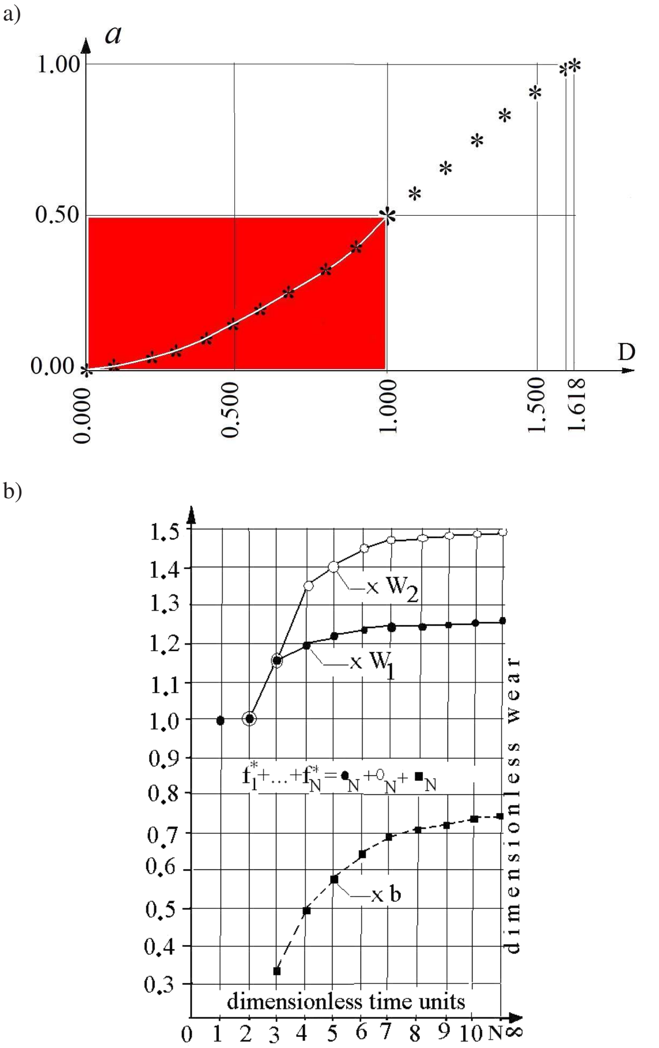


Fig. 2. a) sub-region of the wear convergent process for $a \neq D^2/(D+1)$, $0<a<1/2$, $0<D<1$; b) convergent process of the sum of the wear after N time units if N tends to infinity for $a \neq D^2/(D+1)$, $0<a<1/2$, $0<D<1 \Rightarrow \chi \neq D$, $|\chi|<1$ where, in particular case presented in this picture is $a=1/6$, $D=1/3$. Here are assumed arbitrary dimensional values W_1 [nm], W_2 [nm], b [nm]

8.5. Two dimensional sub-region of divergent micro-bearing wear process. For assumptions:

$$A: a \neq \frac{D^2}{D+1}, \left(\frac{1}{2}<a<1\right)\times(1<D<D_1), \max|\chi|>1,$$
$$B: \left(0<a<\frac{1}{2}\right)\times(1<D<D_1), \max|\chi|<1,$$
$$C: \left(\frac{1}{2}<a<1\right)\times(0<D<1), \max|\chi|>1, \tag{113a}$$

we consider the following two dimensional sub-region of wear process presented in Fig. 3a.

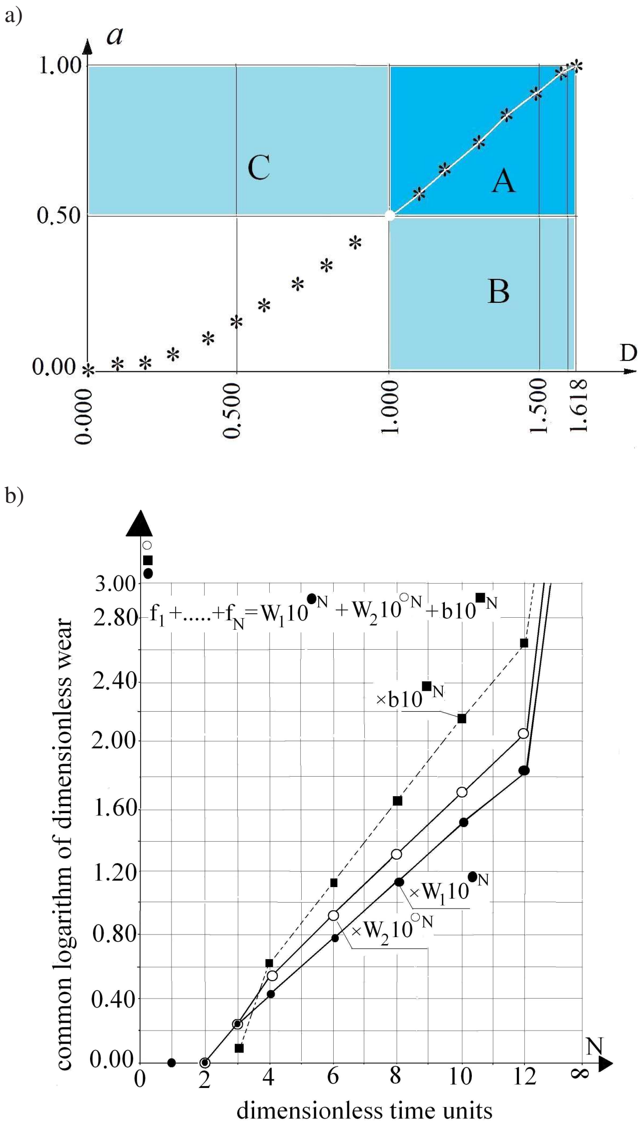


Fig. 3. a) sub-region of the: A) wear divergent process for all components where $a \neq D^2/(D+1)$, $1/2 < a < 1$, $1 < D < 1.618$, B) wear divergent process for stochastic properties function of materials only (coeff. b) where $a \neq D^2/(D+1)$, $0 < a < 1/2$; $1.00 < D < 1.618$, C) wear divergent process for stochastic average tendencies only (coeff. W_1, W_2) where $a \neq D^2/(D+1)$; $0.5 < a < 1.0$; $0 < D < 1$; b) divergent process of the sum of the wear after N time units if N tends to infinity for $a \neq D^2/(D+1)$, $1/2 < a < 1.0$; $1.0 < D < 1.6180 \Rightarrow \chi \neq D, |\chi| > 1$ where in particular case presented in this picture is $a = 0.90$, $D = 1.25$. Here are assumed arbitrary dimensional values W_1 [nm], W_2 [nm], b [nm].

Calculation example. Using measurements and stochastic data presented by $a[1] = 9/10$, $D[1] = 5/4$ for arbitrary b [nm] then we obtain:

$$\chi_1 = \frac{3}{2} > 1, \qquad \chi_2 = -\frac{3}{5}, \qquad a \neq \frac{D^2}{(D+1)} = 25/36. \tag{113b}$$

In this case from formula (110) follows that if time unit $N = 1, 2, 3, \ldots$ increases, then wear always increases. The sequence of sums of wear values in successive time units is divergent and has the following form:

$$\begin{gathered}
\text{for} \quad N = 1, \quad f_1^* = W_1, \\
\text{for} \quad N = 2, \quad f_1^* + f_2^* = W_1 + W_2, \\
\text{for} \quad N = 3, \quad f_1^* + f_2^* + f_3^* = \\
= 1.900W_1 + 1.900W_2 + 1.250b, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\text{for} \quad N = 12, \quad \sum_{n=1}^{12} f_n^* = \\
= 74.0147W_1 + 122.3168W_2 + 466.7707b, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{gathered} \tag{114}$$

The graphical illustrations of results obtained in sequence (114) is presented in Fig. 3b.

8.6. Particular non-homogeneous solution for wear process in curvilinear one dimensional sub-region. For the following assumptions:

$$\begin{gathered}
\left(a = \frac{D^2}{D+1}\right) \cap \left(0 < a < \frac{1}{2}\right) \\
\times (0 < D < 1) \cup \left(\frac{1}{2} < a < 1\right) \times (1 < D < D_1) \\
\Rightarrow \chi_{1,2} = \frac{a}{2} \pm \sqrt{a + \frac{a^2}{4}}, \\
\chi_1 = D, \qquad \chi_2 = -\frac{D}{D+1}
\end{gathered} \tag{115}$$

and by virtue of Eq. (72) the particular solution of non-homogeneous Eq. (100) has the form:

$$f_n^b = \frac{bnD^n}{2D^2 - aD}. \tag{116}$$

The sum of solution (101) i.e. sum of micro-bearing wear values in the N time units has the following form:

$$\begin{gathered}
\sum_{n=1}^{N} f_n^* = \frac{1+D}{2+D} \\
\cdot \left\{ \frac{1-D^N}{1-D^2} + \frac{1+D}{1+2D}\left[1 - \left(\frac{-D}{D+1}\right)^N\right] \right\} W_1 \\
- \frac{1+D}{D(2+D)} \left\{ \frac{1+D}{1+2D}\left[1 - \left(\frac{-D}{D+1}\right)^N\right] - \frac{1-D^N}{1-D} \right\} W_2 \\
+ \frac{b}{D}\left(\frac{1+D}{2+D}\right)^2 \left\{ -\frac{3+2D}{1-D^2}\left(1 - D^N\right) \right. \\
+ \frac{1+D}{1+2D}\left[1 - \left(\frac{-D}{D+1}\right)^N\right] \\
\left. + \left[\frac{D^N(ND - N - 1) + 1}{(1-D)^2(1+D)}\right](2+D) \right\} \\
n = 1, 2, 3, \ldots, N.
\end{gathered} \tag{117}$$

8.7. Curvilinear one dimensional sub-region of divergent wear process. For assumptions:

$$\left(a=\frac{D^2}{D+1}\right)\cap\left(\frac{1}{2}<a<1\right)\times(1<D<D_1)\Rightarrow$$
$$\chi_{1,2}=\frac{a}{2}\pm\sqrt{a+\frac{a^2}{4}},\quad \chi_1=D\quad \chi_2=-\frac{D}{D+1}, \tag{118}$$

we consider the following curvilinear sub-region of wear process presented in Fig. 4.

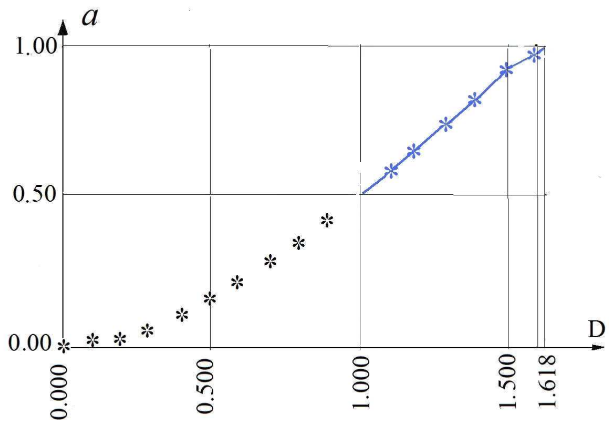


Fig. 4. Sub-region of the wear divergent process for $a=D^2/(D+1)$, $1/2<a<1$; $1.00<D<1.618$

8.8. Curvilinear one dimensional sub-region of convergent wear process. For assumptions:

$$\left(a=\frac{D^2}{D+1}\right)\cap\left(0<a<\frac{1}{2}\right)\times(0<D<1)\Rightarrow$$
$$\chi_{1,2}=\frac{a}{2}\pm\sqrt{a+\frac{a^2}{4}},\quad \chi_1=D,\quad \chi_2=-\frac{D}{D+1}, \tag{119}$$

we consider the following curvilinear sub-region of wear process presented in Fig. 5.

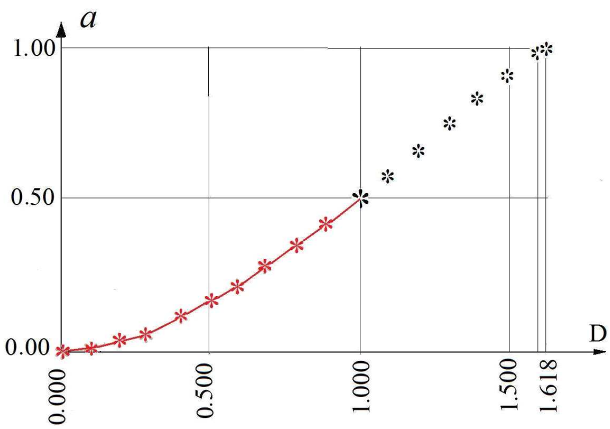


Fig. 5. Sub-region of the wear convergent process for $a=D^2/(D+1)$, $0<a<1/2$, $0<D<1$

Taking into account assumptions:

$$0<a<\frac{1}{2},\quad |\chi_{1,2}|<1,\quad 0<D<1, \tag{120}$$

then after infinitely many time units, sum of wear process (117) is convergent and tends to the following limit value:

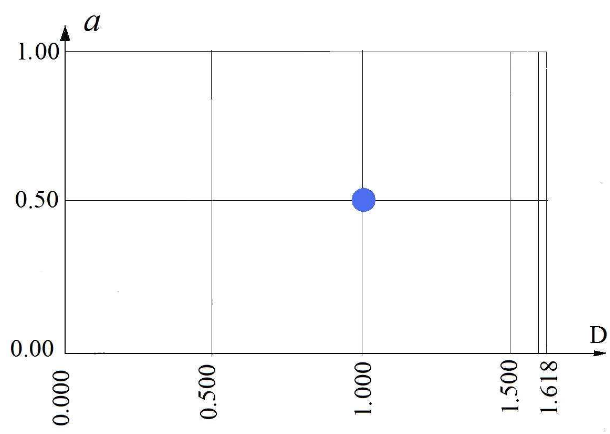
$$\sum_{n=1}^{\infty} f_n=\frac{2+3D-D^2-D^3}{(1-D)(2+D)(1+2D)}W_1+ \\ +\frac{1+D}{(1-D)(1+2D)}W_2+\frac{b(1+D)D}{(D-1)^2(1+2D)}. \tag{121}$$

8.9. Singular point as a sub-region for divergent wear process. For the following assumptions:

$$\left(D=1,\ a=\frac{1}{2}\right)\Rightarrow\left(\chi_1=1,\ \chi_2=-\frac{1}{2}\right), \tag{122}$$

we consider the following one singular point sub-region of wear process presented in Fig. 6a.

a)



b)

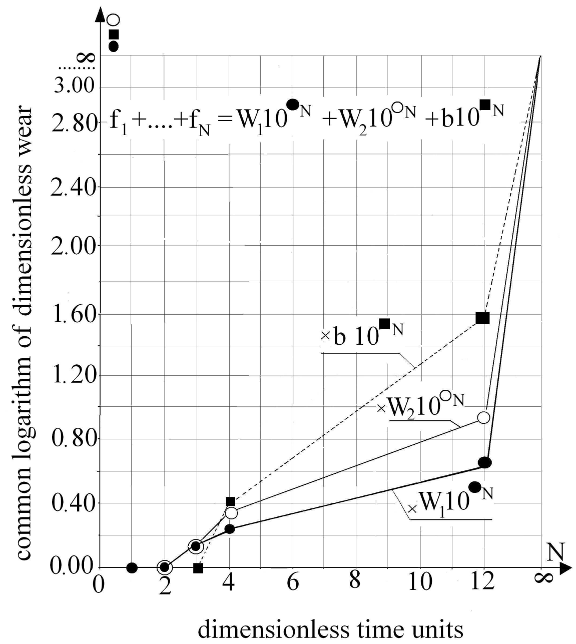


Fig. 6. a) one point sub-region of divergent wear process for $D=1,0$, $a=0,5$; b) divergent process of the sum of the wear after N time units if N tends to infinity for $a=D^2/(D+1)$, $a=0.50$, $D=1.00\Rightarrow\chi_1=D$, $\chi_2=-D/(D+1)$, $|\chi|\leq 1$. Here are assumed arbitrary dimensional values W_1 [nm], W_2 [nm], b [nm]

By virtue of Lemma 2 and Eq. (20a), sum of solution (101) i.e. sum of micro-bearing wear values in the N time units, has the following form:

$$\begin{aligned}\sum_{n=1}^{N} f_n^* &= C_1 \sum_{n=1}^{N} 1 + C_2 \sum_{n=1}^{N} \left(-\frac{1}{2}\right)^n + \frac{b}{2-a} \sum_{n=1}^{N} n \\ &= C_1 N + \frac{1}{3} C_2 \left[\left(-\frac{1}{2}\right)^N - 1\right] + \frac{b\,N(N+1)}{3} \\ &= W_1 \left\{\frac{1}{3} N - \frac{4}{9}\left[\left(-\frac{1}{2}\right)^N - 1\right]\right\} \\ &+ W_2 \left\{\frac{2}{3} N + \frac{4}{9}\left[\left(-\frac{1}{2}\right)^N - 1\right]\right\} \\ &+ b\left\{-\frac{10}{9} N - \frac{8}{27}\left[\left(-\frac{1}{2}\right)^N - 1\right] + \tfrac{N(N+1)}{3}\right\},\end{aligned} \tag{123}$$

for $N = 1, 2, 3, \ldots$.

In this case the increases of the wear has not limit in succeeding time limits during the exploitation and we obtain larger and larger wear values. The graphical illustrations of results obtained in the sum (123) is presented in Fig. 6b.

9. Final remarks

1. The presented paper seeks to join the gap between theoretical ideas of the summation or recurrence equations and the problem of wear determination during the HDD micro-bearing operating time. Moreover, this paper presents the transformation method of summation equations to recurrent equations. This method is constituted by the Lemma and Theorem where in the proof the properties of the solutions of the obtained recurrence equations are indicated.
2. The application of the presented theory in this paper contains the numerical solutions referring to the wear values of the HDD micro-bearing system in the considered period of the operating time.
3. The presented unified algorithm constitutes a useful tool for the solution of summation equations applied to the wear estimation of HDD micro-bearings [13].

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