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# An iterative method for time optimal control of dynamic systems

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An iterative method for time optimal control of a general type of dynamic systems is proposed, subject to limited control inputs. This method uses the indirect solution of open-loop optimal control problem. The necessary conditions for optimality are derived from Pontryagin's minimum principle and the obtained equations lead to a nonlinear two point boundary value problem (TPBVP). Since there are many difficulties in finding the switching points and in solving the resulted TPBVP, a simple iterative method based on solving the minimum energy solution is proposed. The method does not need finding the switching point so that the resulted TPBVP can be solved by usual algorithms such as shooting and collocation. Also, since the solution of TPBVPs is sensitive to initial guess, a short procedure for making the proper initial guess is introduced. To this end, the accuracy and efficiency of the proposed method is demonstrated using time optimal solution of some systems: harmonic oscillator, robotic arm, double spring-mass problem with coulomb friction and F-8 aircraft.

**Key words:** dynamic systems, optimal control, time optimal control, Pontryagin's minimum principle, bounded control inputs

## 1. Introduction

Due to the fact that there are many practical applications where the task of finding the control objective in the shortest possible time is desired, the time optimal control of systems has been of great interest for decades. So, besides researches using the direct method [1]-[4] and other methods [5], many researchers studied the applications of Pontryagin's minimum principle to the time optimal control of linear and nonlinear systems. As regards solving the TPBVP the results from Pontryagin's minimum principle (PMP) are hard. Many numerical methods have been developed to solve time optimal control problems.

Ito and Kunisch [6] regularized the time optimal control problems for a class of linear multi input systems. Time optimal rest-to-rest maneuver of flexible structures, as linear systems, has been investigated by several authors [7]-[10]. The time optimal control of robots is investigated in many researches [11]-[14]. Also, based on the results of applying

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PMP on moon landing system, the problem of time optimal control of manipulators which move on a given manifold, e.g. a prescribed path, has been investigated by many researchers [15]-[17]. Meier and Bryson modified the steepest descent method of optimal programming to find time optimal switch times for bang-bang control systems and the switch time optimization (STO) program has been applied to a two-link manipulator with two control inputs [18]. Kaya and Noakes [19] proposed an efficient algorithm, called the time optimal switching (TOS) algorithm, for the time optimal switching control of nonlinear systems with a single control input. They found a feasible switching control using the switching time computation (STC) method to get from an initial point to a target point with a given number of switchings which was developed in their previous work [20].

Although numerous methods for determining switching times of time optimal problem have been investigated, the number of switching times must be specified at first while the number of switching is usually unknown before the problem is solved. Lee et al. [21] developed a novel problem transformation called the control parameterization enhancing transform (CPET) to address these difficulties in the case of time optimal control problems. Huang and Tseng [22] introduced a numerical two-phase scheme, which combines admissible optimal control problem formulation with enhanced branch-and-bound algorithms, to solve bang-bang control problems in the field of engineering. Xie and Kunisch [23] proposed a procedure to solve time optimal control problems based on Newton's method in combination with a continuous approximation to the discontinuous bang-bang control.

In this paper, the time optimal control of a general type of dynamic systems subject to control inputs limits is investigated. The problem is solved using a simple iterative method based on the indirect optimal control without necessity of finding the switching point, similar to what is proposed in [24] for determining the maximum payload. The necessary conditions for optimality are derived from Pontryagin's minimum principle which is studied in [25]-[27]. The obtained equations lead to a nonlinear two point boundary value problem (TPBVP) which can be solved by usual algorithm of solving TPBVPs such as shooting or collocation method. A residual control based collocation method [28], resulting in a piece of Matlab software called bvp6c, is applied to solve the TPBVP in this work. The TPBVPs are very sensitive to initial guess and choosing the different initial guesses may lead to a different solution which is not a desired one. So, in order to make the proper initial guess, an approximate linear system is considered and the corresponding optimal solution of this system is obtained using the analytical method which is probed in [27]. The proposed method is able to solve the time optimal control of nonlinear systems if its dynamic equations are in desired form or they can be approximated by a linear system. Principal computations are performed, however, using the original system.

To this end, some numerical simulations are studied. At first, the time optimal problem of well-known harmonic oscillator is solved and accuracy of the solution is compared with the analytical one. Then, time optimal control of a robot arm, as a multi-input system in desired form, is investigated. Finally, in order to show the ability of proposed algorithm for solving nonlinear systems which are not in desired form, the time optimal control of double mass-spring system in presence of coulomb friction and F-8 aircraft are studied.

## 2. Problem formulations

### 2.1. Problem statement

A dynamic system whose governing equations are the set of $2n$-first order equations, in the following form, is considered $\left(\mathbf{X} = [\mathbf{x}_1^T,\ \mathbf{x}_2^T]^T = [x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}]^T \in \Re^{2n}\right)$

$$\dot{\mathbf{X}} = \mathbf{A}\left(\mathbf{x}_1(t)\right)\mathbf{X}(t) + \mathbf{B}\left(\mathbf{x}_1(t)\right)\mathbf{u}(t) \tag{1}$$

where $\mathbf{A} \in \Re^{2n\times 2n}$ and $\mathbf{AB} \in \Re^{2n\times m}$ are the coefficient matrices of states and control inputs, respectively. Control inputs belong to the feasible set as follows

$$\mathbf{U} = \left\{\mathbf{u} \in \Re^m : U_i^- \leqslant u_i \leqslant U_i^+, i = 1, \ldots, n\right\}. \tag{2}$$

The problem specifies the admissible control inputs so as the system is changed from the initial states to the final states in minimum time. It should be noted that, although the systems in mentioned form (eqn. (1)) are desired and formulations for such systems are developed, the proposed method is able to find time optimal control of fully nonlinear system if it can be approximated by a linear system.

### 2.2. Optimal control

The optimal control problem determines the admissible control inputs $\mathbf{u} \in \mathbf{U}$, where $\mathbf{U}$ is the set of the admissible control inputs (2), such that the dynamic system described by the differential equation

$$\dot{\mathbf{X}} = f\left(\mathbf{X}(t), \mathbf{u}(t), t\right) \tag{3}$$

is transferred from the initial state at the initial time $(t_0)$ as follows

$$\mathbf{X}(t_0) = \mathbf{X}_0 \quad \in \Re^{2n} \tag{4}$$

to the admissible final state at the final time $(t_f)$ as follows

$$\mathbf{X}(t_f) = \mathbf{X}_f \quad \in \Re^{2n} \tag{5}$$

so that the following performance index be minimized

$$J = \int_{t_0}^{t_f} L\left(\mathbf{X}(t), \mathbf{u}(t), t\right) dt \tag{6}$$

with respect to state vector functions $\mathbf{X}(t)$ and control input functions $\mathbf{u}(t)$.

The indirect method which is based on a generalization of the calculus of variations has been applied here to solve the optimal control problem (PMP). With considering the first variation of the cost function $J$ with dynamic constraints adjoined in the manner of performance index $(L(\mathbf{X}(t),\mathbf{u}(t),t))$, the necessary conditions for optimality are derived. In this method, by introducing the vector of adjoint variables as $\mathbf{P} = [\mathbf{p}_1^T,\ \mathbf{p}_2^T]^T = [p_{11},\ldots,p_{1n},p_{21},\ldots,p_{2n}]^T \in \Re^{2n}$, the Hamiltonian function of the system can be presented as follows

$$H\left(\mathbf{X}\left(t\right),\mathbf{P}\left(t\right),\mathbf{u}\left(t\right),t\right) = L\left(\mathbf{X}\left(t\right),\mathbf{u}\left(t\right),t\right) + \mathbf{P}(t)^T f\left(\mathbf{X}\left(t\right),\mathbf{u}\left(t\right),t\right). \tag{7}$$

According to Pontryagin's minimum principle (PMP), the necessary conditions for optimality are as follows [25]-[27]

$$\dot{\mathbf{X}}^*\left(t\right) = \frac{\partial H\left(\mathbf{X}^*\left(t\right),\mathbf{P}^*\left(t\right),\mathbf{u}^*\left(t\right),t\right)}{\partial \mathbf{P}} \tag{8}$$

$$\dot{\mathbf{P}}^*\left(t\right) = -\frac{\partial H\left(\mathbf{X}^*\left(t\right),\mathbf{P}^*\left(t\right),\mathbf{u}^*\left(t\right),t\right)}{\partial \mathbf{X}} \tag{9}$$

$$\mathbf{u}^* = \arg\min_{\mathbf{u}\in\mathbf{U}} H\left(\mathbf{X}^*\left(t\right),\mathbf{P}^*\left(t\right),t\right). \tag{10}$$

The problem given by equations (8) and (9) is a set of $4n$-first order ordinary differential equations, with considering $m$ input control equations (10), which needs $4n$ conditions to be solved. With considering the $2n$ known initial states $\mathbf{X}_0$ equation (4) and $2n$ final states $\mathbf{X}_f$ equation (5) the required conditions are supplied.

In order to solve the time optimal problem using traditional methods, because of unknown final time, an extra boundary condition, i.e. $H(t_f) = 0$, is required. This condition is usually more complex than the usual conditions and its application to complicated systems make the TPBVP too difficult to solve. Here, similar to what is proposed in [24] for determining the maximum payload, the minimum energy cost function is chosen as the desired fixed final time performance index

$$L\left(\mathbf{X}\left(t\right),\mathbf{u}\left(t\right),t\right) = \mathbf{u}^T\mathbf{R}\mathbf{u} \tag{11}$$

where $\mathbf{R} \in \Re^{m\times m}$ is symmetric, positive semi-definite weighting matrix which expresses the weight of input controls in cost function. In this work, energy weighting matrix is considered as a diagonal matrix $\mathbf{R} = \mathrm{diag}\left(\varepsilon_1,\ldots,\varepsilon_n\right)$. So the minimum time solution is obtained by reducing the terminal time to possible extend. The exact algorithm is elaborated in more.

As mentioned before, in order to provide the proper initial guess, the optimal solution of a linear system (approximated system) is required. If the system is considered as a linear, time-varying plant as

$$\dot{\mathbf{X}}\left(t\right) = \bar{\mathbf{A}}\mathbf{X}\left(t\right) + \bar{\mathbf{B}}\mathbf{u}\left(t\right). \tag{12}$$

With mentioned cost function (11) and boundary conditions given by equation (4) and (5), the optimal states and controls are obtained using the analytical method presented in [27]. This method is based on assuming the vector of adjoint variables as follows

$$\mathbf{P}(t) = \mathbf{Y}(t)\mathbf{X}(t) - \mathbf{Z}(t) \tag{13}$$

where $\mathbf{Y} \in \Re^{2n\times 2n}$ and $\mathbf{Z} \in \Re^{2n}$. Here, this analytical method is summarized to a three-step numerical method as follows

1. Solving the following ordinary differential equations (ODEs)

$$\begin{aligned} \dot{\mathbf{Y}}(t) &= -\mathbf{Y}(t)\bar{\mathbf{A}} - \bar{\mathbf{A}}^T\mathbf{Y}(t) + \mathbf{Y}(t)\bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}^T\mathbf{Y}(t) \quad \in \quad \Re^{2n\times 2n} \\ \dot{\mathbf{Z}}(t) &= \left(\mathbf{Y}(t)\bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}^T - \bar{\mathbf{A}}^T\right)\mathbf{Z}(t) \quad \in \quad \Re^{2n} \end{aligned} \tag{14}$$

   with considering the following initial values

$$\begin{aligned} \mathbf{Y}(t_f) &= \mathbf{I} \quad \in \quad \Re^{2n\times 2n} \\ \mathbf{Z}(t_f) &= \mathbf{X}_f \quad \in \quad \Re^{2n} \end{aligned} \tag{15}$$

   where $\mathbf{I}$ is the identical matrix of $2n$-dimensional space. Since $\mathbf{Y}(t_f)$ is symmetric, the mentioned equations are the set of $n(2n+1)$ first-order equations which could be solved using Euler method or Runge-Kutta method, easily.

2. Computing the optimal states by solving the following ODEs with considering initial states (4) and the solution of $\mathbf{Y}(t)$ and $\mathbf{Z}(t)$ from the previous step.

$$\dot{\mathbf{X}} = \left(\bar{\mathbf{A}} - \bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}^T\mathbf{Y}(t)\right)\mathbf{X}(t) + \bar{\mathbf{B}}\mathbf{R}^{-1}\bar{\mathbf{B}}^T\mathbf{Z}(t). \tag{16}$$

3. Calculating the optimal adjoint variables using $\mathbf{P}(t) = \mathbf{Y}(t)X(t) - \mathbf{Z}(t)$.

### 2.3. Dynamic system formulations

With considering the governing equation of system as equation (1) and performance index of system as equation (11), by substituting equation (1) and (11) into equation (7), (9) and (10), the adjoint differential equation and unlimited control inputs' equations ($\mathbf{u}^\infty \in \Re^m$) are rewritten in following form

$$\begin{aligned} \dot{p}_{1i} &= -\mathbf{P}^T\left(\mathbf{A}\mathbf{e}_i + \frac{\partial \mathbf{A}}{\partial x_{1i}}\mathbf{X} + \frac{\partial \mathbf{B}}{\partial x_{1i}}\mathbf{u}\right) \quad i = 1,\ldots,n \\ \dot{p}_{2i} &= -\mathbf{P}^T\mathbf{A}\mathbf{e}_{i+n} \quad i = 1,\ldots,n \end{aligned} \tag{17}$$

$$\mathbf{u}^\infty = -\frac{1}{2}\mathbf{R}^{-T}\mathbf{B}^T\mathbf{P} \tag{18}$$

where $\mathbf{e}_i$ is the $i$th column (or row) of the identical matrix of $2n$-dimensional space. Also, it is fully clear that equation (1) is the same as equation (8). Since control inputs are usually bounded ($\mathbf{u} \in \mathbf{U}$), control inputs' equations should be modified as follows

$$u_i = \begin{cases} U_i^+ & U_i^+ < u_i^\infty \\ u_i^\infty & U_i^- \leqslant u_i^\infty \leqslant U_i^+ \\ U_i^- & u_i^\infty < U_i^- . \end{cases} \tag{19}$$

It should be noticed that, these formulation is not valid for systems where the governing equations are not in desired form (equation (1)). For such fully nonlinear systems the adjoint differential equations are obtained by applying the main equations (7) and (9). If the governing equations of these systems do not include nonlinear terms in $\mathbf{u}$, the control inputs' equations are obtained by applying the main equations (7) and (10), similarly to computing the adjoint differential equations. Otherwise, if the governing equations of these systems include nonlinear terms in $\mathbf{u}$, the control inputs' equations are obtained using equation (18) with considering the control inputs' coefficient matrix of linear model of system instead of matrix $\mathbf{B}$.

It must be noted that, these assumption impress the minimum energy solution while it does not affect the minimum time solution. In the other words, the results obtained with this assumption (using the control inputs' coefficient matrix of linear model of system for specifying the optimal control inputs) may not be the most minimum energy solution. Since however the consumed energy is unimportant in time optimal solution, this assumption does not affect the time optimal solution.

## 3. Algorithm of minimum time calculation

As mentioned, the presented formulation including $4n$ first-order ordinary differential equation (1) and (17) with considering control inputs' equations (19) and boundary conditions equations (4) and (5) leads to the two point boundary value problem (TP-BVP). In order to find the minimum time used by the system to transfer it from initial state to the final one, the following algorithm should be executed.

- Generating the required equations and boundary conditions.
- Making the proper initial guess by solving the linear system of $\bar{\mathbf{A}} = \mathbf{A}(\bar{\mathbf{x}}_1)$ and $\bar{\mathbf{B}} = \mathbf{B}(\bar{\mathbf{x}}_1)$ using the three-step method without considering bounded controls, where $\bar{\mathbf{x}}_1 = (\mathbf{x}_1(t_0) + \mathbf{x}_1(t_f))/2$. If the governing equations of the system are not in the desired form, the linear model of the system is used to make the initial guess. It should be noted, that the terminal time should be chosen long enough so that the control inputs do not exceed their bounds, but not too long.
- Assuming that a proper initial guess is provided, the optimal time is calculated by applying the algorithm presented in Fig. 1 with considering the limited control

inputs (eqn. (19)). Before that, since the terminal time is unknown and according to following algorithm the final time is varied, it is recommended to transform the time interval into a normalized one. Transformation $t = \eta\,(t_f - t_0) + t_0$ changes $t_0 \leqslant t \leqslant t_f$ to $0 \leqslant \eta \leqslant 1$. Applying this transformation, the set of state space equations are obtained as follows

$$\frac{d}{d\eta}\begin{bmatrix}\mathbf{X}\\ \mathbf{P}\end{bmatrix} = t_f\begin{bmatrix}\dot{\mathbf{X}}\\ \dot{\mathbf{P}}\end{bmatrix}. \tag{20}$$

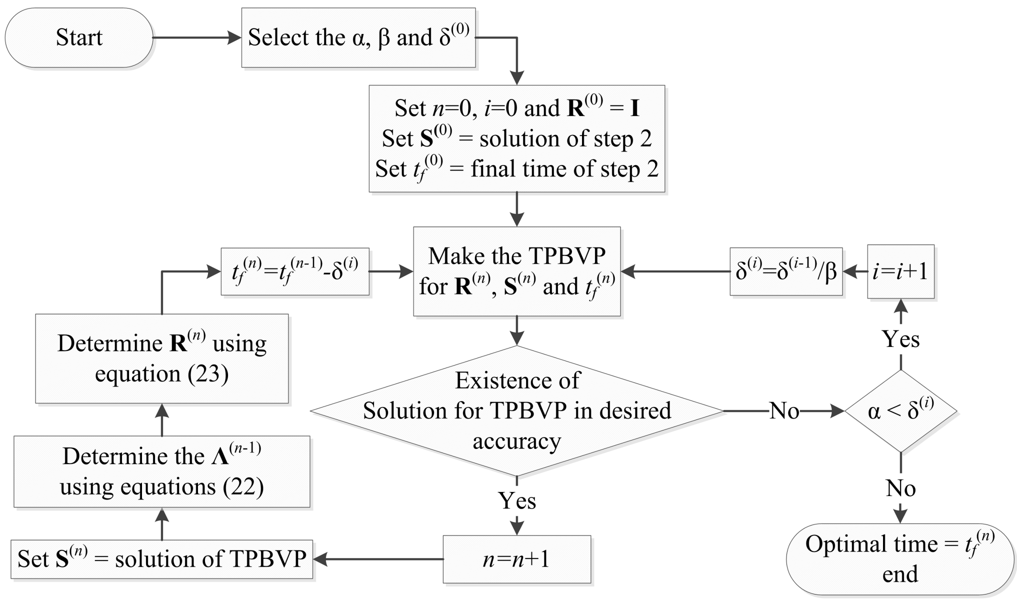


Figure 1. Flowchart of minimum time calculation (last phase).

In presented flowchart, α, δ and β are the accuracies of the minimum time calculation, decrement of the time on each step and the reducing rate of decrement of time, respectively. **S** is the initial guess for solving the TPBVP. **S** is obtained from the solution of the previous iteration. Λ is the vector of absolute maximum rate of control inputs. The vector of control input rates at any instant is computed using the numeric variation of the vector of control inputs as follows

$$\dot{\mathbf{u}}\,(t) = \lim_{\Delta t\to 0}\frac{\mathbf{u}\,(t+\Delta t) - \mathbf{u}\,(t)}{\Delta t}. \tag{21}$$

Therefore, Λ is defined in following form

$$\Lambda = \left[\max\left(|\dot{u}_1\,(t)|\right),\ldots,\max\left(|\dot{u}_n\,(t)|\right)\right]^T \in \Re^n \quad \text{for} \quad t\in[t_0\;\; t_f]\,. \tag{22}$$

As shown in Fig. 1, this step includes two loops. The loop index $(n)$ decreases the terminal time at each iteration and the other one $(i)$ decreases the decrement of time in previous loop. The loop index $(n)$ is iterated till any solution with desired accuracy exists. After that, using the loop index $(i)$ decreases the decrement of time. Then, the loop index $(n)$ is started again with new decrement of time. These iterations are continued till the decrement of time in loop index $(n)$ is greater than the accuracy of the minimum time calculation.

Also, since the control inputs do not have the same efficiency, the energy weighting matrix should be modified in succeeding iterations. In other words, control inputs have different influence and different ability. Thus, the influence of control inputs on performance index should be specified so as their full ability can be used. As regards weight of control inputs and control input rates are in reverse relation, the weighting matrix of energy should be modified in succeeding iterations using the following equation

$$\mathbf{R}^{(n)} = \frac{\mathbf{R}^{(n-1)} \cdot \text{diag}\left(\Lambda^{(n-1)}\right)}{\max\left(\Lambda^{(n-1)}\right)}. \tag{23}$$

It should be pointed out, that in order to decrease computations, there is no need to use modification of weighting matrix of energy at the first loop index $(i)$, $i = 0$. Also, it is fully clear that application of this modification (eqn. (23)) has no merit for single input systems.

## 4. Numerical simulations

In this part, in order to show the accuracy and efficiency of the proposed method, results of simulations are presented. Firstly, the time optimal problem of harmonic oscillator as a single input linear system is studied. Then, the time optimal control of robot arm as a multi input system in desired form is investigated. Also, the time optimal control of two fully nonlinear systems, which are not in desired form, is probed. The first one is a double mass-spring system in presence of coulomb friction and the other one is the F-8 aircraft.

For all simulations, the accuracy of minimum time calculation, initial decrement of time on each step and reducing rate of decrement of time are chosen, respectively, as $\alpha = 10^{-5}$, $\delta^{(0)}(0) = 6^6 \cdot 10^{-5}$ and $\beta = 6$.

### 4.1. Harmonic oscillator

The time optimal control of harmonic oscillator is considered as the first case of investigation to compare the solution of proposed method with analytical one. The governing equation of system is as follows

$$\dot{\mathbf{X}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}. \tag{24}$$

The control input is bounded ($u \in [-1,\ 1]$). The initial point and target point are

$$\mathbf{X}_0 = \begin{bmatrix} -5 & 5 \end{bmatrix}^T \text{ and } \mathbf{X}_f = \begin{bmatrix} 0 & 0 \end{bmatrix}^T. \tag{25}$$

By applying the proposed algorithm for $t_f^{(0)} = 20$s, after 41 iterations, the minimum time is calculated. The computed optimal time is $t_f = 10.58715$s which is in a good agreement with the exact solution, $t_f = 10.5871$s, and the solution presented in [6], $t_f = 10.588$s. As shown in Fig. 2 (the error is defined by $\left(t_f^{i-\text{th iteration}} - t_f^{\text{exact}}\right)/t_f^{\text{exact}}$, after 21 iterations no effective change is happened while the computational time is increased (the computations are done using an entertainment PC of Pentium 3, CPU 2.26GHz and RAM 1.5GB).

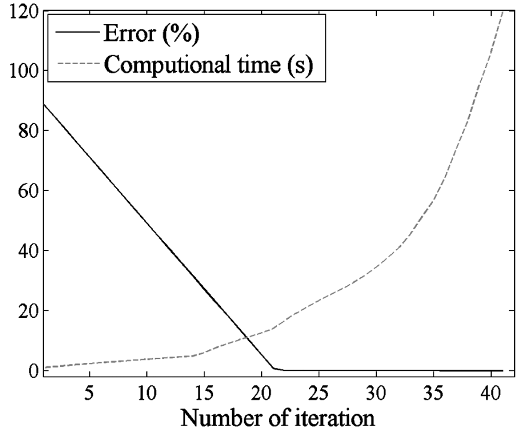


Figure 2. Pecentage of error and computational time versus iterations.

The optimal input control and phase plane, respectively, have been shown in Fig. 3 and Fig. 4 for $t_f^{(0)} = 20$s, $t_f^{(21)} = 10.6688$s and $t_f^{(41)} = 10.58715$s.

As illustrated in Fig. 3, the phase plane of $t_f^{(41)}$ and $t_f^{(21)}$ are nearly the same, while their input controls, as shown in Fig. 4, is not exactly the same. With considering the limitation of industrial actuators, which never can act like control of $t_f^{(41)}$, and required computation for such little improvement (-0.77%), it is not useful to perform more than 21 iterations.

### 4.2. Robot arm

This simulation case is a robot arm which has been investigated in [4]. The dynamic equations of this system, as well as the robot parameters and boundary conditions, are taken from this reference:

$$L\ddot{\rho} = u_1 \quad I_\theta\ddot{\theta} = u_2 \quad I_\varphi\ddot{\varphi} = u_3 \tag{26}$$

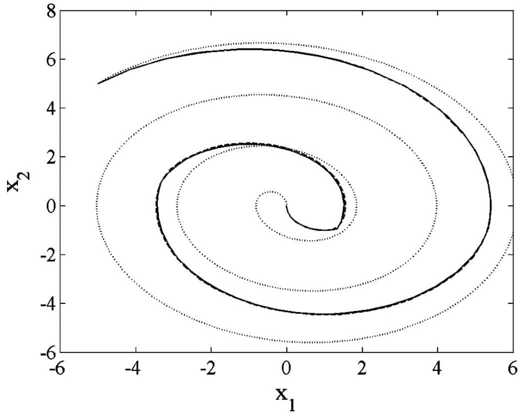


Figure 3. Optimal phase planes for different final times (solid line: $t_f^{(41)}$, dashed line: $t_f^{(21)}$, dotted line: $t_f^{(0)}$).

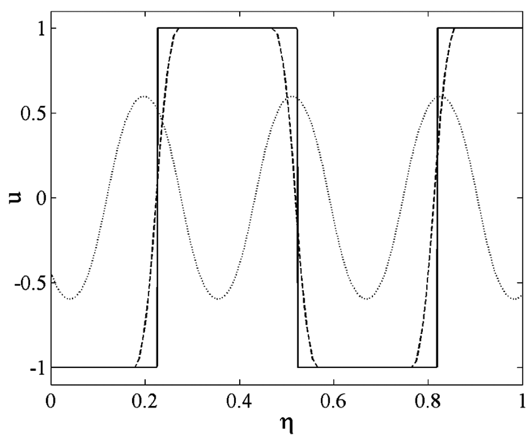


Figure 4. Optimal input controls for different final times (solid line: $t_f^{(41)}$, dashed line: $t_f^{(21)}$, dotted line: $t_f^{(0)}$).

where $L$ is the length of rigid robotic arm $(L = 5)$, $\rho$ is the length of the arm from pivot point, $\theta$ and $\varphi$ are the horizontal and vertical angles from the horizontal plane, respectively. $I_\theta$ and $I_\varphi$ show moment of inertia of the arm which are defined as follows

$$I_\varphi = \left((L-\rho)^3 + \rho^3\right)/3 \quad , \quad I_\theta = I_\varphi \sin^2\varphi. \tag{27}$$

The actuation constraints on control variables are

$$u_i \in \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad i = 1,2,3. \tag{28}$$

Also, by defining the vector of state variables $\mathbf{X} = \begin{bmatrix} \rho & \theta & \varphi & \dot{\rho} & \dot{\theta} & \dot{\varphi} \end{bmatrix}^T$, the initial and terminal conditions are

$$\mathbf{X}_0 = \begin{bmatrix} 4.5 & 0 & \pi/4 & 0 & 0 & 0 \end{bmatrix}^T \tag{29}$$

$$\mathbf{X}_f = \begin{bmatrix} 4.5 & 2\pi/3 & \pi/4 & 0 & 0 & 0 \end{bmatrix}^T. \tag{30}$$

By applying the proposed algorithm, with considering $t_f^{(0)} = 20$s, the minimum time is computed. The computed optimal time is $t_f = 9.14123$s which is in good compatibility with the solution presented in [4], $t_f = 9.14101$s. The optimal states and input controls are shown in Fig. 5 and Fig. 6, respectively.

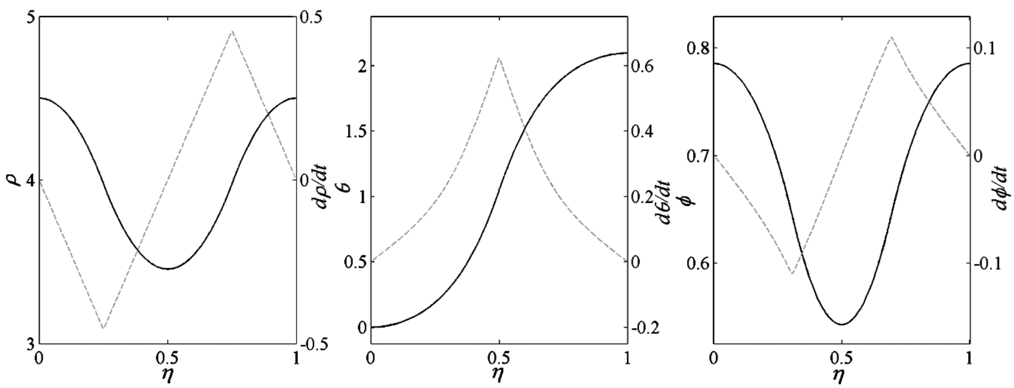


Figure 5. Time optimal states versus non-dimensional time. (dashed line refers to the right y-axis, solid line refers to the left y-axis.

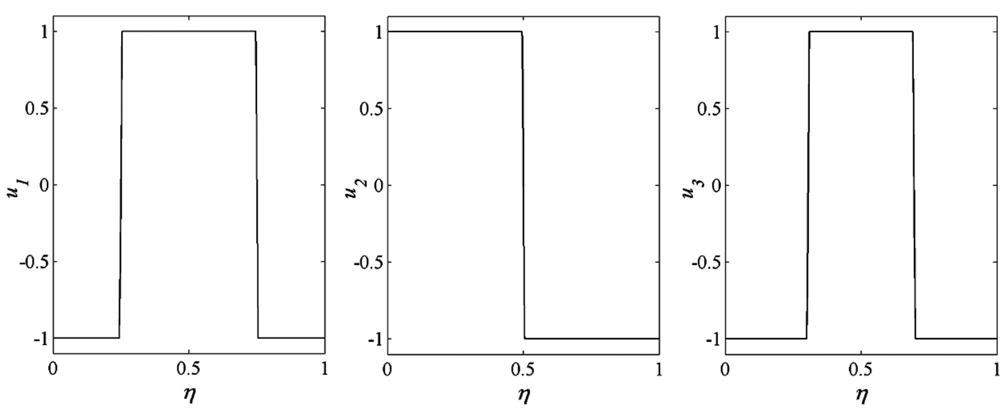


Figure 6. Time optimal controls versus non-dimensional time.

As shown in Fig. 7, the energy coefficient of the third actuator is decreased more than the other ones. It means that, because of the situation of this actuator, its energy coefficient should be less than the other ones to use its whole ability. Likewise, similar situation is occurred for the first actuator while energy coefficient of the second actuator has no change.

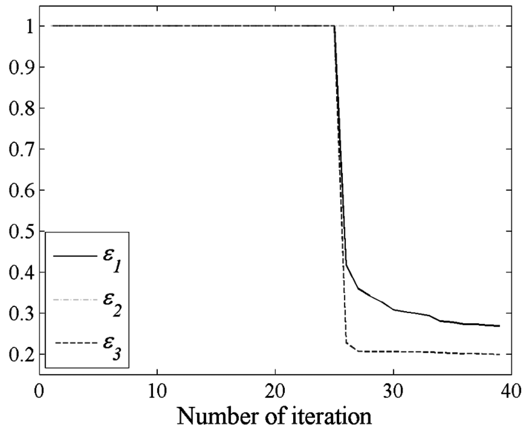


Figure 7. The coefficients of energy weighting matrix versus iterations.

### 4.3. Double spring-mass problem with coulomb friction

As mentioned above, the proposed method can solve the fully nonlinear system which is not in desired form, if a proper linearized model exists. Here, a double spring-mass system with coulomb friction, which was investigated in [3] using a linear programming method, is considered. A schematic image of system is illustrated in Fig. 8.

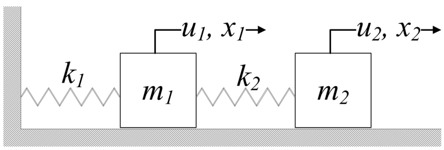


Figure 8. Schematic image of double spring-mass system.

The governing equations of system with considering the coulomb friction are as follows (sgn(·) shows the sign function of (·))

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{-k_1-k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{1}{m_1} & 0 \\ 0 & \dfrac{1}{m_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \mu \begin{Bmatrix} \dfrac{\mathrm{sgn}(\dot{x}_1)}{m_1} \\ \dfrac{\mathrm{sgn}(\dot{x}_2)}{m_2} \end{Bmatrix}. \tag{31}$$

The physical parameters, initial conditions and final conditions of system are taken from [3] and are noted in Tab. 1. Without considering the last term, the term is caused by coulomb friction, the dynamic equations of system is a set of linear ones. So, the proper initial guess is obtained by solving the optimal problem of this linear system. In order to solve the TPBVP using usual algorithm, the sign function is approximated in following form to provide smoother frictional force

Table 1. Physical parameters and boundary conditions of system.

| $i$ | $m_i$ | $k_i$ | $\mu$ | Control input bounds | $x_i(t_0)$ | $\dot{x}_i(t_0)$ | $x_i(t_f)$ | $\dot{x}_i(t_f)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.1 | 0.95 | 1.0 | $u_1 \in [-4\ \ 4]$ | 0 | -1.0 | 1.0 | 0 |
| 2 | 1.2 | 0.85 | 1.0 | $u_2 \in [-4\ \ 4]$ | 0 | -2.0 | 2.0 | 0 |

$$\operatorname{sgn}(\dot{x}_i) \approx \frac{2}{\pi} \arctan\left(100\dot{x}_i^+ \cdot \dot{x}_i\right) \qquad i = 1,2 \tag{32}$$

where $\dot{x}_i^+$ is the maximum absolute of $\dot{x}_i$ ($\dot{x}_i^+ = \max(|\dot{x}_i|)$) obtained from the solution of linear system (initial guess). For this case, with considering $t_f^{(0)} = 5$s, these values are $\dot{x}_1^+ = 1.11$ and $\dot{x}_2^+ = 2.97$. By applying the proposed method, using the equations (7)-(10), the time optimal trajectories are obtained. The minimum time is $t_f = 2.11394$s and optimal states are shown in Fig. 9.

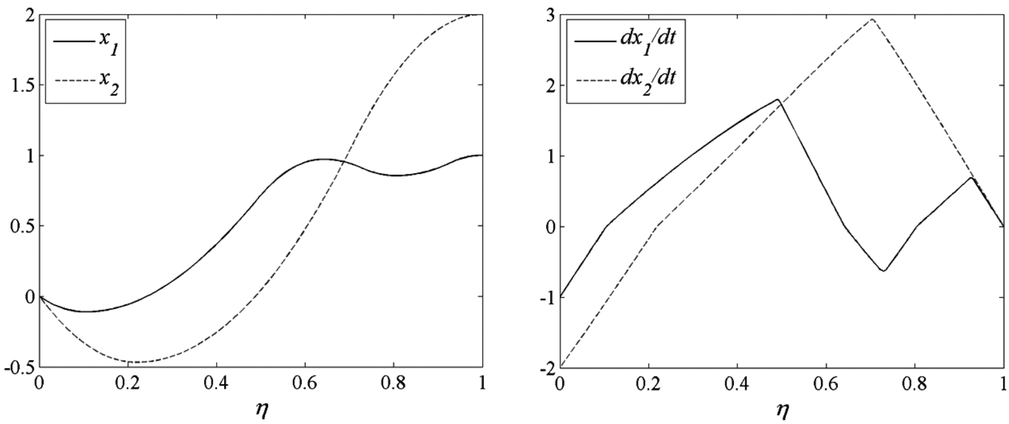


Figure 9. Time optimal states versus non-dimensional time.

The optimal input controls and frictional force with considering mentioned smooth approximation ($f_i = -\dfrac{2\mu}{\pi} \cdot \arctan\left(100\dot{x}_i^+ \cdot \dot{x}_i\right)$) are illustrated in Fig. 10 to show the accuracy of this approximation.

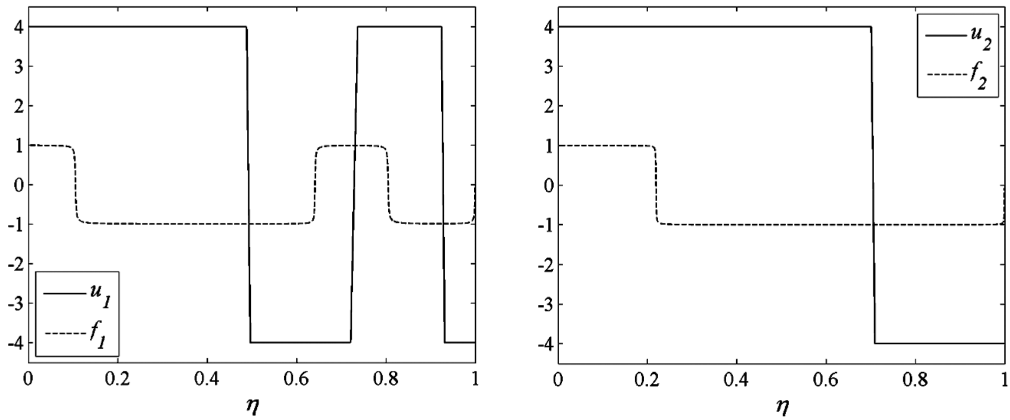


Figure 10. Time optimal controls and frictional forces versus non-dimensional time.

### 4.4. F-8 aircraft

As the last case, the time optimal control of the F-8 aircraft is investigated. The F-8 aircraft has been considered in various control studies and the time optimal control of this nonlinear model has been investigated in many works. The dynamic model of this system is taken from [29] and it is formulated by the following nonlinear equations

$$\begin{aligned} \dot{x}_1 &= -0.877x_1 + x_3 - 0.0088x_1x_3 + 0.47x_1^2 - 0.019x_2^2 - x_1^2x_3 \\ &+ 3.846x_1^3 - 0.215u + 0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \end{aligned} \tag{33}$$

$$\dot{x}_2 = x_3 \tag{34}$$

$$\begin{aligned} \dot{x}_3 &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 - 20.967u \\ &+ 6.265x_1^2u + 46x_1u^2 + 61.4u^3 \end{aligned} \tag{35}$$

where $x_1$ is the angle of attack in radians, $x_2$ is the pitch angle, $x_3$ is the pitch rate in radians per second and the control input $u$ represents the tail deflection angle in radians. The tail deflection angle is bounded $-0.05236 \leqslant u \leqslant 0.05236$ and should be found that brings the system from the initial state to the final one in minimum time.

The boundary conditions, by defining the vector of state variables $\mathbf{X} = [x_1,\, x_2,\, x_3]^T$, are as follows

$$\mathbf{X}_0 = \begin{bmatrix} 0.4655 & 0 & 0 \end{bmatrix}^T \text{and}\, \mathbf{X}_f = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T. \tag{36}$$

As mentioned above, a linearized model for such nonlinear systems is needed. The algorithm presented in [30] is applied to provide the required linear system. The linear model of the desired system is taken from this reference in following form

$$\dot{\mathbf{X}} = \begin{bmatrix} -0.891 & 0 & 0.954 \\ 0 & 0 & 1 \\ -4.230 & 0 & -0.400 \end{bmatrix} \mathbf{X} + \begin{bmatrix} -0.424 \\ 0 \\ -20.895 \end{bmatrix} u. \tag{37}$$

Using the linear system lead to a non-acceptable errors [30], but since the quality of initial guess is valid (as mentioned in results of [30]), the final results are in proper accuracy and non-acceptable errors are vanished by solving the TPBVP which is made of original system. Thus, the proper initial guess is provided by applying the three-step method for linear system with considering $t_f^{(0)} = 10$s. Then, similar to the last case, the adjoint differential equations are obtained using equations (7) and (9). The control input equation is computed from equation (18) by using the control input's coefficient matrix of linear model for $\mathbf{B}$ and, as mentioned before, this assumption does not impress the time optimal solution. As justified in [29] and [30], neglecting the nonlinear terms in $u$ has no effective influence on the results. Nevertheless, in this work, these terms are neglected only for computing the control input equations (not for dynamic equations of system and adjoint differential equations).

The optimal states and control are shown in Fig. 11 and Fig. 12, respectively.

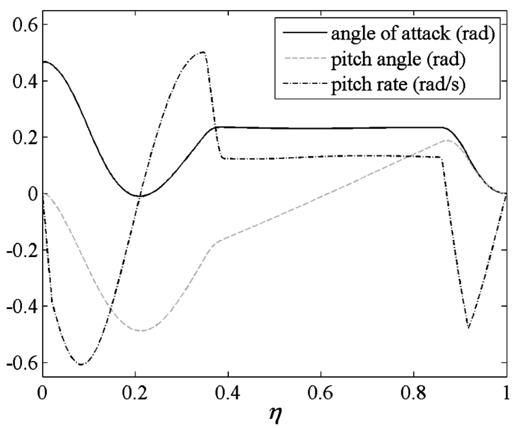


Figure 11. Time optimal states versus non-dimensional time.

Terminal time computed by various methods for this problem in comparison with final time calculated by proposed method is cited in Table 2. Comparing the illustrated results in Table 2 shows the ability of proposed method.

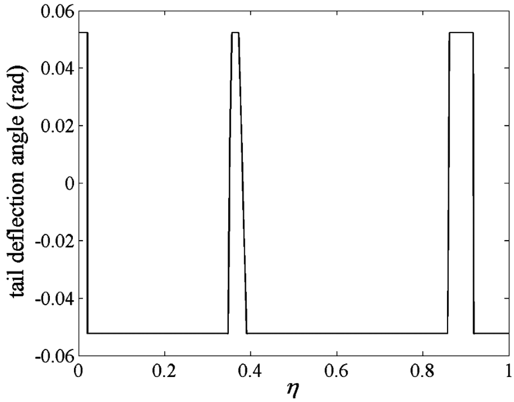


Figure 12. Time optimal control versus non-dimensional time.

Table 2. Results of various methods for the time optimal problem of F-8 aircraft.

| Method | Terminal time (s) |
|---|---|
| STC (Kaya and Noakes, 1996, [20]) | 6.3867 |
| CPET (Lee et al., 1997, [21]) | 6.0350 |
| TOS (Kaya and Noakes, 2003, [19]) | 5.74217 |
| Combined method (Xie and Kunisch, 2005, [23]) | 5.72994 |
| Two-Phase Scheme (Huang and Tseng, 2006, [22]) | 5.7422 |
| Proposed method | 5.71949 |

## 5. Conclusions

In this paper, an iterative method (based on PMP) for the time optimal open loop control of a general type of dynamic systems subject to limited control inputs is proposed. This method is free of necessity of finding the switching point and leads to a TPBVP which can be solved by usual algorithm of solving TPBVPs. Due to the fact that TPBVPs are very sensitive to initial guess, a simple procedure for generating the proper initial guess, based on unique analytical solution of the optimal control of linear systems, is proposed. Simulations including linear system, nonlinear system in desired form and fully nonlinear systems are performed to demonstrate the accuracy and efficiency of the proposed method in comparison with other methods.

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