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THE INF-SUP CONDITION TESTS FOR SHELL/PLATE FINITE ELEMENTS

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Development of high-performance finite elements for thick, moderately thick, as well as thin shells and plates, was one of the active areas of the finite element technology for 40 years, followed by hundreds of publications. A variety of shell elements exist in the FE codes, but "the best" finite element is still to be discovered. The paper deals with an evaluation of some existing shell finite elements, from the point of view of the third of three requirements to be satisfied by the element: ellipticity, consistency and inf-sup condition. It is difficult to prove the inf-sup condition analytically, so, a numerical verification is proposed. A set of numerical tests is considered for shell and plate problems. Two norm matrices and a selection of the stiffness matrices (bending, shear and membrane dominated) are analysed. Finite elements from various computer systems can be evaluated and compared with the use of the proposed tests.

Key words: shells, plates, evaluation, inf-sup tests.

1. Introduction

Shells/plates are widely considered in engineering applications. The corresponding discretization procedures are not yet sufficiently reliable, in particular as regards shell structures. A major cause of these difficulties lies in the numerical locking phenomena that arise in such formulations (Hughes, Hinton [15], Kardestuncer, Norrie [17], Taylor, Zienkiewicz [20], Radwańska [18]). It is extremely difficult to obtain a shell finite element that is optimal. In the optimal formulation we should aim to satisfy the following conditions (Arnold et all [1], Chapelle, Bathe [9], Chapelle, Paris Suarez [11], Bathe et all. [3],[4]), Brezzi, Bathe [6], Brezzi, Fortin [7], Gilewski [13]):

Ellipticity. This condition ensures that the finite element model is solvable, and physically means there are no spurious zero energy modes. This condition can easily be verified by studying the zero eigenvalues and corresponding eigenvectors of the stiffness matrix of a single unsupported finite element (Brezzi, Fortin [7], Gilewski [13], Taylor, Zienkiewicz [20] for example).

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Consistency. The finite element solution must converge to the solution of a mathematical problem with the element size h close to zero. The bilinear forms used in the finite element discretization must approach the exact bilinear forms of the mathematical model as h approaches zero (Brezzi, Fortin [7], Gilewski [12,13]).

Inf-sup condition. Satisfying this condition implies uniform and optimal convergence in bending-dominated shell problems (Bathe [2], . The shell element is released from shear and membrane locking with solution accuracy independent of the shell thickness parameter. In general it is very difficult to prove analytically whether a shell or plate finite element satisfies this condition, and numerical tests are to be employed (Bathe [2], Bathe et all. [3,4,5], Brezzi, Bathe [6], Chapelle, Bathe [8,9,10], Hiller, Bathe [14], Iosilevich at all [16], Sitek [19]).

A variety of shell finite elements exist in commercial FE codes and, according to the authors opinion, it is the right time to compare and evaluate them. It is not easy because the theoretical details of the formulations are not always available. The tests presented below can be used for this evaluation and comparison.

The present paper is dedicated to the numerical verification of the inf-sup condition (Sitek [19]) for shell/plate finite elements in general. A detailed evaluation of standard plate finite elements, as well as the elements available in the commercial programme Abaqus, is presented. The advantage of the inf-sup tests is that they can be used for displacement based finite elements, as well as for mixed models. Finite element models classified as "tricks" (strain energy multipliers, reduced integration, etc.) can be also evaluated by the tests.

2. The inf-sup condition

Shell elements in the finite element systems are based on the displacement, mixed, assumed strain or other formulation. Details of the inf-sup conditions, difficulties to fulfil it and the concept of the inf-sup test can be found in the papers of Bathe [2], Bathe et all. [3,4,5], Brezzi, Bathe [6], Chapelle, Bathe [8,9,10], Hiller, Bathe [14], Iosilevich at all [16], Sitek [19]).

Effective numerical implementation of the inf-sup test is based on the following steps:

- A sequence of N finite element meshes should be chosen for a selection of bending dominated problems, with decreasing characteristic element size h^k ($k=1,2,\ldots,N$). It is preferably to use the element sides not aligned on the asymptotic lines of the mid-surface.
- For every k in the sequence, establish the stiffness matrix $\hat{\mathbf{K}}^k$ and the norm matrix $\mathbf{S}^k$ and calculate the smallest eigenvalue $\lambda^k_{\min}$ of the generalized eigenvalue problem

$$\hat{\mathbf{K}}^k\mathbf{q} = \lambda^k\mathbf{S}^k\mathbf{q}$$

- Plot $\log\left(\lambda^k_{\min}\right)$ versus $\log(1/N)$.

- If the curve clearly flattens out as h^k decreases, and the $\lambda_{\min}$ value stabilizes at some positive level, then the inf-sup condition is satisfied.
- If the right behaviour is observed for all test problems, the element passes the inf-sup test.

There are three crucial decisions in the above evaluation procedure:
1. Selection of shell/plate geometry for the test,
2. Selection of the stiffness matrix $\hat{\mathbf{K}}^k$ (bending or shear/membrane dominated),
3. Selection of the norm matrix $\mathbf{S}^k$.

These aspects will be discussed below.

3. A SET OF PLATE/SHELL TESTS

In order to perform the inf-sup analysis of a particular shell/plate finite element, a suitable set of well-chosen bending dominated problems should be selected (Bathe [2], Bathe et all. [3,4,5], Brezzi, Bathe [6], Chapelle, Bathe [8,9,10], Hiller, Bathe [14], Iosilevich at all [16]). Five shell and three plate tests are proposed (Fig. 1-8), from the point of view of bending energy value (versus membrane/shear energy) corresponding to the first eigenvector of the structure, as well as of Gaussian curvature sign (positive, negative or zero) for shells. The balance of strain energy is calculated for the first eigenvalue (with the use of the most effective finite elements. Rectangular plate element with the so called physical shape functions is used for plate problems (Gilewski [13]) and the S4 Abaqus shell element for shell problems. The balances are presented in Fig. 9,10 (h is plate thickness, a is a typical mid-plane dimension).

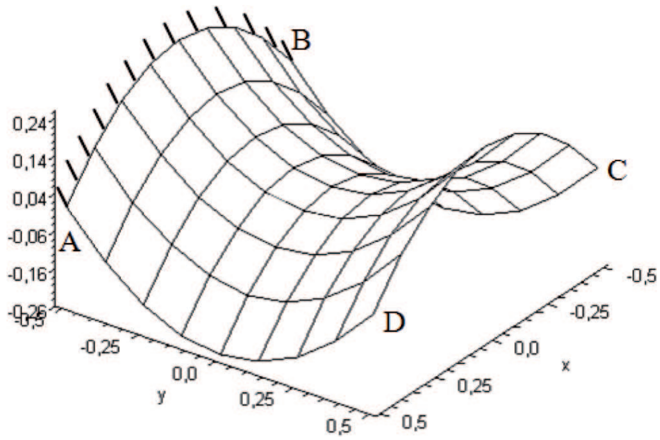

Fig. 1. Cantilever hyperbolic paraboloid.
Rys. 1. Wspornikowa paraboloida hiperboliczna

The proposed set of tests is an extension of the tests proposed in the literature of the subject.

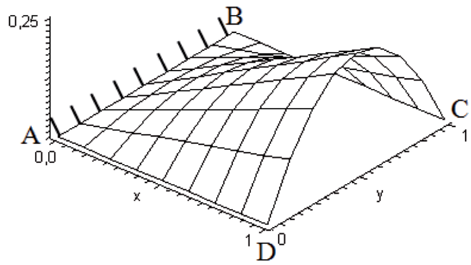

Fig. 2. Cantilever conoid.
Rys. 2. Wspornikowa konoida

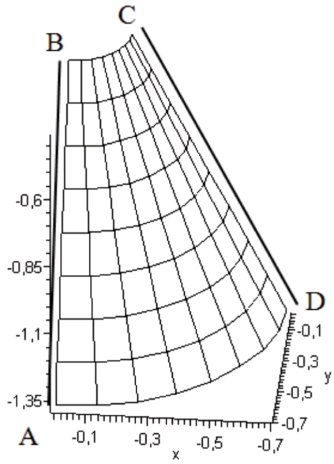

Fig. 3. Cone with symmetry boundary conditions: u_x= f_y=f_z=0 on AB; u_y= f_x= f_z=0 on CD.
Rys. 3. Stożek z warunkami brzegowymi symetrii: u_x= f_y=f_z=0 na AB; u_y= f_x= f_z=0 ona CD

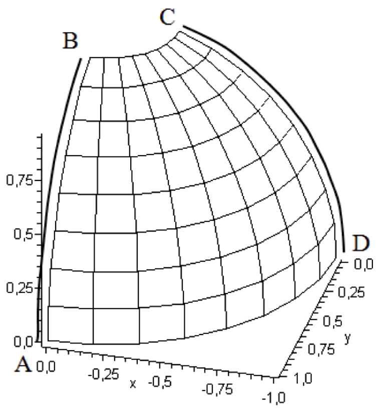

Fig. 4. Sphere with symmetry boundary conditions: u_x= f_y=f_z=0 on AB; u_y= f_x= f_z=0 on CD.
Rys. 4. Sfera z warunkami brzegowymi symetrii: u_x= f_y=f_z=0 na AB; u_y= f_x= f_z=0 ona CD

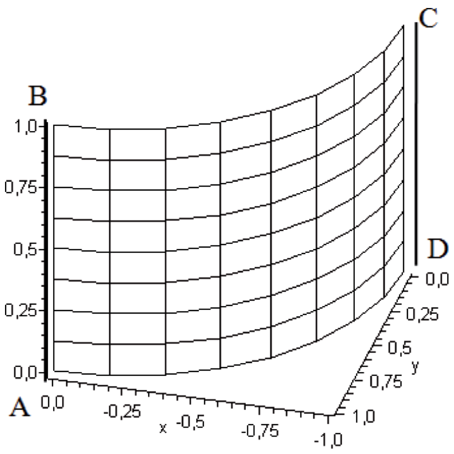

Fig. 5. Cylindrical shell with symmetry boundary conditions: $u_x = f_y = f_z = 0$ on AB; $u_y = f_x = f_z = 0$ on CD.
Rys. 5. Powłoka walcowa z warunkami brzegowymi symetrii: $u_x = f_y = f_z = 0$ na AB; $u_y = f_x = f_z = 0$ na CD

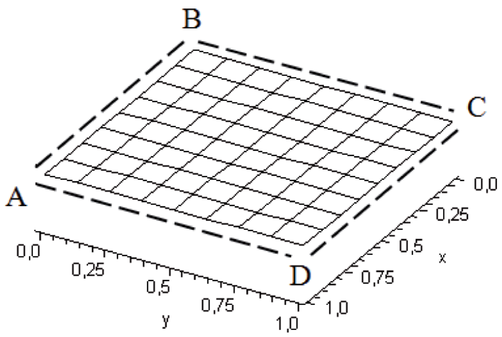

Fig. 6. Simply supported (pinned) plate.
Rys. 6. Płyta swobodnie podparta

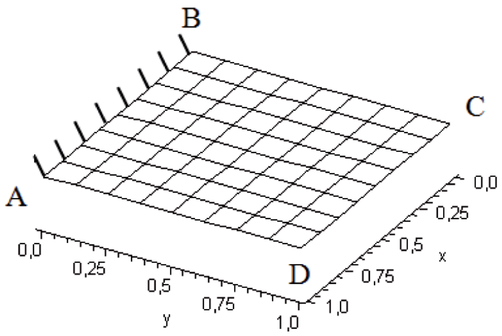

Fig. 7. Cantilever plate.
Rys. 7. Płyta wspornikowa

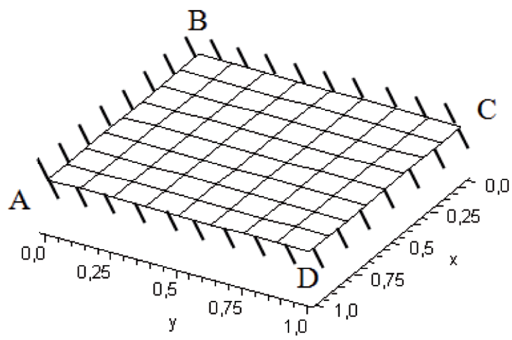

Fig. 8. Clamped plate.
Rys. 8. Płyta utwierdzona

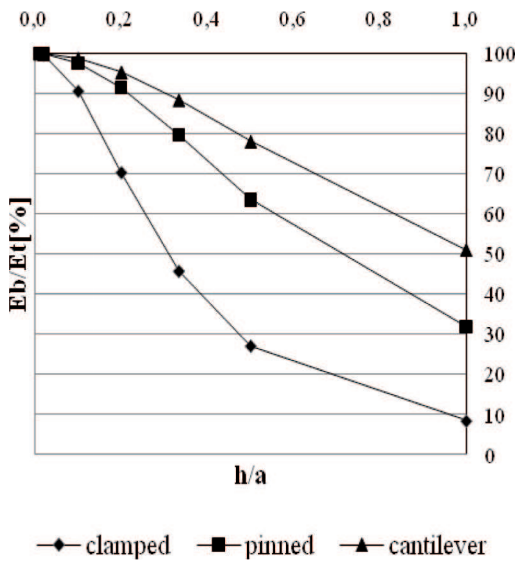

Fig. 9. Bending energy contribution in total energy depending on plate thickness (h/a=1/100, 1/50, 1/10, 1/5, 1/3, 1/2, ν=0.3).
Rys. 9. Udział energii zgięciowej w energii całkowitej w zależności of grubości płyty (h/a=1/100, 1/50, 1/10, 1/5, 1/3, 1/2, ν=0.3)

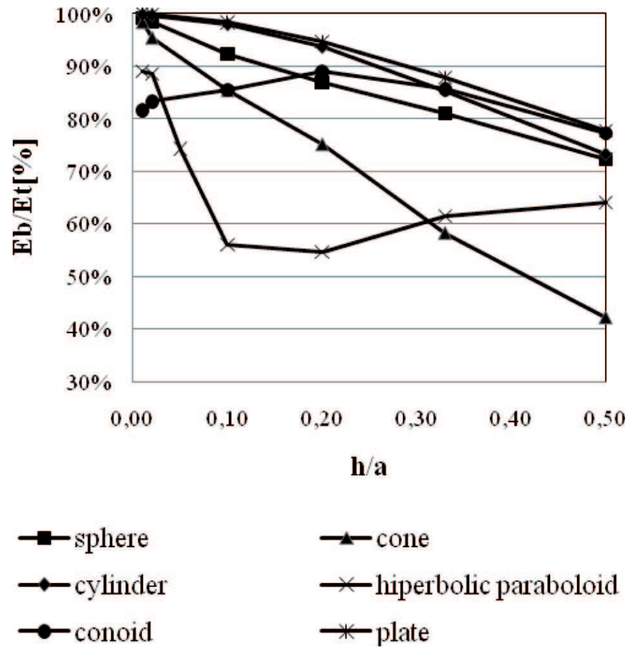

Fig. 10. Bending energy contribution in total energy depending on shell thickness (h/a=1/100, 1/50, 1/10, 1/5, 1/3, 1/2, ν=0.3).
Rys. 10. Udział energii zgięciowej w energii całkowitej w zależności of grubości powłoki (h/a=1/100, 1/50, 1/10, 1/5, 1/3, 1/2, ν=0.3)

4. STIFFNESS AND NORM MATRICES FOR THE INF-SUP TESTS

Two displacement norms are considered: L_2 – norm (associated with the displacements) and H_1 – semi-norm (associated with the first derivatives of displacements):

(4.1) $$||\mathbf{u}||^2_{L_2} = \iiint\limits_{\Omega} \mathbf{u}^T (x_1, x_2, x_3)\, \mathbf{u}\, (x_1, x_2, x_3)^d\, \Omega,$$

(4.2) $$||\mathbf{u}||^2_{H_1} = \iiint\limits_{\Omega} (\Delta\mathbf{u}\, (x_1, x_2, x_3))^T\, (\Delta\mathbf{u}\, (x_1, x_2, x_3))^d\, \Omega.$$

Following the plate/shell theory assumptions (2D models) and standard finite element approximation $\mathbf{u} = \mathbf{Nq}$ (with the use of displacement shape functions matrix) one can receive quadratic forms

(4.3) $$||\mathbf{u}||^2_{L_2} = \mathbf{q}^T \cdot \mathbf{S}0 \cdot \mathbf{q},$$

(4.4) $$||\mathbf{u}||^2_{H_1} = \mathbf{q}^T \cdot \mathbf{S}1 \cdot \mathbf{q}.$$

with corresponding norm matrices named **S**0 and **S**1. The norm matrix **S**0 is equivalent to the mass matrix (**M**) of a structure with unit material density and is easy to use in commercial programs. The use of complex norm matrix

$$\mathbf{S}01 = \mathbf{S}0 + a^2\mathbf{S}1, \tag{4.5}$$

where a is characteristic dimension of plate/shell mid-surface, it is more complicated and needs separate programming of the matrices.

The second crucial aspect of the inf-sup tests is the form of the stiffness matrix $\hat{\mathbf{K}}$**.** Full stiffness matrices (**K**) as well as shear or membrane oriented ($\mathbf{K}_s$, $\mathbf{K}_m$) matrices are considered in the examples presented in the paper.

5. EVALUATION OF SOME PLATE/SHELL FINITE ELEMENTS

The first set of examples is the analysis of well known plate rectangular finite elements with 4 corner nodes and 12 degrees-of-freedom. Full, selectively reduced and uniformly reduced integration is applied (see Gilewski [13], Taylor, Zienkiewicz [20] for detailed description of the finite elements). All matrices are calculated by the authors in a separate programme and included to the Abaqus commercial code. The objectives of those examples are to compare the use of **S**0 and **S**01 norm matrices as well as the evaluation of those finite elements. The results are presented on Fig. 11-19.

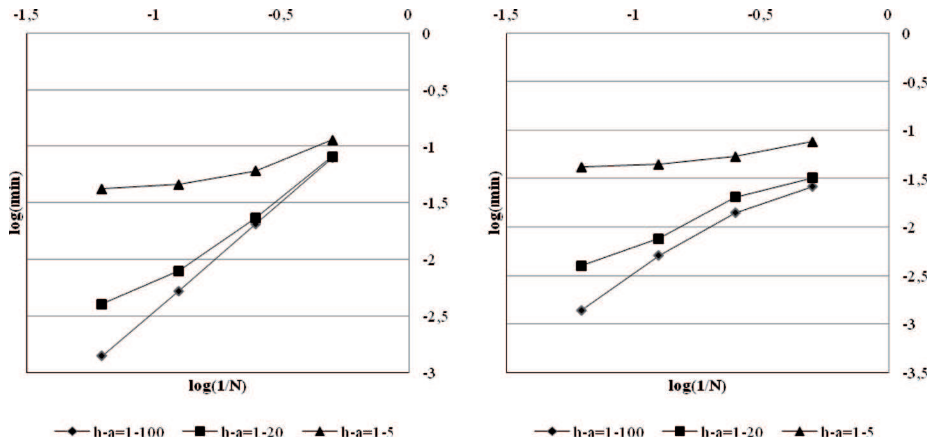

Fig. 11. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and full integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).

Rys. 11. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu pełnym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

Rectangular plate finite element with linear shape functions and full integration does not satisfy the inf-sup condition for three plate test problems. Locking effect is

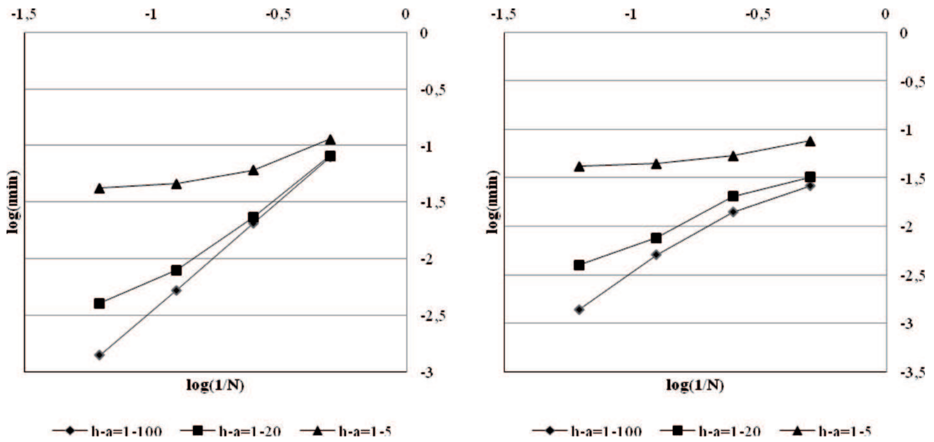

Fig. 12. Inf-sup test – clamped plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and full integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).

Rys. 12. Test inf-sup – płyta utwierdzona $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu pełnym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

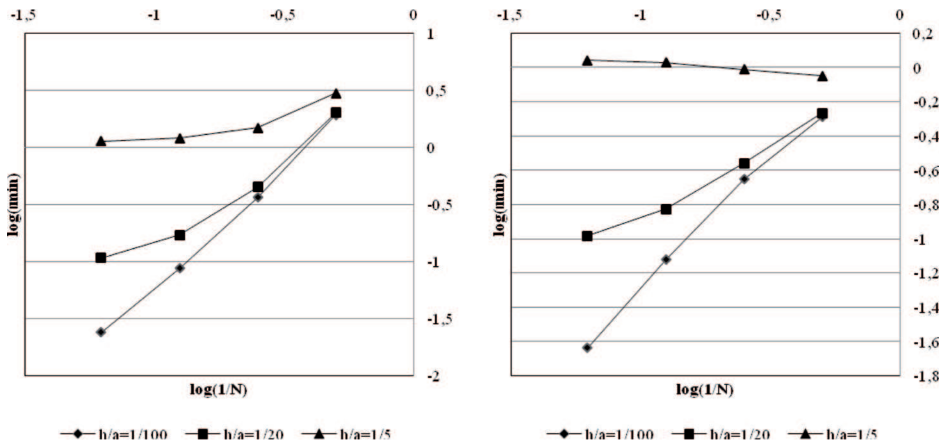

Fig. 13. Inf-sup test – simply supported plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and full integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).

Rys. 13. Test inf-sup – płyta swobodnie podparta $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu pełnym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

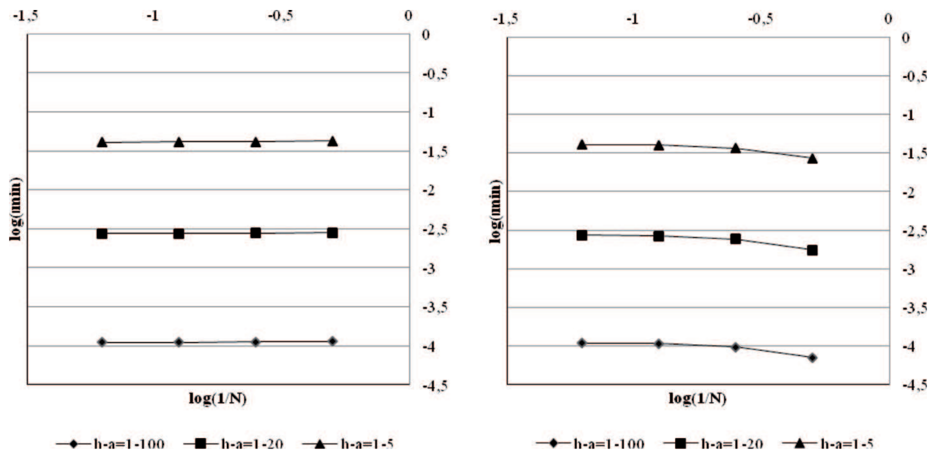

Fig. 14. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and selective/reduced integration. Norm matrices $\mathbf{S}0 = \mathbf{M}$(left) and $\mathbf{S}01$ (right).

Rys. 14. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu selektywnie zredukowanym. Macierze normowe $\mathbf{S}0 = \mathbf{M}$ (po lewej) i $\mathbf{S}01$ (po prawej)

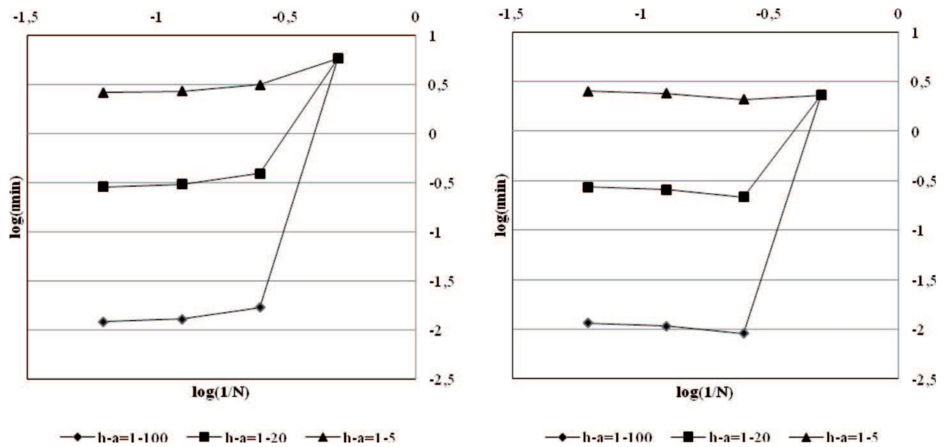

Fig. 15. Inf-sup test – clamped plate $h/a = 0.01, 0.05, 0.2$. Plate element with linear shape functions and selective/reduced integration. Norm matrices $\mathbf{S}0 = \mathbf{M}$(left) and $\mathbf{S}01$ (right).

Rys. 15. Test inf-sup – płyta utwierdzona $h/a = 0.01, 0.05, 0.2$. Element płytowy o liniowych funkcjach kształtu i całkowaniu selektywnie zredukowanym. Macierze normowe $\mathbf{S}0 = \mathbf{M}$ (po lewej) i $\mathbf{S}01$ (po prawej)

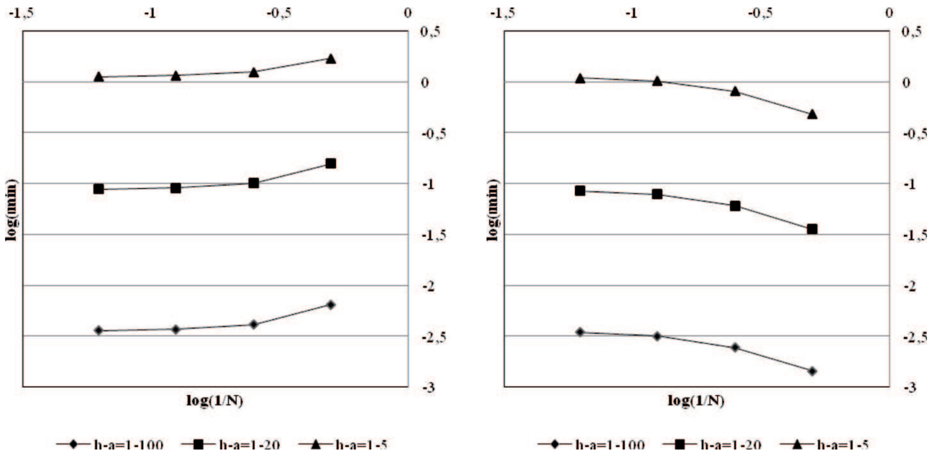

Fig. 16. Inf-sup test – simply supported plate h/a = $1/100$, $1/20$, $1/5$. Plate element with linear shape functions and selective/reduced integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).
Rys. 16. Test inf-sup – płyta swobodnie podparta h/a = $1/100$, $1/20$, $1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu selektywnie zredukowanym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

observed for thin plates. Qualititatively similar graphs are received for **S**0 = **M** and **S**01 norm matrices.

Rectangular plate finite element with linear shape functions and selective/reduced integration satisfy the inf-sup condition for three plate test problems. No locking effect is observed for thin or moderately thick plates. Qualititatively similar graphs are received for **S**0 = **M** and **S**01 norm matrices.

Rectangular plate finite element with linear shape functions and uniformly reduced integration satisfy the inf-sup condition for three plate test problems. No locking effect is observed for thin or moderately thick plates. Qualititatively similar graphs are received for **S**0 = **M** and **S**01 norm matrices.

To conclude the first group of tests: looking for the results it is well seen that the norm matrices **S**0=**M** can be used for the inf-sup tests.

After careful selection of the norm matrices, the following finite elements are analysed in this chapter: S4, S4R and S4R5 ABAQUS/Standard shell elements which are based on assumed strain concept. The norm matrices **S0=M** are used for all tests and full (**K**) or restricted shear/membrane ($\mathbf{K}_s$, $\mathbf{K}_m$) stiffness matrices are used for the shell . The results for various test problems and several geometrical parameters are presented in Fig. 20-28 for plate problems and in Fig. 29-34 for shell problems.

Abaqus shell/plate finite element S4R5 satisfy the inf-sup condition for three plate test problems. No locking effect is observed for thin or moderately thick plates.

Abaqus shell/plate finite element S4R satisfy the inf-sup condition for three plate test problems. No locking effect is observed for thin or moderately thick plates.

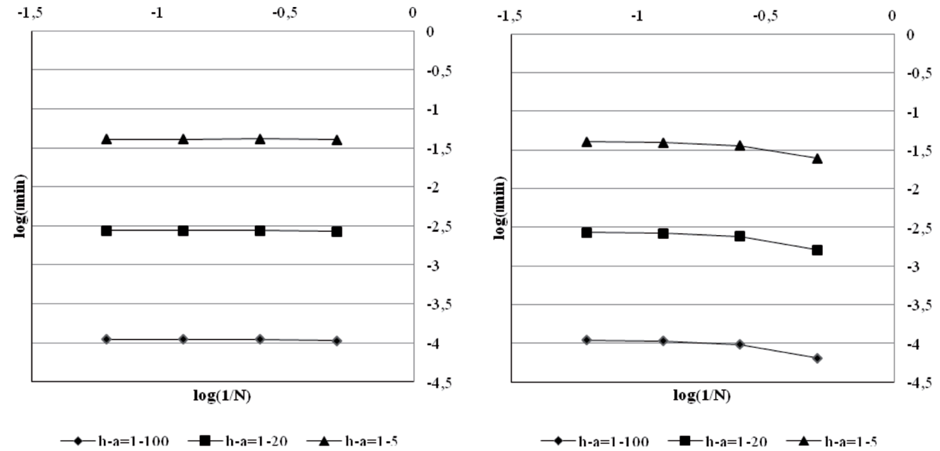

Fig. 17. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and uniformly reduced integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).

Rys. 17. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu zredukowanym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

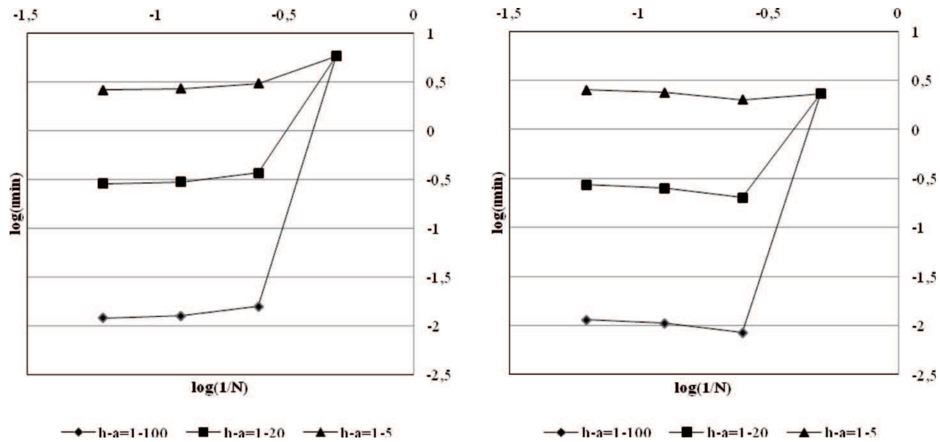

Fig. 18. Inf-sup test – clamped plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and uniformly reduced integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).

Rys. 18. Test inf-sup – płyta utwierdzona $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu zredukowanym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

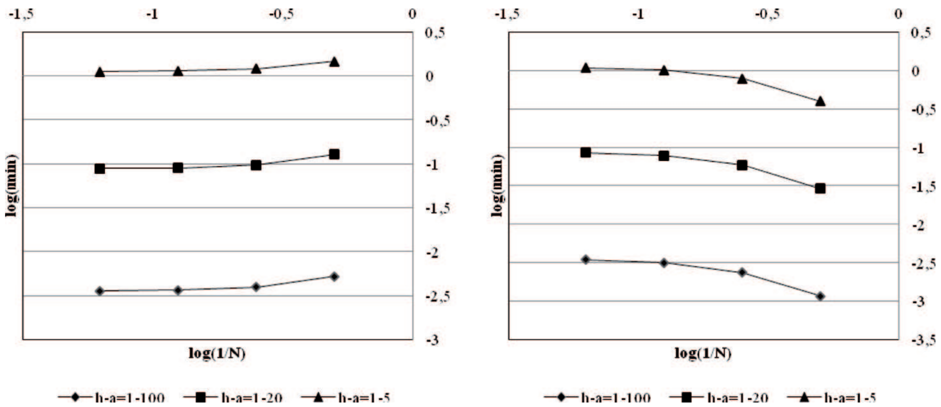

Fig. 19. Inf-sup test – simply supported plate $h/a = 1/100, 1/20, 1/5$. Plate element with linear shape functions and uniformly reduced integration. Norm matrices **S**0 = **M**(left) and **S**01 (right).
Rys. 19. Test inf-sup – płyta swobodnie podparta $h/a = 1/100, 1/20, 1/5$. Element płytowy o liniowych funkcjach kształtu i całkowaniu zredukowanym. Macierze normowe **S**0 = **M** (po lewej) i **S**01 (po prawej)

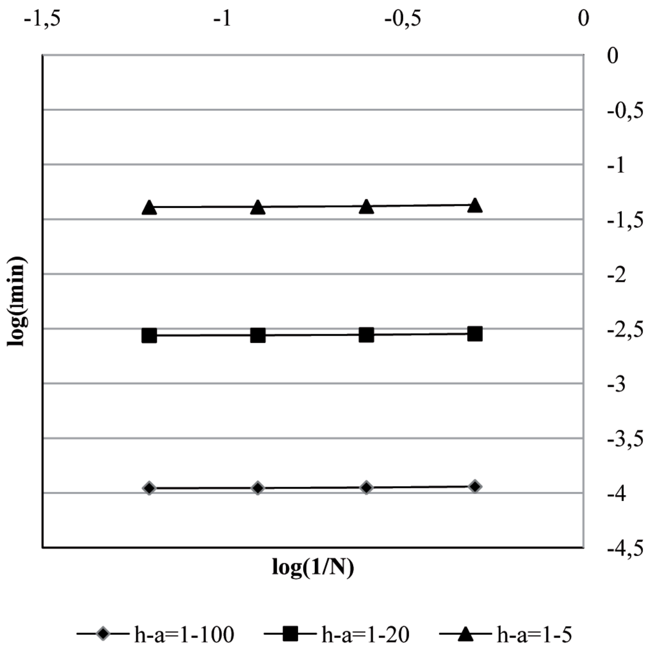

Fig. 20. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. S4R5 Abaqus element. Norm matrix **S**0 = **M**.
Rys. 20. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R5. Macierz normowa **S**0 = **M**

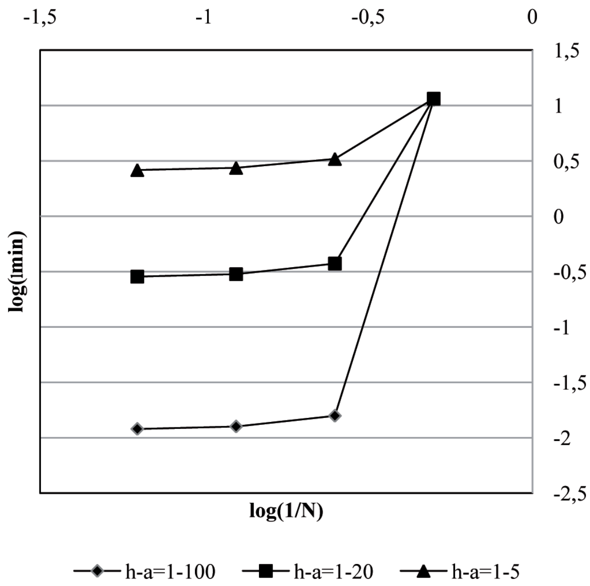

Fig. 21. Inf-sup test – clamped plate $h/a = 1/100, 1/20, 1/5$. S4R5 Abaqus element. Norm matrix **S**0 = **M**.
Rys. 21. Test inf-sup – płyta utwierdzona $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R5. Macierz normowa **S**0 = **M**

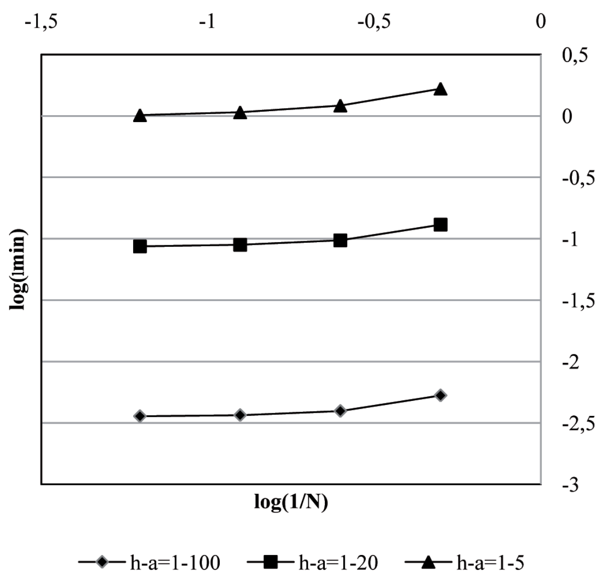

Fig. 22. Inf-sup test – simply supported plate $h/a = 1/100, 1/20, 1/5$. S4R5 Abaqus element. Norm matrix **S**0 = **M**.
Rys. 22. Test inf-sup – płyta swobodnie podparta $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R5. Macierz normowa **S**0 = **M**

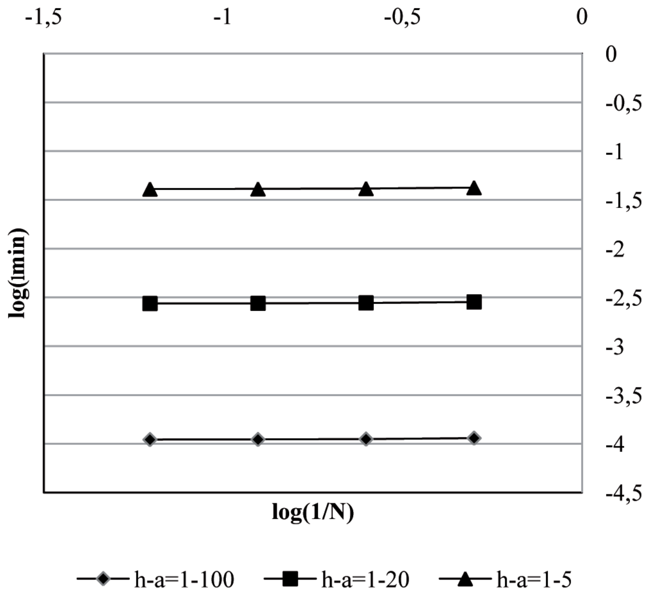

Fig. 23. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. S4R Abaqus element. Norm matrix **S**0 = **M**.
Rys. 23. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R. Macierz normowa **S**0 = **M**

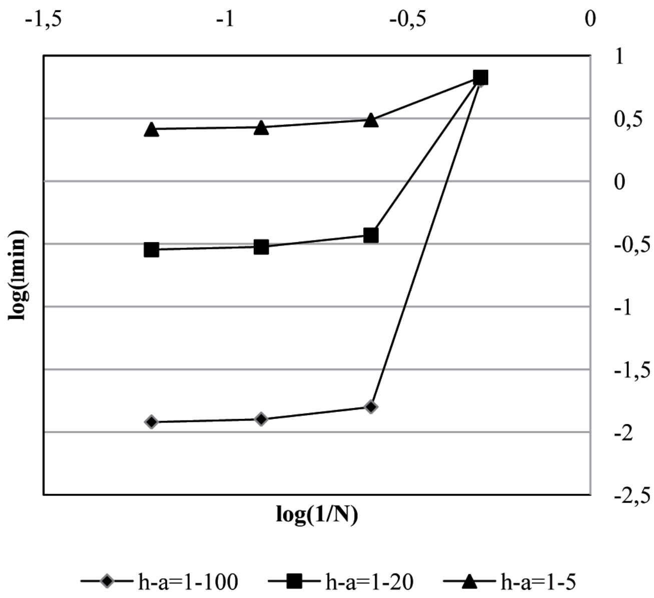

Fig. 24. Inf-sup test – clamped plate $h/a = 1/100, 1/20, 1/5$. S4R Abaqus element. Norm matrix **S**0 = **M**.
Rys. 24. Test inf-sup – płyta utwierdzona $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R. Macierz normowa **S**0 = **M**

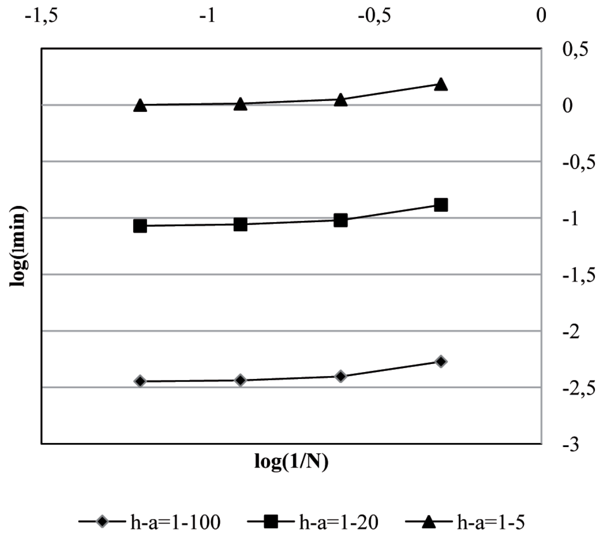

Fig. 25. Inf-sup test – simply supported plate $h/a = 1/100, 1/20, 1/5$. S4R Abaqus element. Norm matrix **S0** = **M**.

Rys. 25. Test inf-sup – płyta swobodnie podparta $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4R. Macierz normowa **S0** = **M**

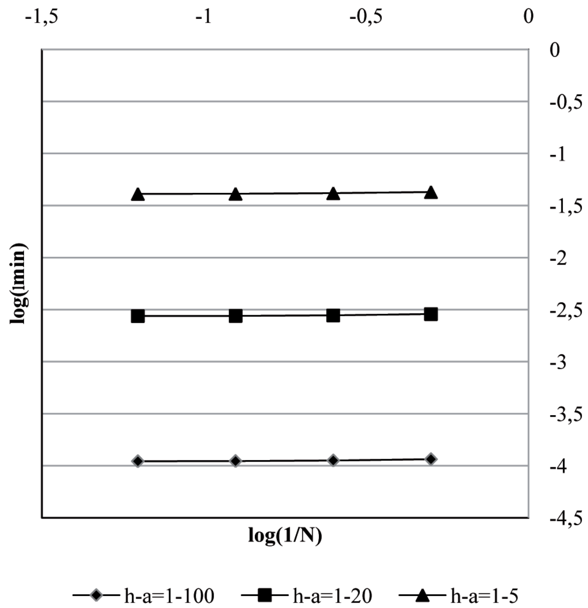

Fig. 26. Inf-sup test – cantilever plate $h/a = 1/100, 1/20, 1/5$. S4 Abaqus element. Norm matrix **S0** = **M**.

Rys. 26. Test inf-sup – płyta wspornikowa $h/a = 1/100, 1/20, 1/5$. Element Abaqus S4. Macierz normowa **S0** = **M**

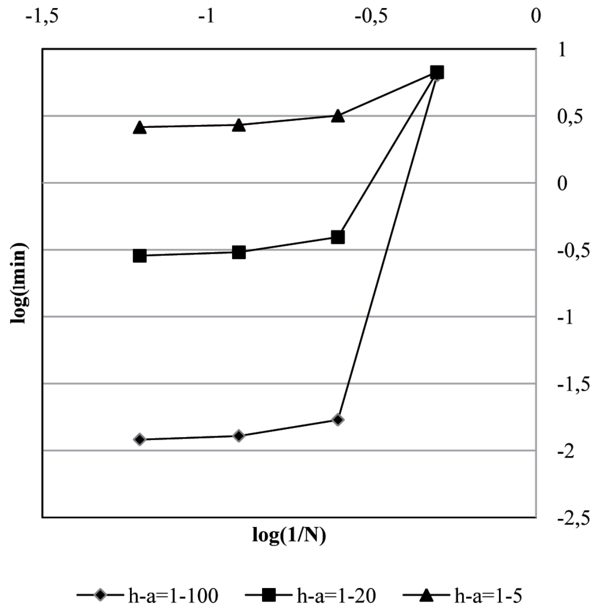

Fig. 27. Inf-sup test – clamped plate h/a = $1/100$, $1/20$, $1/5$. S4 Abaqus element. Norm matrix **S**0 = **M**.
Rys. 27. Test inf-sup – płyta utwierdzona h/a = $1/100$, $1/20$, $1/5$. Element Abaqus S4. Macierz normowa **S**0 = **M**

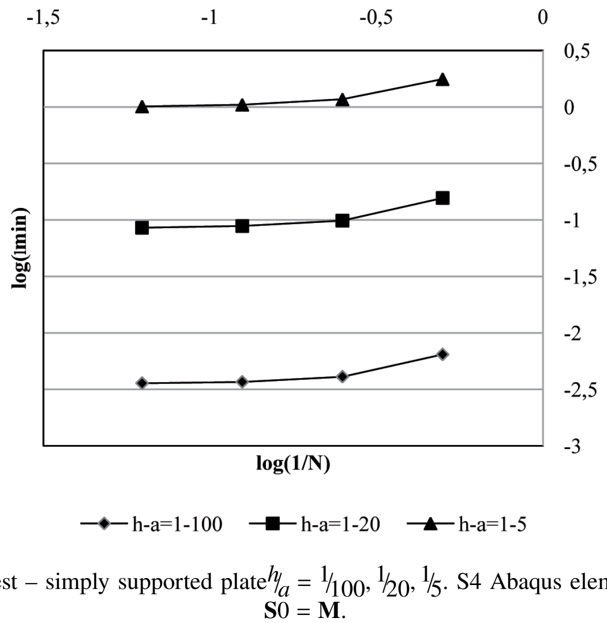

Fig. 28. Inf-sup test – simply supported plate h/a = $1/100$, $1/20$, $1/5$. S4 Abaqus element. Norm matrix **S**0 = **M**.
Rys. 28. Test inf-sup – płyta swobodnie podparta h/a = $1/100$, $1/20$, $1/5$. Element Abaqus S4. Macierz normowa **S**0 = **M**

Abaqus shell/plate finite element S4 satisfy the inf-sup condition for three plate test problems. No locking effect is observed for thin or moderately thick plates.

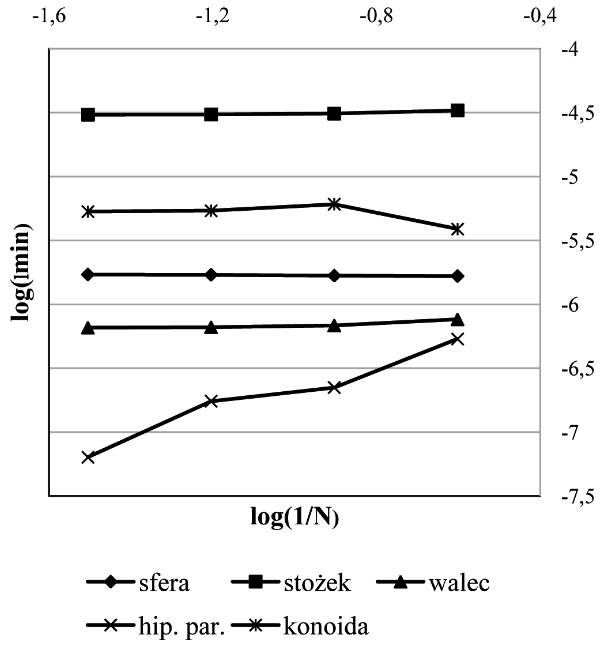

Fig. 29. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/1000$. S4R5 Abaqus element. Norm matrix **S0** = **M**. Full stiffness matrix**K**.
Rys. 29. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/1000$. Element Abaqus S4R5. Macierz normowa **S0** = **M**. Pełna macierz sztywności **K**

Abaqus shell/plate finite element S4R5 satisfy the inf-sup condition for sphere, cone, cylinder and conoid if full stiffness matrix is used. Locking effect is observed for hyperbolic paraboloid geometry.

Abaqus shell/plate finite element S4R satisfy the inf-sup condition for sphere, cone and cylinder. Locking effect is observed for hyperbolic paraboloid geometry as well as for conoid, if the shear oriented matrix $\mathbf{K}_s$ is used.

Abaqus shell/plate finite element S4 satisfy the inf-sup condition for sphere, cone and cylinder. Locking effect is observed for hyperbolic paraboloid geometry as well as for conoid, if the shear/membrane oriented matrix $\mathbf{K}_{sm}$ is used.

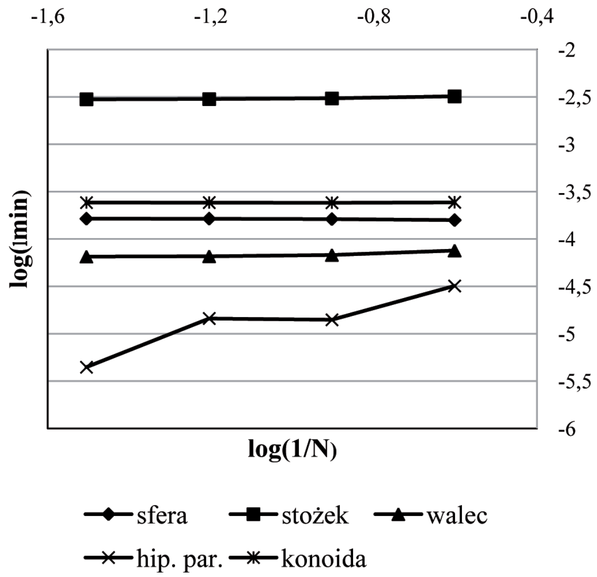

Fig. 30. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/100$. S4R5 Abaqus element. Norm matrix **S**0 = **M**. Full stiffness matrix **K**.

Rys. 30. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/100$. Element Abaqus S4R5. Macierz normowa **S**0 = **M**. Pełna macierz sztywności **K**

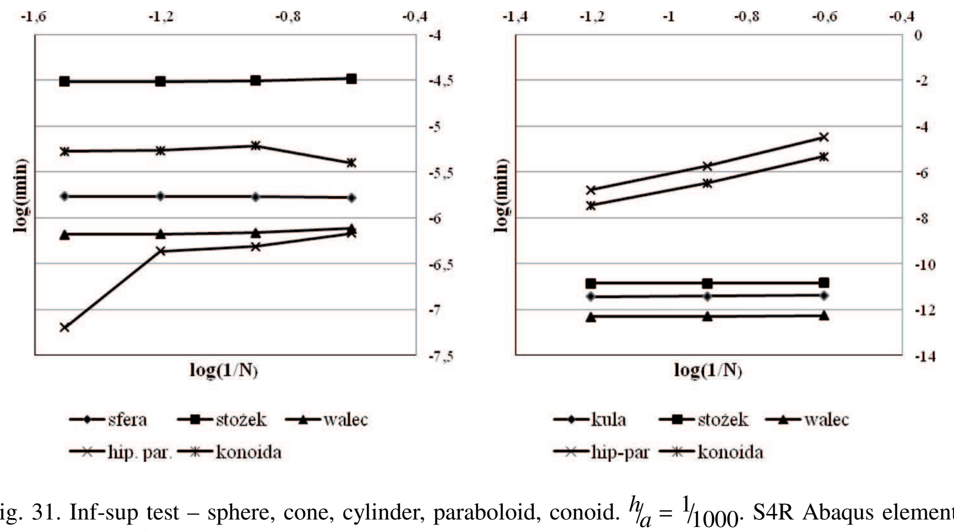

Fig. 31. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/1000$. S4R Abaqus element. Norm matrix **S**0 = **M**. Full stiffness matrix **K**(left) and shear oriented stiffness matrix $\mathbf{K}_s$ (right).

Rys. 31. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/1000$. Element Abaqus S4R. Macierz normowa **S**0 = **M**. Pełna macierz sztywności **K** (po lewej) i macierz sztywności na ścinanie $\mathbf{K}_s$ (po prawej)

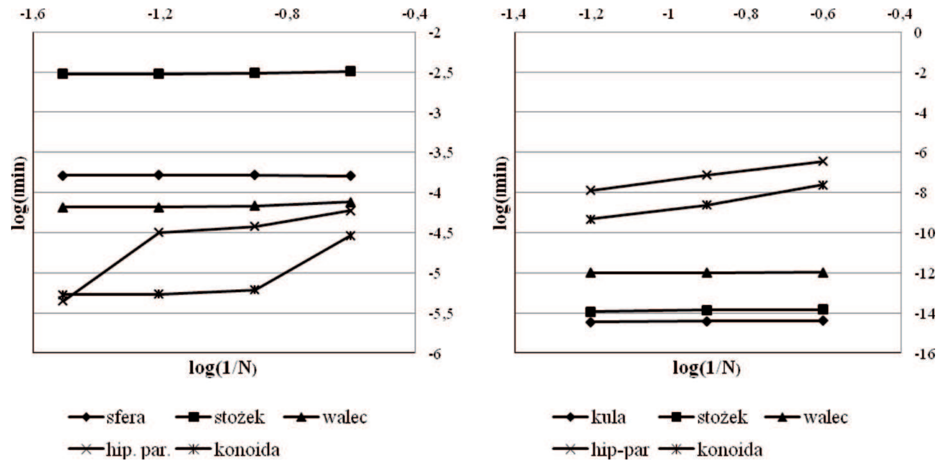

Fig. 32. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/100$. S4R Abaqus element. Norm matrix **S**0 = **M**. Full stiffness matrix **K**(left) and shear oriented stiffness matrix $\mathbf{K}_s$ (right).
Rys. 32. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/100$. Element Abaqus S4R. Macierz normowa **S**0 = **M**. Pełna macierz sztywności **K** (po lewej) i macierz sztywności na ścinanie $\mathbf{K}_s$ (po prawej)

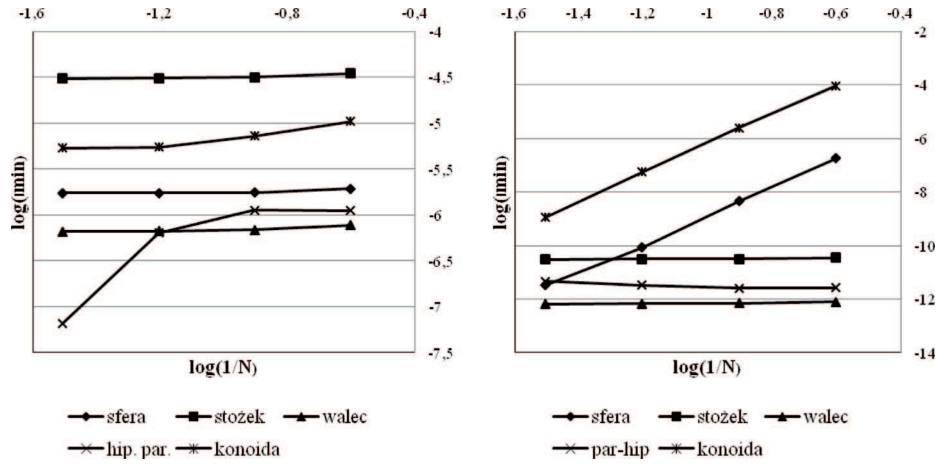

Fig. 33. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/1000$. S4 Abaqus element. Norm matrix **S**0 = **M**. Full stiffness matrix **K**(left) and shear/membrane oriented stiffness matrix $\mathbf{K}_{sm}$ (right).
Rys. 33. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/1000$. Element Abaqus S4. Macierz normowa **S**0 = **M**. Pełna macierz sztywności **K** (po lewej) i macierz sztywności na ścinanie/rozciąganie $\mathbf{K}_{sm}$ (po prawej)

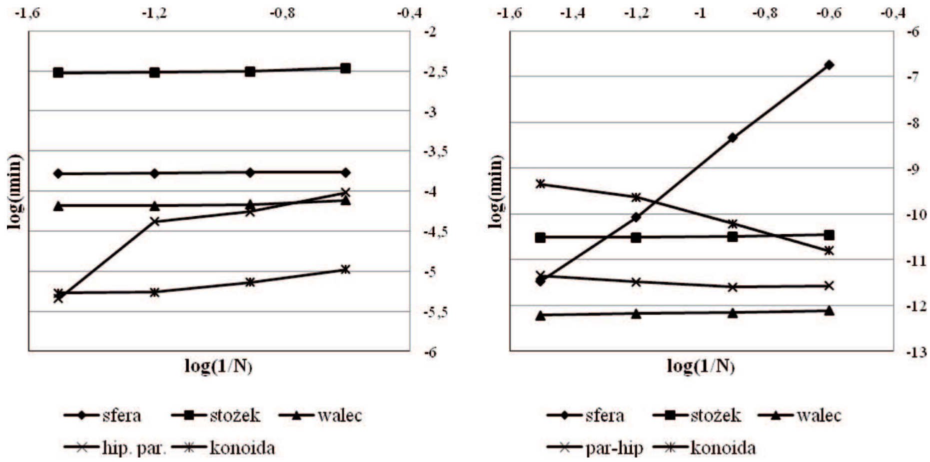

Fig. 34. Inf-sup test – sphere, cone, cylinder, paraboloid, conoid. $h/a = 1/100$. S4 Abaqus element. Norm matrix **S**0 = **M**. Full stiffness matrix **K**(left) and shear/membrane oriented stiffness matrix $\mathbf{K}_{sm}$ (right).
Rys. 34. Test inf-sup – sfera, stożek, walec, paraboloida, konoida. $h/a = 1/100$. Element Abaqus S4. Macierz normowa **S**0 = **M**. Pełna macierz sztywności **K** (po lewej) i macierz sztywności na ścinanie/rozciąganie $\mathbf{K}_{sm}$ (po prawej)

6. Concluding remarks

Three requirements should be satisfied during development or evaluation of shell/plate finite elements: ellipticity, consistency, and inf-sup condition. The inf-sup condition is the most difficult to evaluate. The authors followed and extended the concept of numerical verification of this condition developed by Bathe et al.

A set of shell and plate problems for this evaluation is proposed and analysed: rectangular plate with various boundary conditions, spherical shell, cylindrical shell, hyperbolic paraboloid, conoidal shell, and conical shell. The most demanding test is based on the hyperbolic paraboloid shell geometry.

Two norm matrices are analyzed – based on the L_2 displacement norm and H_1 semi-norm. It is proved that the norm matrix based on the L_2 norm is sufficient for the inf-sup tests. This matrix is equivalent to the mass matrix with unit density of the material, and is easy to evaluate shell/plate finite elements used in commercial codes.

It is also shown that it is possible to use a part of a stiffness matrix related to membrane and shear strains (for non-displacement models), as well as full stiffness matrix for evaluation of the inf-sup condition numerically with no qualitative differences of results.

S4, S4R and S4R5 ABAQUS/Standard shell elements which are based on assumed strain concept, and four-noded rectangular plate elements with linear shape functions, and full reduced or selective integration are evaluated.

The only finite element that do not pass the tests for plates is four noded plate element with linear shape functions and full integration. The other finite elements pass the tests.

The locking effects are observed on S4R5, S4R, S4 Abaqus element for thin hyperbolic paraboloid geometry if the full stiffness matrix **K** as well as for conoid, if the shear/membrane oriented matrix $\mathbf{K}_s$ or $\mathbf{K}_{sm}$ is used. No locking effects are observed for other shell geometries. No differencies in convergence the inf-sup parameter are observed for plate problems if full o restricted stiffnes matrices are used.

REFERENCES

1. D.N. ARNOLD, F. BREZZI, R.S. FALK, *Locking-free Reissner-Mindlin elements without reduced integration*. Comp. Meth. Appl. Mech. Eng., **196**, 3660-3671, 2007.
2. K.J. BATHE, *The inf-sup condition and its evaluation for mixed finite element methods*. Comp. Struct., **79**, 243-252, 2001.
3. K.J. BATHE, M.L. BUCALEM, F. BREZZI, *Displacement and stress convergence of our MITC plate bending elements*. Eng. Comp., **7**, 291-302, 1990.
4. K.J. BATHE, A. IOSILEVICH, D. CHAPELLE, *An evaluation of the MITC shell elements*, Comp. Struct., **75**, 1-30, 2000.
5. K.J. BATHE, A. IOSILEVICH, D. CHAPELLE, *An inf-sup test for shell finite elements*, Comp. Struct., **75**, 439-456, 2000.
6. F. BREZZI, K.J. BATHE, *A discourse on stability conditions for mixed finite element formulations*. Comp. Meth. Appl. Mech. Eng., **82**, 27-57, 1990.
7. F. BREZZI, M. FORTIN, *Mixed and Hybrid Finite Element Methods*. Springer-Verlag 1991.
8. D. CHAPELLE, K.J. BATHE, *The inf-sup test*. Comp. Struct., **47**, 537-545, 1993.
9. D. CHAPELLE, K.J. BATHE, *Optimal consistency errors for general shell elements*. Math. Prob. Mech. (Num. Anal.), C. R. Acad. Sci. Paris, **332**, serie I, 771-776, 2001.
10. D. CHAPELLE, K.J. BATHE, *The Finite Element Analysis of Shells – Fundamentals,* Springer, Berlin, Heidelberg, NewYork 2003.
11. D. CHAPELLE, I. PARIS SUAREZ, *Detailed reliability assessment of triangular MITC elements for thin shells*. Comp. Struct., **86**, 2192-2202, 2008.
12. W. GILEWSKI, *Correctness of plate-bending element with physical shape functions*. Finite Element News, **3**, 29-34, 1993.
13. W. GILEWSKI, *On the Criteria for Evaluation of Finite Elements – From Timoshenko Beam to Hencky-Bolle Plate* [in Polish], Oficyna Wydawnicza Politechniki Warszawskiej, Warsaw 2005.
14. J.F. HILLER, K.J. BATHE, *Measuring convergence of mixed finite element discretization: an application to shell structures*. Comp. Struct., **81**, 639-654, 2003.
15. T.J.R. HUGHES, E. HINTON eds., *Finite Element Methods for Plate and Shell Structures*. Vol. 1: Element Technology, Pineridge Press Int., Swensea, 1986.
16. A. IOSILEVICH, K.J. BATHE, F. BREZZI, *On evaluating the inf-sup condition for plate bending elements*, Int. Journ. Num. Methods Engng., **40**, 3639-3663, 1997.
17. H. KARDESTUNCER, D.H. NORRIE eds., *Finite Element Handbook*. Mac Graw-Hill, New York 1987.
18. M. RADWAŃSKA, *An overview of selected plate and shell FE models with graphic presentation of governing equations*. Comp. Ass. Mech. Eng. Sci., **14**, 431-456, 2007.
19. M. SITEK, *Correctness of rod, plate and shell finite elements* [in Polish], Ph. D. Thesis, Warsaw University of Technology 2005.

20. R.L. Taylor, O.C. Zienkiewicz, *The Finite Element Method*. Vol. 2: Solid Mechanics. Butterworth-Heinemann 2000.

TESTY WARUNKU INF-SUP DO OCENY PŁYTOWYCH I POWŁOKOWYCH ELEMENTÓW SKOŃCZONYCH

S t r e s z c z e n i e

Elementy skończone płyt i powłok powinny spełniać trzy warunki poprawności sformułowania: warunek eliptyczności, warunek zgodności i warunek inf-sup. Najtrudniejszy do sprawdzenia jest warunek inf-sup. W pracy rozwinięto i zweryfikowano koncepcję numerycznej weryfikacji tego warunku, zaproponowanej w pracach Bathego i współpracowników.

Przedstawiono i zweryfikowano szereg testów płyt o różnych warunkach brzegowych, oraz powłok o różnych krzywiznach Gaussa, w kształcie sfery, walce, paraboloidy hiperbolicznej, stożka i konoidy. Najbardziej wymagającym testem okazała się wspornikowa paraboloida hiperboliczna.

Zaproponowano i przeanalizowano dwie macierze normowe, bazujące na normie przemieszczeń w sensie L_2 i semi-normie H_1. Wykazano, że macierz zbudowana na bazie normy L_2 jest wystarczająca do badania warunku inf-sup. Macierz ta jest równoważna macierzy mas elementu skończonego, przy jednostkowej gęstości materiału, co oznacza ułatwione badania elementów skończonych w programach komercyjnych.

Przedstawiono także stosowność wykorzystania pełnej macierzy sztywności, oraz macierzy ograniczonych do opisu stanu poprzecznego ścinania lub stanu membranowego.

Zweryfikowano poprawność klasycznych prostokątnych elementów płytowych o liniowych funkcjach kształtu i całkowaniu pełnym lub zredukowanym, oraz elementów powłokowych S4, S4R and S4R5 statemu ABAQUS/Standard.

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