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An analytical technique for solving general linear integral equations of the second kind and its application in analysis of flash lamp control circuit

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Abstract. In this paper an application of the homotopy perturbation method for solving the general linear integral equations of the second kind is discussed. It is shown that under proper assumptions the considered equation possesses a unique solution and the series obtained in the homotopy perturbation method is convergent. The error of approximate solution, received by taking only the partial sum of the series, is also estimated. Moreover, there is presented an example of applying the method for approximate solution of an equation which has a practical application for charge calculation in supply circuit of the flash lamps used in cameras.

Key words: homotopy perturbation method, integral equation, convergence, error estimations.
MSC 2010: 45A05, 45L05, 65R20.

1. Introduction

In recent times the methods enabling to determine the solutions of operator equations modeling physical and technical problems of different kind have found a number of applications. One of these methods is the homotopy perturbation method [1–3]. This method can be used, among others, for solving differential equations [4–12]. Ganji and his colleagues [13–17] applied this method for considering various problems connected with the heat transfer processes. In papers [18, 19] an application of the homotopy perturbation method for determining the exact (or approximate) solution of the one-phase and two-phase inverse Stefan problem is shown. Whereas, in work the [20] this method is used for reconstructing the missing boundary condition in the inverse heat conduction problem. Furthermore, in the paper [21] an application of the discussed method for determining the temperature distribution in the cast-mould heterogeneous domain is presented. Convergence of the considered method in case of differential equations is investigated in papers [22–24].

The homotopy perturbation method is also very often used for seeking the solution of integral equations of different kind [25–38]. Convergence of this method in case of integral equations is considered only in few papers. Authors of the paper [32] prove the convergence of method with the so-called convex homotopy for Fredholm and Volterra integral equations of the second kind. Convergence and estimation of the error for the piecewise homotopy perturbation method used for solving the weakly singular Volterra integral equations is discussed in paper [36]. The convergence conditions of the homotopy perturbation method for Fredholm and Volterra integral equations of the second kind are formulated and proven in paper [37]. Moreover, the formulae for estimating the error of approximate solution are elaborated in this paper. Similar results in case of the Volterra-Fredholm integral equations of the second kind are presented in paper [38].

In the current paper we apply the homotopy perturbation method for solving the general linear integral equations of the second kind. Impulse for discussing this type of equations was given by its special case (considered in Sec. 4) which has practical application for charge calculation in supply circuit of the flash lamps used in cameras [39]. Moreover, we prove in this paper that under proper assumptions the discussed equation possesses the unique solution and the series obtained in the homotopy perturbation method is convergent. Estimation of the error of approximate solution, received by taking only partial sum of the series, is given as well. We also present examples of applying this method for considered equations.

2. Homotopy perturbation method

The homotopy perturbation method enables to seek a solution of the following operator equation

$$L(u) + N(u) = F, \tag{1}$$

where L denotes the linear operator, N can be either linear or nonlinear operator, F is the given function and u is the sought function.

Let us introduce a new operator H, called the homotopy operator, in the following way

$$H(v,p) = L(v) - L(u_0) + p\,(L(u_0) + N(v) - F), \tag{2}$$

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where $p \in [0,1]$ is the so-called homotopy parameter, $v(z,p) : \Omega \times [0,1] \to \mathbb{R}$, and u_0 defines the initial approximation of the solution of Eq. (1). Notice that $H(v,0) = L(v) - L(u_0)$, which means that for $p = 0$ the solution of operator equation $H(v,0) = 0$ is equivalent to the solution of a trivial problem $L(v) - L(u_0) = 0$. On the other hand, for $p = 1$ the solution of operator equation $H(v,1) = 0$ is equivalent to the solution of Eq. (1). It means that together with the change of parameter p (from zero to one) the solution v of $H(v,p) = 0$ changes from the trivial solution of $L(v) - L(u_0) = 0$ to the solution of the given equation (i.e. the solution v changes from u_0 to u).

In the homotopy perturbation method the solution of equation $H(v,p) = 0$ is searched in the form of power series

$$v = \sum_{j=0}^{\infty} p^j \, v_j. \tag{3}$$

If the above series possesses the radius of convergence not smaller than one and the series $\sum_{j=0}^{\infty} v_j$ is absolutely convergent, then according to Abel's Theorem the solution of Eq. (1) is obtained

$$u = \lim_{p \to 1^-} v = \sum_{j=0}^{\infty} v_j. \tag{4}$$

In many cases the series (3) is fast convergent, therefore even the partial sum of this series gives a very good approximation of the sought solution. The first $n+1$ components create the so-called n-order approximate solution in the form

$$\widehat{u}_n = \sum_{j=0}^{n} v_j. \tag{5}$$

In order to find the function v_j, relation (3) is substituted into equation $H(v,p) = 0$ and the expressions with the same powers of parameter p are compared. In this way we obtain the sequence of operator equations which enables to determine the successive functions v_j. Thus, determining the solution of considered problem can be reduced to solving the sequence of problems, solutions of which are easy to find.

3. General linear integral equation of the second kind

We consider the equation of the form

$$u(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(u(t))\, dt = F(x), \tag{6}$$

where $x \in [a,b]$, $R : C[a,b] \to C[a,b]$ is bounded linear operator, $f, g \in C[a,b]$, $a \le f(x) \le g(x) \le b$, $K \in C([a,b] \times [a,b])$ and $F \in C[a,b]$, whereas the function u is sought. Particular cases of the above equation are, obviously, the Fredholm and Volterra integral equations of the second kind.

Operators L and N can be define in the following way

$$L(v) = v, \qquad N(v) = -\int_{f(x)}^{g(x)} K(x,t)\, R(v(t))\, dt. \tag{7}$$

By using the above definitions and relation (2), we obtain the homotopy operator

$$H(v,p) = v(x) - u_0(x) + p\left(u_0(x) - \int_{f(x)}^{g(x)} K(x,t)\, R(v(t))\, dt - F(x) \right). \tag{8}$$

According to the method, in the next step we search for the solution of operator equation $H(v,p) = 0$ in the form of power series

$$v(x) = \sum_{j=0}^{\infty} p^j \, v_j(x). \tag{9}$$

In order to determine the functions v_j we substitute relation (9) into equation $H(v,p) = 0$ and we get (under assumption that the series is convergent, which will be discussed later, and by using the linearity of operator R):

$$\sum_{j=0}^{\infty} p^j \, v_j(x) = u_0(x) + p(F(x) - u_0(x)) + \sum_{j=1}^{\infty} p^j \int_{f(x)}^{g(x)} K(x,t)\, R(v_{j-1}(t))\, dt. \tag{10}$$

By comparing the expressions with the same powers of parameter p we receive the relations

$$v_0(x) = u_0(x), \tag{11}$$

$$v_1(x) = F(x) - u_0(x) + \int_{f(x)}^{g(x)} K(x,t)\, R(v_0(t))\, dt, \tag{12}$$

$$v_j(x) = \int_{f(x)}^{g(x)} K(x,t)\, R(v_{j-1}(t))\, dt, \qquad j \ge 2. \tag{13}$$

Now we proceed to discussing the convergence of series (9).

Since functions $K(x,t)$ and $F(x)$, appearing in Eq. (6), are continuous, then K and F are certainly bounded. It means, there exist the positive numbers M_1 and N_1 such that

$$|K(x,t)| \le M_1 \;\wedge\; |F(x)| \le N_1 \quad \forall\, x,t \in [a,b]. \tag{14}$$

In order to simplify the notation we denote the norm of operator R by M_0.

Theorem 1. If the following inequality

$$M_0 \, M_1 \, (b-a) < 1 \tag{15}$$

is satisfied and as the initial approximation u_0 any function, continuous in interval $[a,b]$, is chosen, then series (9), in which the functions v_j are determined by means of relations (11)–(13), is uniformly convergent in interval $[a,b]$ for each $p \in [0,1]$.

Proof. Let $u_0 \in C[a,b]$. Therefore there exists a positive number N_0 such that

$$|u_0(x)| \le N_0 \qquad \forall\, x \in [a,b].$$

Then we get the following estimations

$$|v_0(x)| = |u_0(x)| \le N_0,$$

$$|v_1(x)| = \left| F(x) - u_0(x) + \int_{f(x)}^{g(x)} K(x,t)\, R(v_0(t))\, dt \right|$$

$$\le |F(x)| + |u_0(x)| + \int_{f(x)}^{g(x)} |K(x,t)|\, |R(v_0(t))|\, dt$$

$$\le N_1 + N_0 + \int_{f(x)}^{g(x)} M_1\, M_0\, N_0\, dt$$

$$= N_0 + N_1 + N_0\, M_0\, M_1 \big(g(x) - f(x)\big)$$
$$\le N_0 + N_1 + N_0\, M_0\, M_1\, (b-a) =: B,$$

$$|v_2(x)| = \left| \int_{f(x)}^{g(x)} K(x,t)\, R(v_1(t))\, dt \right|$$

$$\le \int_{f(x)}^{g(x)} |K(x,t)|\, |R(v_1(t))|\, dt$$

$$\le \int_{f(x)}^{g(x)} M_1\, M_0\, B\, dt \le B\, M_0\, M_1\, (b-a),$$

where $B := N_0 + N_1 + N_0\, M_0\, M_1\, (b-a)$. In general we obtain

$$|v_j(x)| \le B\, \big(M_0\, M_1\, (b-a)\big)^{j-1}, \qquad x \in [a,b], \quad j \ge 1.$$

In this way, for considered series (9) we get for $p \in [0,1]$:

$$\sum_{j=0}^{\infty} p^j\, v_j(x) \le \sum_{j=0}^{\infty} |v_j(x)| \le N_0$$

$$+ B \sum_{j=1}^{\infty} \big(M_0\, M_1\, (b-a)\big)^{j-1} \overset{(15)}{=} N_0 + B \frac{1}{1 - M_0 M_1 (b-a)}.$$

It means that considered series (9) is uniformly convergent in interval $[a,b]$ for each $p \in [0,1]$.

Remark 1. From the above theorem it results that Eq. (6) possesses the solution in class $C[a,b]$.

Theorem 2. If inequality (15) holds, then the solution of Eq. (6) is unique.

Proof. Assume that integral Eq. (6) has two solutions u_1 and u_2, $u_1, u_2 \in C[a,b]$. Then for any $x \in [a,b]$, by using the properties of operator R, we get

$$|u_1(x) - u_2(x)| = \left| \int_{f(x)}^{g(x)} K(x,t)\, R(u_1(t) - u_2(t))\, dt \right|$$
$$\le M_0\, M_1\, (b-a)\, \|u_1 - u_2\|,$$

where $\|u_1 - u_2\| := \sup_{x \in [a,b]} |u_1(x) - u_2(x)|$. Hence we obtain

$$\|u_1 - u_2\| \le M_0\, M_1\, (b-a)\, \|u_1 - u_2\|.$$

And since $M_0\, M_1\, (b-a) < 1$, therefore it is possible only if $u_1 = u_2$.

Remark 2. Construction of the method implies that the sum of series (9) for $p = 1$ satisfies Eq. (6). If condition (15) is fulfilled, then the integral Eq. (6) possesses exactly one solution in the class of continuous functions. Hence, it follows that, in this case, series (9) for $p = 1$ is convergent to the only one solution of Eq. (6), independently on the selected initial approximation $u_0 \in C[a,b]$.

Remark 3. If we deal with the generalized Volterra equation of the second kind, it means if $f(x) = a$ and $g(x) = x$ for $x \in [a,b]$, then series (9) is always convergent (even if inequality (15) is not fulfilled). It results from the fact that in this case we obtain the following estimation (see also [37]):

$$|v_j(x)| \le B\, \frac{\big(M_0\, M_1\, (x-a)\big)^{j-1}}{(j-1)!}, \qquad x \in [a,b], \quad j \ge 1.$$

If we are not able to determine the sum of series (9) (for $p = 1$), then as the approximate solution of considered equation we can take the partial sum of this series. If we take the first $n+1$ terms, we obtain the so-called n-order approximate solution

$$\widehat{u}_n(x) := \sum_{j=0}^{n} v_j(x). \tag{16}$$

Now let us proceed to estimate the error of approximate solution constructed in this way.

Theorem 3. Error of the n-order approximate solution can be estimated in the following way

$$E_n \le B\, \frac{\big(M_0\, M_1\, (b-a)\big)^n}{1 - M_0\, M_1\, (b-a)}, \tag{17}$$

where $E_n := \sup_{x \in [a,b]} |u(x) - \widehat{u}_n(x)|$, moreover M_0, M_1 and B are the constants determined above.

Proof. By using the estimations of functions v_j we get for any $x \in [a,b]$:

$$|u(x) - \widehat{u}_n(x)| = \left| \sum_{j=0}^{\infty} v_j(x) - \sum_{j=0}^{n} v_j(x) \right|$$

$$= \left| \sum_{j=n+1}^{\infty} v_j(x) \right|$$

$$\le \sum_{j=n+1}^{\infty} |v_j(x)| \le B \sum_{j=n+1}^{\infty} \big(M_0\, M_1\, (b-a)\big)^{j-1}$$

$$= B\, \frac{\big(M_0\, M_1\, (b-a)\big)^n}{1 - M_0\, M_1\, (b-a)}.$$

Remark 4. In case of the generalized Volterra equation of the second kind, it means if $f(x) = a$ and $g(x) = x$ for $x \in [a,b]$, by using the estimation given in Remark 3 one can elaborate the following estimation of error (see also [37]):

$$E_n \le B\left(e^{M_0\,M_1\,(b-a)} - \sum_{j=0}^{n-1}\frac{\left(M_0\,M_1\,(b-a)\right)^j}{j!}\right) \le B\,\frac{(M_0\,M_1\,(b-a))^n}{(n+1)!}\left(n+e^{M_0\,M_1\,(b-a)}\right). \tag{18}$$

Example. In this example we apply the described method for solving the following equation

$$u(x) - \int_{x/2}^{x}\frac{x\,t}{2}\,u(t)\,dt = x - \frac{7}{48}\,x^4, \tag{19}$$

for $x \in [0,1]$. In considered equation $R(u) = u$ and

$$M_1 = \max_{x,t\in[0,1]}|K(x,t)| = \frac{1}{2}, \qquad M_0 = \|R\| = 1.$$

Hence

$$M_0\,M_1\,(b-a) = \frac{1}{2},$$

which means that the homotopy perturbation method will be convergent if we will select as u_0 the function continuous in interval $[a,b]$.

Let us set $u_0(x) = 0$. Then by calculating the successive functions v_j, determined by relations (11)–(13), we receive successively

$$\begin{aligned}
v_0(x) &= u_0(x) = 0,\\
v_1(x) &= x - \frac{7}{48}\,x^4,\\
v_2(x) &= \frac{7}{48}\,x^4 - \frac{49}{4096}\,x^7,\\
v_3(x) &= \frac{49}{4096}\,x^7 - \frac{25039}{37748736}\,x^{10},\\
v_4(x) &= \frac{25039}{37748736}\,x^{10} - \frac{11392745}{412316860416}\,x^{13},\\
v_5(x) &= \frac{11392745}{412316860416}\,x^{13} - \frac{74661215083}{81064793292668928}\,x^{16}.
\end{aligned}$$

Calculating the partial sums of series (9), that is the n-order approximate solutions $\widehat{u}_n$, we get

$$\begin{aligned}
\widehat{u}_5(x) &= x - 9.210066670181957\cdot 10^{-7}\,x^{16},\\
\widehat{u}_{10}(x) &= x - 3.916722852435405\cdot 10^{-15}\,x^{31},\\
\widehat{u}_{15}(x) &= x - 1.3977516439016215\cdot 10^{-24}\,x^{46},\\
\widehat{u}_{20}(x) &= x - 9.661593342504271\cdot 10^{-35}\,x^{61},\\
\widehat{u}_{25}(x) &= x - 1.948818708474067\cdot 10^{-45}\,x^{76},
\end{aligned}$$

where $x \in [0,1]$. Since $|\widehat{u}_n(x) - x| \to 0$, so $u(x) = x$.

Formula (17) gives the error estimation of the approximate solution (the worst possible case). In fact, the errors of approximate solutions are in general much smaller than the value determined in the right hand side of inequality (17). Since in the considered example we have

$$M_0 = 1, \qquad M_1 = \frac{1}{2}$$

and

$$N_0 = \max_{x\in[0,1]}|u_0(x)| = 0, \qquad N_1 = \max_{x\in[0,1]}|F(x)| = \frac{41}{48},$$

thus inequality (17) takes in this case the form

$$E_n \le O_n = \frac{41}{24}\left(\frac{1}{2}\right)^n.$$

Comparison of the errors of approximate solutions and estimations of these errors resulting from the above inequality are collected in Table1. In each case the real errors are much smaller than their theoretical estimations.

Table 1
Errors of approximate solution (E_n) and estimation (O_n) of these errors resulting from inequality (17)

n	E_n	O_n
5	$9.210\cdot 10^{-7}$	$5.339\cdot 10^{-2}$
10	$3.917\cdot 10^{-15}$	$1.668\cdot 10^{-3}$
15	$1.398\cdot 10^{-24}$	$5.213\cdot 10^{-5}$
20	$9.662\cdot 10^{-35}$	$1.629\cdot 10^{-6}$
25	$1.949\cdot 10^{-45}$	$5.091\cdot 10^{-8}$

4. Practical application in flash lamp control circuit

Now we deal with an equation having practical application for the charge calculation in supply circuit of flash lamps used in cameras. Supply circuit of flash lamps may be represented (for illustrative purposes) by the simple electrical circuit which consists of the source and, series connected with ideal switch, the resistor and the capacitor. Charging or discharging process may be represented by the first convolution integral. Part of the above mentioned Eq. (6), connected with second integral, may represent the simplified measurement circuit (with characteristic described by the integral function) which calculates charge collected on capacitor in some time interval. But the whole above mentioned equation will represent charge referred to some value which gives the input signal for main controller to stop, for example, the charging process of the capacitor or discharging the capacitor by connected flash lamp.

The main charging circuit, presented in Fig. 1, consists of: DC/DC converter [40, 41] (which converts small voltage from battery to relatively high voltage app. 300 V), switch s_1, charging current limiting resistor R_1 and main capacitor C_1. The rest of circuit after switch s_2 does not work during charging process of capacitor C_1. For simplicity all physical values of real elements shown in circuit presented in Fig. 1 are as follows: $R_1 = 100\ \Omega$, $R_2 = 10\ R_1$, $C_1 = 220\ \mu$F, $C_2 = 0.1\ C_1$. The charging and discharging process of capacitor C_1 is controlled by microcontroller (input and output signals of microcontroller are denoted in Fig. 1 by μC) with the help of switches s_1, s_2 and by transistor T_1. The xenon lamp ignition results from high voltage pulse generated by high voltage transformers T_{HV} after switching transistor T_1.

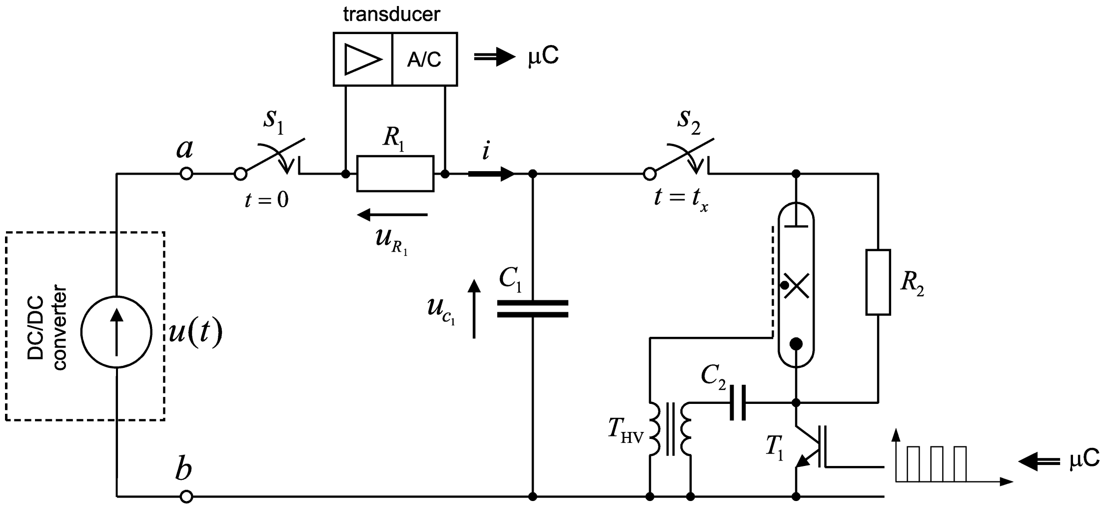


Fig. 1. Simplified driving circuit of xenon flash lamp

Let us consider two charging phases (of capacitor C_1) in two different cases:

Case 1. The batteries are charged to maximum value of voltage and it will be discharged for the first time. So time plot of supply voltage (applied to series connected resistance R_1 and capacitance C_1) may be regarded as the following function

$$u(t) = u_0\,1(t), \tag{20}$$

where u_0 – maximum DC/DC converter output voltage app. 300 V.

Case 2. The batteries are partially discharged (after some period of utilization – after for example 100 charging process of main capacitor C_1). So time plot of supply voltage (applied to series connected resistance R_1 and capacitance C_1) may be regarded as the following function

$$u(t) = u_0(1-t), \quad \text{for} \quad t \in [0,1]. \tag{21}$$

After finishing the processes related to case 1 it is possible to calculate reference charge collected at capacitance C_1, according to formula

$$Q_{ref} = \int\limits_0^{t_1} \frac{u_0\, e^{-t/(R_1 C_1)}}{R_1}\, dt, \tag{22}$$

where t_1 denotes capacitance C_1 charging time.

Case 2 occurs after some period of time, after intensive charging and discharging process of battery, its capacity decreased and also the internal resistances increased. It results in faster discharging the battery and in accordance output voltage of DC/DC inverter may vary strongly until main capacitor C_1 is charging. That state may be described (in simplified way) by function (21), so in case 2 calculation of charge will flow in following way:

– voltage drop calculation on R_1 resistor during supply by voltage described by function (21) (for solving that issue the Duhamel integral will be used),
– charging current calculation based on voltage drop on R_1 calculation,
– charge collected in capacitance C_1 calculation.

Duhamel integral is given by the following expression

$$y(t) = h(t)\,x(0) + \int\limits_0^t x'(t)\,h(t-\tau)\,d\tau, \tag{23}$$

where $h(t) = u_{R_1}(t)$ – voltage drop on R_1 resistances in case 1, $x(0) = 1$ – normalized (to nominal voltage value u_0) initial output voltage of DC/DC inverter, $x'(t) = -1$ – normalized (to nominal voltage value u_0) time derivative from Eq. (21), $h(t-\tau)$ – voltage drop on R_1 in case 2.

Voltage drop, in new supply condition according to (21), on limiting resistor R_1 is calculated as follows (basing on (23)):

$$\begin{aligned} u_{R_1}(t) &= u_0\, e^{-t/(R_1 C_1)} - \int\limits_0^t u_0\, e^{-(t-\tau)/(R_1 C_1)}\, d\tau \\ &= u_0 \left((1 + R_1 C_1)\, e^{-t/(R_1 C_1)} - R_1 C_1\right) \qquad \text{for } t \in [0,1]. \end{aligned} \tag{24}$$

In general the charge difference between two supply cases (case 1 and case 2) may be written in following form

$$\begin{aligned} \Delta Q = Q_{ref} - \int\limits_0^x \cos(x-t)\, \frac{1}{R_1} \\ \cdot \left(u_0\, e^{-t/(R_1 C_1)} - \int\limits_0^t u_0\, e^{-(t-\tau)/(R_1 C_1)}\, d\tau \right) dt, \end{aligned} \tag{25}$$

where x – limit of integration time of the transducer (which is shown in Fig. 1).

For examination purposes the part of presented in Fig. 1 circuit, representing charging circuit of capacitor, was implemented in MATLAB/Simulink program. The block diagram presented in Fig. 2 allows for calculating the charge collected in C_1 and the charging current and also allows for controlling the charging process.

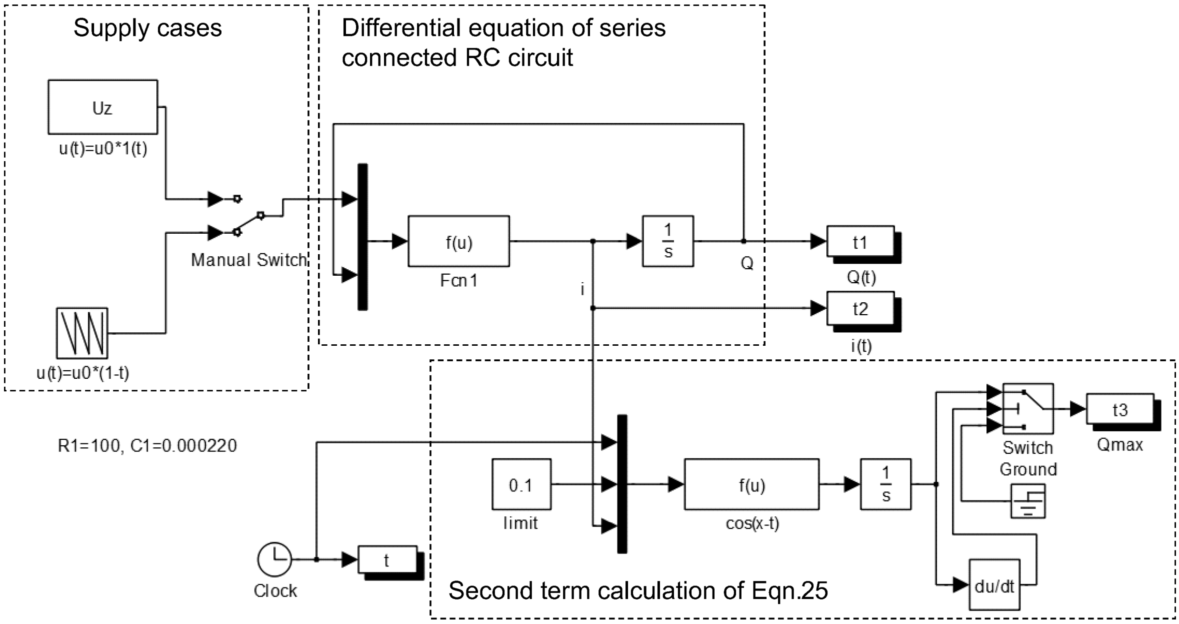


Fig. 2. Simulink block diagram of charging process

In Figs. 3 and 4 there are presented the results of simulation of the above diagram showing: changing in time the charge collected in C_1 during voltage supply conditions according to cases 1 and 2, charging currents during voltage supply conditions according to cases 1 and 2.

Studding the above figures (Figs. 3 and 4) it may be spotted that charge collected on capacitance C_1 in case 2 during charging process reaches the maximum value after some period of time, and after that gradually decreases. At this moment the current started flowing in opposite direction and discharging process of C_1 begins. It is undesired phenomenon and charging process should be stopped when the above mentioned situation occurs. The charge waiting for xenon lamp ignition may be calculated as follows

$$Q_x = Q_{ref} - \Delta Q. \tag{26}$$

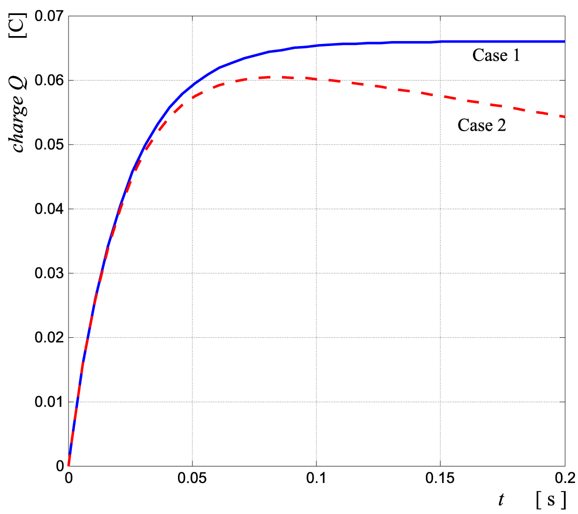


Fig. 3. Time plot of charge collected during 0.2 s in capacitor C_1 under supply condition related to cases 1 and 2

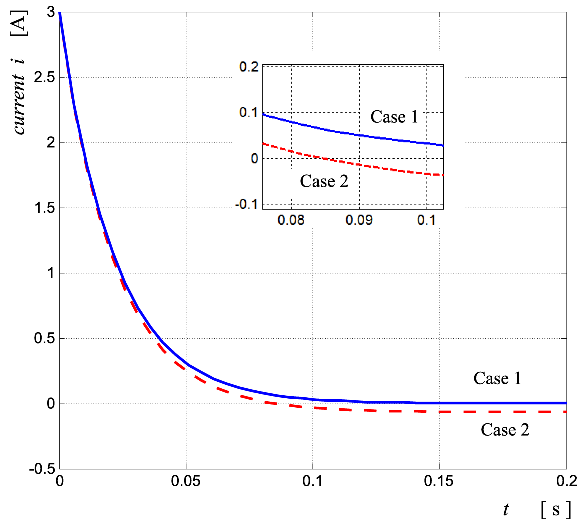


Fig. 4. Time plot of current flown in $R_1 C_1$ circuit under supply condition related to cases 1 and 2

5. Approximate solution obtained by using the homotopy perturbation method

Now we use the homotopy perturbation method for determining the approximate solution of the following integral equation of convolution type basing on Eq. (25):

$$u(x) - \int_0^x \cos(x-t) \left(\int_0^t h(\tau)\, u(t-\tau)\, d\tau \right) dt = F(x), \tag{27}$$

for $x \in [0,1]$, where (see Fig. 5):

$$h(t) = \begin{cases} \dfrac{1}{2}, & t \in \left[0, \dfrac{1}{4}\right) \cup \left[\dfrac{3}{4}, 1\right], \\ 1, & t \in \left[\dfrac{1}{4}, \dfrac{3}{4}\right), \end{cases}$$

and

$$F(x)=\begin{cases} x+\sin x, & x\in\left[0,\dfrac{1}{4}\right), \\ \dfrac{1}{4}-\sin\left(\dfrac{1}{4}-x\right)+\sin x, & x\in\left[\dfrac{1}{4},\dfrac{1}{2}\right), \\ \dfrac{5}{4}-2x-\sin\left(\dfrac{1}{4}-x\right)+2\sin\left(\dfrac{1}{2}-x\right)+\sin x, & \\ & x\in\left[\dfrac{1}{2},\dfrac{3}{4}\right), \\ -1+x-\sin\left(\dfrac{1}{4}-x\right)+2\sin\left(\dfrac{1}{2}-x\right) & \\ \qquad +32\sin\left(\dfrac{3}{4}-x\right)+\sin x, & x\in\left[\dfrac{3}{4},1\right]. \end{cases}$$

Because of the figure scale the plot of function F makes an illusive impression to be a polygonal chain. Certainly it is not in real (see the formula above).

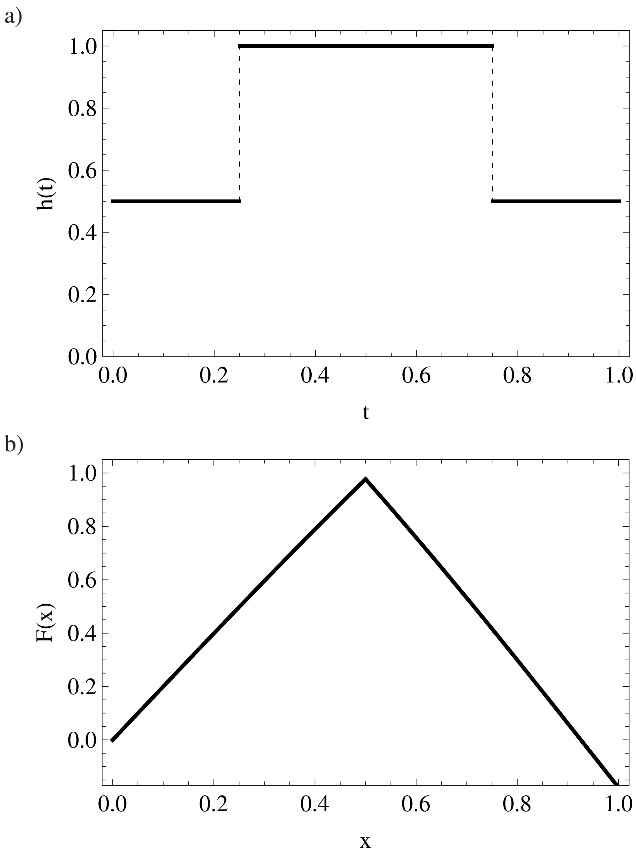


Fig. 5. Functions h and F

Exact solution of Eq. (27) is given by function (see Fig. 6):

$$u_e(x)=\begin{cases} 2\,x, & x\in\left[0,\dfrac{1}{2}\right), \\ 2-2\,x, & x\in\left[\dfrac{1}{2},1\right]. \end{cases}$$

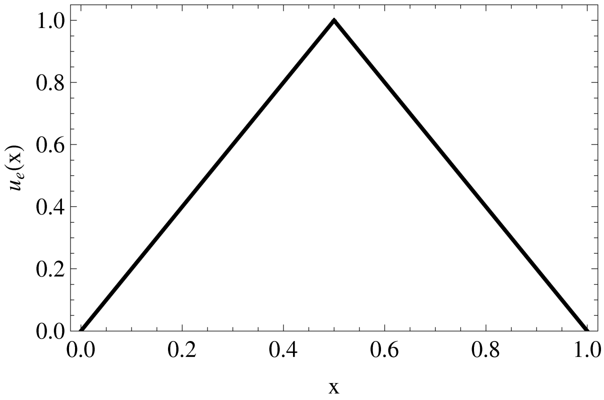


Fig. 6. Exact solution of Eq. (27)

Considered equation is the generalized Volterra equation of the second kind, therefore by applying the homotopy perturbation method we receive the convergent series (see Remark 3). By taking $u_0(x)=0$ we get successively

$$v_0(x)=0,$$

$$v_1(x)=F(x),$$

$$v_2(x)=\begin{cases} \dfrac{1}{4}(2x-x\cos x-\sin x), & x\in\left[0,\dfrac{1}{4}\right), \\ \dfrac{1}{8}\Bigg(4x+(1-4x)\cos\left(\dfrac{1}{4}-x\right)-2x\cos x & \\ \quad -4\sin\left(\dfrac{1}{4}-x\right)-2\sin x\Bigg), & \\ & x\in\left[\dfrac{1}{4},\dfrac{1}{2}\right), \\ \dfrac{1}{8}\Bigg(6-8x+(1-4x)\cos\left(\dfrac{1}{4}-x\right) & \\ \quad +(2x-1)\cos\left(\dfrac{1}{2}-x\right) & \\ \quad -2x\cos x-4\sin\left(\dfrac{1}{4}-x\right) & \\ \quad -10\sin\left(\dfrac{1}{2}-x\right)-2\sin x\Bigg), & \\ & x\in\left[\dfrac{1}{2},\dfrac{3}{4}\right), \\ \dfrac{1}{8}\Bigg(6-8x+(1-4x)\cos\left(\dfrac{1}{4}-x\right) & \\ \quad +(2x-1)\cos\left(\dfrac{1}{2}-x\right) & \\ \quad +(12x-9)\cos\left(\dfrac{3}{4}-x\right)-2x\cos x & \\ \quad -6\sin\left(\dfrac{1}{4}-x\right)-9\sin\left(\dfrac{1}{2}-x\right) & \\ \quad +2\left(6-\cos\dfrac{1}{4}+2\cos\dfrac{1}{2}+\cos\dfrac{3}{4}\right)\sin & \\ \quad \cdot\left(\dfrac{3}{4}-x\right)+\sin(1-x)-2\sin\left(\dfrac{5}{4}-x\right) & \\ \quad -\sin\left(\dfrac{3}{2}-x\right)-\sin x\Bigg), & x\in\left[\dfrac{3}{4},1\right]. \end{cases}$$

$$\vdots$$

In Table 2 there are compiled the errors with which the approximate solutions $\widehat{u}_n$ approximate the exact solution u_e, where

$$\Delta = \left(\int_0^1 \left(u_e(x) - \widehat{u}_n(x) \right)^2 dx \right)^{1/2},$$

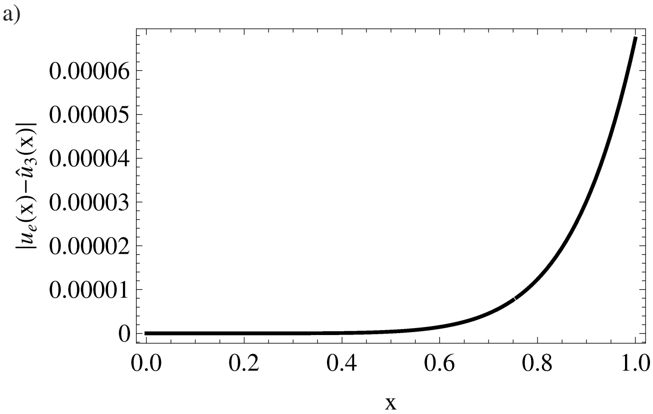
$$\delta = \frac{\Delta}{\left(\int_0^1 u_e^2(x)\, dx \right)^{1/2}} \cdot 100\ [\%].$$

Differences $|u_e(x) - \widehat{u}_n(x)|$ for $n = 3$ and $n = 6$ are displayed in Fig. 7. Obtained results indicate that the method is very fast convergent, therefore calculation of only few first components of series (9) ensures a very good approximation of exact solution.

Table 2
Errors of the exact solution approximation

n	Δ	δ [%]
1	$7.15441\,10^{-2}$	12.39181
2	$1.67791\,10^{-3}$	0.29062
3	$1.67199\,10^{-5}$	$2.89596\,10^{-3}$
4	$9.68642\,10^{-8}$	$1.67774\,10^{-5}$
5	$3.75990\,10^{-10}$	$6.51234\,10^{-8}$
6	$1.05772\,10^{-12}$	$1.83203\,10^{-10}$

a)



b)

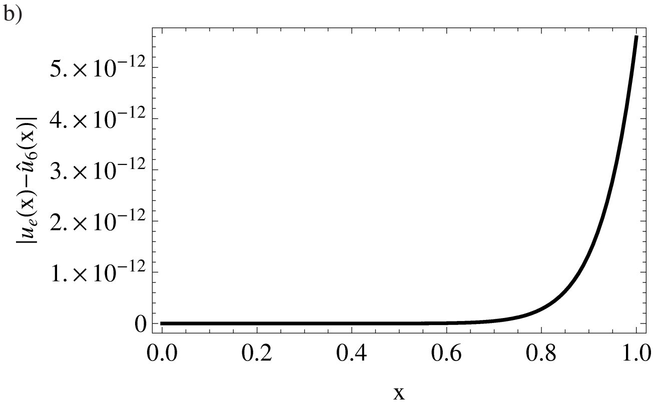


Fig. 7. Distribution of error of the exact solution approximation for $n = 3$ (a) and $n = 6$ (b)

6. Conclusions

In the paper we apply the homotopy perturbation method for solving the general linear integral equations of the second kind. Impulse for discussing this type of equations was given by its special case which has practical application for charge calculation in a supply circuit of the flash lamps used in cameras. We prove that under proper assumptions the discussed equation possesses the unique solution.

Using the homotopy perturbation method we obtain the function series, terms of which are determined iteratively. We prove that under proper assumptions the series is convergent. In many cases it is possible to determine the sum of received series and thereby to obtain the exact solution of the considered problem. In cases when the sum of series is impossible to find, one can use the initial terms of obtained series and form the approximate solution. Estimation of the error of approximate solution, received by taking only the partial sum of the series, is given in the paper. Executed calculations, the part of which is only presented in the paper, show that the created series is mostly very fast convergent. In such a case using only few initial terms of the series ensures a very small error of the exact solution reconstruction.

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