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Steady-state analysis for a class of hyperbolic systems with boundary inputs

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Results of a steady-state analysis performed for a class of distributed parameter systems described by hyperbolic partial differential equations defined on a one-dimensional spatial domain are presented. For the case of the system with two state variables and two boundary inputs, the analytical expressions for the steady-state distribution of the state variables are derived, both in the exponential and in the hyperbolic form. The influence of the location of the boundary inputs on the steady-state response is demonstrated. The considerations are illustrated with a practical example of a shell and tube heat exchanger operating in parallel- and countercurrent-flow modes.

Key words: distributed parameter system, partial differential equation, hyperbolic system, steady-state solution, heat exchanger

1. Introduction

Distributed parameter systems (DPSs), also known as systems with spatiotemporal dynamics, comprise a large class of dynamical systems in which state variables depend not only on time but also on the spatial variable(s). In fact, almost all industrial processes belong into this category, while the existence of the so-called lumped parameter systems (LPS) results from the adoption of a simplified model of the reality, in which spatial effects are neglected or averaged. Typical examples of DPSs include heat transfer and fluid flow phenomena, as well as processes occurring in chemical reactors, semiconductor manufacturing, polymer processing, bioreactors and many others [4, 5, 14, 20, 25, 32, 34]. Based on the phenomenological models of the processes, established usually on the basis of the mass or energy conservation balance laws, one obtains their mathematical description, mostly in the form of partial differential equations (PDEs) [19, 28]. Depending on the nature of the phenomena modeled, the equations can be of parabolic type (which are typical for the unsteady heat conduction and for the diffusion problems), hyperbolic type (representing convection, advection and wave propagation phenomena) or elliptic type (describing steady-state physical phenomena, e.g. elec-

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trostatic, magnetostatic or gravitational fields). Mathematical models of DPSs obtained on the basis of the PDEs are described in an infinite-dimensional state space, usually in a Hilbert space, and their transfer functions have the form of irrational functions, as opposed to the rational ones describing the dynamic properties of LPS [7, 10, 13, 26, 35].

The concept of the *steady state* of a dynamical system is essential in many areas, particularly in thermodynamics, chemistry, economics, electrical and mechanical engineering. From the control theory viewpoint, the knowledge of the behavior of a dynamical system in the steady-state conditions is very important e.g. in the context of the steady-state responses which describe the input-output relationships of the system after the transient responses are terminated. The information about the steady-state properties is also important in connection with the problem of the steady-state optimization of the operating point of a dynamical system [31]. In the case of the DPS, due to the above mentioned spatial dependence of the state variables, a mathematical steady-state model provides not only information about the static input-output mappings but also describes the spatial distribution of the state variables for the steady-state conditions. This fact is very important from a practical and technological point of view since it allows not only to determine e.g. the outlet temperature of the heated fluid in a heat exchanger or the voltage at the endpoint of an electrical transmission line, but also enables an analysis of their distribution along the geometrical axis of the system.

The paper presents the results of the steady-state analysis for a certain class of DPSs in which the mass, heat and energy transport phenomena take place. This class of systems, among which one can mention e.g. heat exchangers, transport pipelines, irrigation channels or electrical transmission lines, is usually described by PDEs of hyperbolic type and known under the common name of hyperbolic systems of conservation laws [3, 4, 5, 6, 9, 8, 15, 20, 24]. The present paper can be considered as a complement to our recent work [7], where a general transfer function representation for this class of systems has been analyzed. Its structure is as follows: After the introduction, Section 2 reviews the mathematical model of the considered class of DPSs in the form of a set of PDEs and formulates its hyperbolicity conditions. Next, the considerations are focused on the often encountered in the industrial practice systems with two spatially distributed state variables and boundary-type control, represented by a system of two PDEs with Dirichlet boundary conditions. Two different typical configurations of boundary inputs are introduced here. Section 3 starts with the definition of the steady-state solution of the considered initial-boundary value problem. The analytical expressions for the steady-state distribution of the state variables are derived for the two considered boundary input configurations, both in the exponential and in the hyperbolic form. In Section 4 a shell and tube heat exchanger operating in parallel- and countercurrent-flow modes is introduced as a typical example of the considered hyperbolic DPS with boundary inputs. Selected steady-state distributions of the fluid temperatures for the both flow arrangements are presented and analyzed here based on the analytical expressions derived in Section 3. The article concludes with a summary of new results and directions for further research.

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2. Weakly coupled hyperbolic systems

2.1. General case

Some of the above-mentioned DPSs can be described by the following system of linear homogeneous PDEs of the first order (see [7, 11, 12, 16, 19, 28]):

$$\frac{\partial x(l,t)}{\partial t} + \Lambda \frac{\partial x(l,t)}{\partial l} = Kx(l,t), \tag{1}$$

where $x(l,t) : Q \to \mathbb{R}^n$ is a vector function representing the spatio-temporal distribution of the n state variables

$$x(l,t) = \begin{bmatrix} x_1(l,t) & x_2(l,t) & \dots & x_n(l,t) \end{bmatrix}^T, \tag{2}$$

defined on a set $Q = \Omega \times \Theta$, where $\Omega = [0, L] \subset \mathbb{R}$ is the domain of the spatial variable l, $\Theta = [0, +\infty) \subset \mathbb{R}$ is the domain of the time variable t, $K \in \mathbb{R}^{n \times n}$ is a matrix with constant entries and Λ is a diagonal matrix of the following form:

$$\Lambda = diag(\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_n), \tag{3}$$

with $\lambda_i \in \mathbb{R} \setminus 0$ and

$$\lambda_1 > \dots > \lambda_p > 0 > \lambda_{p+1} > \dots > \lambda_n, \tag{4}$$

where $p \leqslant n$ represents the number of positive elements λ_i.

Remark 4 Owing to the diagonal form of Λ, each equation of the system (1) contains both temporal and spatial derivatives of the same state variable $x_i(l,t)$, for $i = 1, 2, ..., n$. Therefore, this system is commonly referred to as *decoupled* or *weakly coupled*, i.e. coupled only through the terms that do not contain derivatives.

Definition 3 *The system* (1) *is said to be* hyperbolic *iff all elements of* Λ *given by* (3) *are real and different from zero as assumed above. Additionally, if all the elements are distinct then system* (1) *is said to be* strictly hyperbolic.

Remark 5 In the case of the hyperbolic PDEs describing physical phenomena mentioned in Section 1, the elements of Λ usually represent the mass and energy transport rates.

2.2. Initial and boundary conditions

In order to obtain a unique solution of (1), one must specify the appropriate *initial* and *boundary* conditions. The initial conditions represent the initial (i.e. determined for $t = 0$) distribution of the values of all n state variables for the whole set Ω

$$x(l,0) = x_0(l), \tag{5}$$

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where $x_0(l) : \Omega \rightarrow \mathbb{R}^n$ is a given vector function.

On the other hand, the boundary conditions represent the requirements to be met by the solution $x(l,t)$ at the boundary points of Ω. In general, these conditions may take the form of a linear combination of the Dirichlet and Neumann boundary conditions, as the so-called boundary conditions of the third kind [1, 17, 30]. For the considered class of hyperbolic systems we assume the Dirichlet boundary conditions which can be written in the following compact way [16, 33]

$$\begin{bmatrix} x^+(0,t) \\ x^-(L,t) \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \begin{bmatrix} x^+(L,t) \\ x^-(0,t) \end{bmatrix} + \begin{bmatrix} R_0 \\ R_1 \end{bmatrix} u(t), \tag{6}$$

with

$$x^+ = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}, \quad x^- = \begin{bmatrix} x_{p+1} \\ \vdots \\ x_n \end{bmatrix}. \tag{7}$$

The vector function $u(t) : [0,+\infty) \rightarrow \mathbb{R}^r$ in (6) expresses the inhomogeneity of the boundary conditions which can be identified with r external inputs to the system, including control signals as well as external disturbances. The constant matrices $P_{00} \in \mathbb{R}^{p\times p}$, $P_{01} \in \mathbb{R}^{p\times(n-p)}$, $P_{10} \in \mathbb{R}^{(n-p)\times n}$, $P_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ express boundary feedbacks and reflections, whereas $R_0 \in \mathbb{R}^{p\times r}$ and $R_1 \in \mathbb{R}^{(n-p)\times r}$ represent the effect of the external inputs $u(t)$ on the boundary conditions $x^+(0,t)$ and $x^-(L,t)$, respectively.

2.3. Second-order systems

Among many different kinds of DPSs, an important class is constituted by the systems with two distributed state variables which can be described, after appropriate assumptions, by the second-order hyperbolic PDE. The following typical examples can be mentioned here [2, 5, 18, 21, 22, 23, 29]:

- the temperatures $\vartheta_1(l,t)$ and $\vartheta_2(l,t)$ of the heating and the heated fluid in the case of a coaxial heat exchanger,
- the voltage $u(l,t)$ and the current $i(l,t)$ in the electrical transmission line,
- the pressure $p(l,t)$ and the flow $q(l,t)$ of the medium transported through the pipeline.

Remark 6 Some of the above-mentioned systems, such as e.g. heat exchanger, are described directly by the weakly coupled hyperbolic PDEs, while the equations of the others, such as electrical transmission line or transport pipeline, are strongly coupled. In order to express them in the form of Eqn. (1), the decoupling procedure has to be carried out.

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In the case of the above mentioned systems, Eqn. (1) takes, after possible decoupling (see [7, 19, 28]), the form of the following two PDEs:

$$\frac{\partial x_1(l,t)}{\partial t}+\lambda_1\frac{\partial x_1(l,t)}{\partial l}=k_{11}x_1(l,t)+k_{12}x_2(l,t)\,, \tag{8}$$

$$\frac{\partial x_2(l,t)}{\partial t}+\lambda_2\frac{\partial x_2(l,t)}{\partial l}=k_{21}x_1(l,t)+k_{22}x_2(l,t)\,, \tag{9}$$

where $k_{11}, k_{12}, k_{21}, k_{22}$ and λ_1, λ_2 are constant elements of the matrices K and Λ in (1), respectively.

Therefore, it is assumed here that the only external influence on the state variables x_1 and x_2 is given by the boundary conditions (6). Two cases often occurring in practice are considered here: in the first one, both boundary conditions are given for the same edge ($l = 0$) of Ω and in the second – the input function $u(t)$ acts on the two different edges, $l = 0$ and $l = L$, respectively. Further analysis is based on the additional assumption that no boundary feedback nor reflection are present in the system, i.e. P_{00}, P_{01}, P_{10}, P_{11} in (6) are all zero matrices. The next assumption is that the system is given directly by the two weakly coupled PDEs (8) and (9). Such a situation occurs e.g. in the case of shell and tube heat exchangers [4, 8, 15, 23, 27, 34]. Therefore, two definitions originally proposed in [7] are recalled below in order to discriminate between two above mentioned classes of boundary inputs.

Definition 4 *The external inputs of the system* (8) *and* (9) *will be referred to as* congruent *boundary inputs for the following parameter values of* (6)*:* $n = l = p = 2$ *and* $R_0 = I_2$*, which leads to the following expressions on the boundary values of the state variables:*

$$x_1\,(0,t) = u_1(t), \tag{10}$$

$$x_2\,(0,t) = u_2(t). \tag{11}$$

Definition 5 *The external inputs to the system* (8) *and* (9) *will be referred to as* incongruent *boundary inputs for the following parameter values of* (6)*:* $n = l = 2$*,* $p = 1$*,* $R_0 = [1\ 0]$ *and* $R_1 = [0\ 1]$*, which leads to the following expressions on the boundary values of the state variables:*

$$x_1\,(0,t) = u_1(t), \tag{12}$$

$$x_2\,(L,t) = u_2(t). \tag{13}$$

Remark 7 Taking into account (4) it can be noticed that the congruent boundary inputs should be imposed for $\lambda_1 > 0$ and $\lambda_2 > 0$, while the incongruent ones – for $\lambda_1 > 0$ and $\lambda_2 < 0$.

The above assumptions about the form of the boundary conditions representing the external influences on the system have its practical reasons. For example, in the case of

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the above mentioned shell and tube heat exchanger operating in the so-called *parallel-flow* mode, the temperatures of the heated and the heating medium are given for the same geometric point of the exchanger. On the other hand, the temperatures of the fluids flowing into the exchanger operating in the *countercurrent-flow* mode are specified for its two opposite sides.

3. Steady-state equations

In this section the definition of the steady-state solution is formulated for the considered class of DPSs. Next, the expressions describing the steady-state distribution of the state variables x_1 and x_2 are derived for the system (8) and (9), both for the congruent and incongruent boundary conditions adopted in Definitions 4 and 5, respectively. Furthermore, the simplified form of the solutions is obtained for the case of the matrix K having some zero elements.

Definition 6 *A* steady state *solution* $\bar{x}(l):\Omega\rightarrow\mathbb{R}^n$ *of the initial-boundary value problem described by* (1)-(7) *is a solution that does not depend on time, i.e. the one which can be obtained by assuming all time derivatives in* (1) *equal to zero*

$$\frac{\partial x_i(l,t)}{\partial t}=0 \quad \text{for} \quad i=1,2,\ldots,n. \tag{14}$$

3.1. Congruent boundary inputs

Result 1 *The steady-state distribution of the state variables x_1 and x_2 of the system* (8) *and* (9) *can be described for the case of the constant congruent boundary inputs $x_1(0,t)=u_{10}$ and $x_2(0,t)=u_{20}$ by the following equations:*

$$\bar{x}_1(l)=\left(\frac{\lambda_2\phi_1-k_{22}}{2\lambda_2\beta}\mathrm{e}^{\phi_1 l}-\frac{\lambda_2\phi_2-k_{22}}{2\lambda_2\beta}\mathrm{e}^{\phi_2 l}\right)u_{10}+\frac{k_{12}}{2\lambda_1\beta}\left(\mathrm{e}^{\phi_1 l}-\mathrm{e}^{\phi_2 l}\right)u_{20}, \tag{15}$$

$$\bar{x}_2(l)=\frac{k_{21}}{2\lambda_2\beta}\left(\mathrm{e}^{\phi_1 l}-\mathrm{e}^{\phi_2 l}\right)u_{10}+\left(\frac{\lambda_1\phi_1-k_{11}}{2\lambda_1\beta}\mathrm{e}^{\phi_1 l}-\frac{\lambda_1\phi_2-k_{11}}{2\lambda_1\beta}\mathrm{e}^{\phi_2 l}\right)u_{20}, \tag{16}$$

where

$$\alpha=\frac{1}{2}\left(\frac{k_{11}}{\lambda_1}+\frac{k_{22}}{\lambda_2}\right),\quad \beta=\frac{1}{2}\sqrt{\left(\frac{k_{11}}{\lambda_1}-\frac{k_{22}}{\lambda_2}\right)^2+4\frac{k_{12}}{\lambda_1}\frac{k_{21}}{\lambda_2}}, \tag{17}$$

$$\phi_1=\alpha+\beta,\quad \phi_2=\alpha-\beta. \tag{18}$$

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Proof After setting the time derivatives in (8) and (9) to zero, one obtains the following system of two ODEs:

$$\lambda_1 \frac{dx_1(l,t)}{dl} = k_{11}x_1(l,t) + k_{12}x_2(l,t), \tag{19}$$

$$\lambda_2 \frac{dx_2(l,t)}{dl} = k_{21}x_1(l,t) + k_{22}x_2(l,t), \tag{20}$$

with the boundary conditions $x_1(0) = u_{10}$, $x_2(0) = u_{20}$. The solution of (19) and (20) is the given by (15)-(18). □

Lemma 9 *For any $x, y, z \in \mathbb{R}$ such that $z \neq 0$ and $z \neq y$, the following relationship holds:*

$$e^x - \frac{y}{z}e^{-x} = \frac{z-y}{z}\left(\cosh x + \frac{z+y}{z-y}\sinh x\right). \tag{21}$$

Proof By using the well known identities

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \tag{22}$$

the right-hand side of (21) can be transformed in the following way:

$$\begin{aligned} &\frac{z-y}{z}\left(\cosh x + \frac{z+y}{z-y}\sinh x\right) = \frac{z-y}{2z}\left(e^x + e^{-x} + \frac{z+y}{z-y}\left(e^x - e^{-x}\right)\right) = \\ &= \frac{z-y}{2z}\left(\frac{2z}{z-y}e^x - \frac{2y}{z-y}e^{-x}\right) = e^x - \frac{y}{z}e^{-x}. \end{aligned} \tag{23}$$

□

Result 2 *The steady-state distribution given by* (15) *and* (16) *can be expressed in the following equivalent form using the hyperbolic functions:*

$$\bar{x}_1(l) = e^{\alpha l}\left(\cosh \beta l + \frac{\lambda_2\alpha - k_{22}}{\lambda_2\beta}\sinh \beta l\right)u_{10} + \frac{k_{12}}{\lambda_1\beta}e^{\alpha l}\sinh \beta l \cdot u_{20}, \tag{24}$$

$$\bar{x}_2(l) = \frac{k_{21}}{\lambda_2\beta}e^{\alpha l}\sinh \beta l \cdot u_{10} + e^{\alpha l}\left(\cosh \beta l + \frac{\lambda_1\alpha - k_{11}}{\lambda_1\beta}\sinh \beta l\right)u_{20}. \tag{25}$$

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Proof The expression at u_{10} on the right-hand side of (15) can be transformed using (18) and Lemma 9 in the following way:

$$\begin{aligned}
&\frac{\lambda_2\phi_1-k_{22}}{2\lambda_2\beta}e^{\phi_1 l}-\frac{\lambda_2\phi_2-k_{22}}{2\lambda_2\beta}e^{\phi_2 l}= \\
&=\frac{\phi_1}{2\beta}e^{\alpha l}e^{\beta l}-\frac{k_{22}}{2\lambda_2\beta}e^{\alpha l}e^{\beta l}-\frac{\phi_2}{2\beta}e^{\alpha l}e^{-\beta l}+\frac{k_{22}}{2\lambda_2\beta}e^{\alpha l}e^{-\beta l}= \\
&=\frac{\phi_1}{2\beta}e^{\alpha l}\left(e^{\beta l}-\frac{\phi_2}{\phi_1}e^{-\beta l}\right)-\frac{k_{22}}{2\lambda_2\beta}e^{\alpha l}\left(e^{\beta l}-e^{-\beta l}\right)= \\
&=e^{\alpha l}\left(\cosh\beta l+\frac{\alpha}{\beta}\sinh\beta l\right)-\frac{k_{22}}{\lambda_2\beta}e^{\alpha l}\sinh\beta l= \\
&=e^{\alpha l}\left(\cosh\beta l+\frac{\lambda_2\alpha-k_{22}}{\lambda_2\beta}\sinh\beta l\right).
\end{aligned} \tag{26}$$

Similarly, the expression at u_{20} in (15) can be transformed as follows:

$$\frac{k_{12}}{2\lambda_1\beta}\left(e^{\phi_1 l}-e^{\phi_2 l}\right)=\frac{k_{12}}{2\lambda_2\beta}e^{\alpha l}\left(e^{\beta l}-e^{-\beta l}\right)=\frac{k_{12}}{\lambda_2\beta}e^{\alpha l}\sinh\beta l. \tag{27}$$

Due to the obvious symmetry, the hyperbolic version (25) of (16) can be obtained in the similar manner. □

Remark 8 Assuming $l=0$ in (15) and (16) one obtains $e^{\phi_1 l}=e^{\phi_2 l}=1$ and finally $\bar{x}_1(0)=u_{10}$, $\bar{x}_2(0)=u_{20}$. Analogous results can be obtained based on the hyperbolic form (24) and (25) of the steady-state equations. In this case one obtains for $l=0$: $e^{\alpha(s)l}=1$, $\sinh\beta(s)l=0$ and $\cosh\beta(s)l=1$ which leads to the same result.

3.2. Incongruent boundary inputs

Result 3 *The steady-state distribution of the state variables x_1 and x_2 of the system* (8) *and* (9) *can be described for the case of the constant incongruent boundary inputs $x_1(0,t)=u_{10}$ and $x_2(L,t)=u_{2L}$ by the following equations:*

$$\begin{aligned}
\bar{x}_1(l)=&\frac{e^{\phi_2 L}e^{\phi_1 l}(\lambda_2\phi_1-k_{22})-e^{\phi_1 L}e^{\phi_2 l}(\lambda_2\phi_2-k_{22})}{e^{\phi_2 L}(\lambda_2\phi_1-k_{22})-e^{\phi_1 L}(\lambda_2\phi_2-k_{22})}u_{10}+ \\
&+\frac{k_{12}\left(e^{\phi_2 l}-e^{\phi_1 l}\right)}{e^{\phi_2 L}(\lambda_1\phi_2-k_{11})-e^{\phi_1 L}(\lambda_1\phi_1-k_{11})}u_{2L},
\end{aligned} \tag{28}$$

$$\begin{aligned}
\bar{x}_2(l)=&\frac{k_{21}\left(e^{\phi_2 L}e^{\phi_1 l}-e^{\phi_1 L}e^{\phi_2 l}\right)}{e^{\phi_2 L}(\lambda_2\phi_1-k_{22})-e^{\phi_1 L}(\lambda_2\phi_1-k_{22})}u_{10}+ \\
&+\frac{e^{\phi_2 l}(\lambda_1\phi_2-k_{11})-e^{\phi_1 l}(\lambda_1\phi_1-k_{11})}{e^{\phi_2 L}(\lambda_1\phi_2-k_{11})-e^{\phi_1 L}(\lambda_1\phi_1-k_{11})}u_{2L},
\end{aligned} \tag{29}$$

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where the parameters α, β, ϕ_1 *and* ϕ_2 *are given by* (17) *and* (18).

Proof By solving equation (19) and (20) with the boundary conditions $x_1(0) = u_{10}$ and $x_2(L) = u_{2L}$. □

Result 4 *The steady-state distribution given by* (28) *and* (29) *can be expressed in the following equivalent form using the hyperbolic functions:*

$$\begin{aligned}\bar{x}_1(l) = &\frac{e^{\alpha l}\left(\lambda_2\beta\cosh\beta(l-L)+(\lambda_2\alpha-k_{22})\sinh\beta(l-L)\right)}{\lambda_2\beta\cosh\beta L-(\lambda_2\alpha-k_{22})\sinh\beta L}x_{10}+\\ &+\frac{k_{12}e^{\alpha(l-L)}\sinh\beta l}{\lambda_1\beta\cosh\beta L+(\lambda_1\alpha-k_{11})\sinh\beta L}x_{2L},\end{aligned} \tag{30}$$

$$\begin{aligned}\bar{x}_2(l) = &\frac{k_{21}e^{\alpha l}\sinh\beta(l-L)}{\lambda_2\beta\cosh\beta L-(\lambda_2\alpha-k_{22})\sinh\beta L}x_{10}+\\ &+\frac{e^{\alpha(l-L)}\left(\lambda_1\beta\cosh\beta l+(\lambda_1\alpha-k_{11})\sinh\beta l\right)}{\lambda_1\beta\cosh\beta L+(\lambda_1\alpha-k_{11})\sinh\beta L}x_{2L}.\end{aligned} \tag{31}$$

Proof As in the case of the proof of Result 2, i.e. using (18) and Lemma 9. □

Remark 9 In the case of the incongruent boundary conditions one obtains: $\bar{x}_1(0) = u_{10}$ and $\bar{x}_2(L) = u_{2L}$.

3.3. The case of zero-valued elements in matrix K

Assuming in (8) and (9) $k_{12} = k_{21} = 0$ one obtains two fully decoupled ODEs. In this case the expressions for β, ϕ_1 and ϕ_2 in (17) and (18) simplify to

$$\beta = \frac{1}{2}\left(\frac{k_{11}}{\lambda_1} - \frac{k_{22}}{\lambda_2}\right), \tag{32}$$

$$\phi_1 = \frac{k_{11}}{\lambda_1}, \quad \phi_2 = \frac{k_{22}}{\lambda_2}. \tag{33}$$

Taking into account (32) and (33) one obtains from (15) and (16) the greatly simplified steady-state expressions for the congruent boundary inputs

$$\bar{x}_1(l) = e^{\frac{k_{11}}{\lambda_1}l}u_{10}, \quad \bar{x}_2(l) = e^{\frac{k_{22}}{\lambda_2}l}u_{20}, \tag{34}$$

and

$$\bar{x}_1(l) = e^{\frac{k_{11}}{\lambda_1}l}u_{10}, \quad \bar{x}_2(l) = e^{\frac{k_{22}}{\lambda_2}(l-L)}u_{2L}, \tag{35}$$

for the incongruent ones.

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The above representation means that the steady-state value of the given state variable x_i depends for all $l \in \Omega$ solely on the boundary input value u_i without any cross interaction from the second boundary input.

Assuming also $k_{11} = k_{22} = 0$, one obtains from (8) and (9) two pure time-delay systems with the extremely simplified form of the steady-state solutions

$$\bar{x}_1(l) = u_{10}, \quad \bar{x}_2(l) = u_{20}, \tag{36}$$

for the congruent boundary inputs, and

$$\bar{x}_1(l) = u_{10}, \quad \bar{x}_2(l) = u_{2L}, \tag{37}$$

for the incongruent ones.

4. Example: Parallel- and countercurrent-flow heat exchanger

For a practical illustration of the issues discussed above, this chapter performs the steady-state analysis of a shell and tube heat exchanger, which can be considered as a typical DPS whose mathematical description, after some assumptions, takes the form (8) and (9). The analysis is performed for both the exchanger operating in the parallel-flow mode, for which the boundary conditions have the form specified in the Definition 4, and for the countercurrent-flow configuration with boundary conditions as given by Definition 5.

Under some simplifying assumptions, the dynamic properties of a shell and tube heat exchanger can be described, based on the thermal energy balance equations, by the following PDE system [4, 8, 15, 21, 23, 27, 34]:

$$\frac{\partial \vartheta_1(l,t)}{\partial t} + v_1 \frac{\partial \vartheta_1(l,t)}{\partial l} = \alpha_1 \left(\vartheta_2(l,t) - \vartheta_1(l,t)\right), \tag{38}$$

$$\frac{\partial \vartheta_2(l,t)}{\partial t} + v_2 \frac{\partial \vartheta_2(l,t)}{\partial l} = \alpha_2 \left(\vartheta_1(l,t) - \vartheta_2(l,t)\right), \tag{39}$$

where the 1- and 2- sub-indexed figures represent the tube-side and shell-side fluid variables/coefficients, respectively; specifically $\vartheta_1(l,t)$ and $\vartheta_2(l,t)$ – the temperatures, v_1 and v_2 – the velocities, α_1 and α_2 – the heat transfer coefficients.

Assuming $v_1 = 1\,\mathrm{m/s}, v_2 = 0.2\,\mathrm{m/s}, \alpha_1 = \alpha_2 = 0.05\,1/\mathrm{s}$ in (38) and (39), one obtains the following matrices of the system (1):

$$\Lambda = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, \tag{40}$$

$$K = \begin{bmatrix} -\alpha_1 & \alpha_1 \\ \alpha_2 & -\alpha_2 \end{bmatrix} = \begin{bmatrix} -0.05 & 0.05 \\ 0.05 & -0.05 \end{bmatrix} \tag{41}$$

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and the vector of the state variables (2) given by

$$x(l,t) = \left[\vartheta_1(l,t) \quad \vartheta_2(l,t)\right]^T. \tag{42}$$

The fluid inlet temperatures ϑ_{1i}, ϑ_{2i} can be taken as the input signals, which in the considered case of the parallel-flow corresponds the following congruent boundary conditions (see Definition 4):

$$\vartheta_1(0,t) = \vartheta_{1i}(t), \tag{43}$$

$$\vartheta_2(0,t) = \vartheta_{2i}(t). \tag{44}$$

Figure 1 shows the steady-state distributions of the temperatures of the tube- and shell-side fluids, calculated based on the equations (15)-(16) for the constant values of the inlet temperatures: $\vartheta_{1i} = 100$ °C, $\vartheta_{2i} = 50$ °C. Additionally, the temperature profiles for $v_2 = 0.1$ m/s are marked by dashed line.

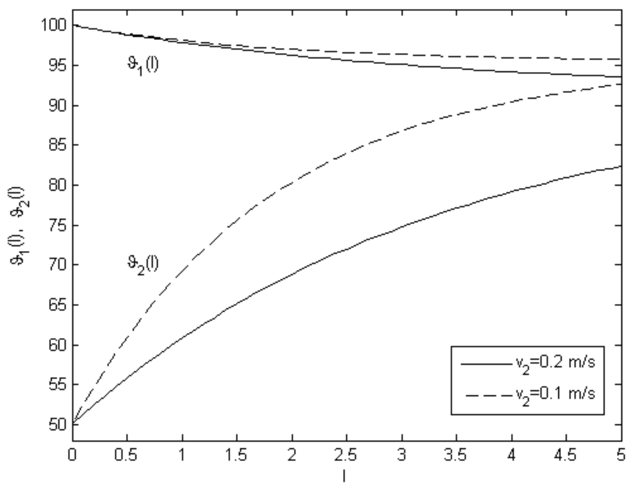

Figure 1. The steady state temperature distributions $\bar{\vartheta}_1(l)$ and $\bar{\vartheta}_2(l)$ for the parallel-flow heat exchanger ($\vartheta_{1i} = 100$ °C, $\vartheta_{2i} = 50$ °C, $v_1 = 1$ m/s, $v_2 = 0.2$ m/s and $v_2 = 0.1$ m/s).

In the countercurrent mode of operation, the fluids involved in the heat exchange enter the exchanger from its opposite ends. The PDEs describing the dynamics of the heat exchanger have the same form (38)-(39) as for the parallel-flow mode, and the difference in the mathematical description consists in the opposite signs of fluid velocities ($v_1 > 0$, $v_2 < 0$) as well as in the different boundary conditions

$$\vartheta_1(0,t) = \vartheta_{1i}(t), \tag{45}$$

$$\vartheta_2(L,t) = \vartheta_{2i}(t). \tag{46}$$

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This situation represents the case of the incongruent boundary inputs given by Definition 5. Figure 2 shows the steady-state temperature profiles for the countercurrent-flow heat exchanger calculated based on (28)-(29) assuming $\vartheta_{1i} = 100$ °C, $\vartheta_{2i} = 50$ °C and $v_1 = 1$ m/s, $v_2 = -0.2$ m/s. Additionally, the temperature profiles for $v_2 = -0.1$ m/s are shown here by dashed line.

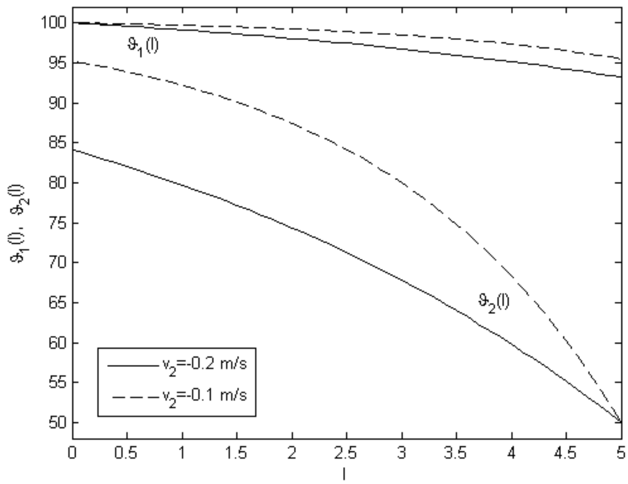

Figure 2. The steady state temperature distributions $\bar{\vartheta}_1(l)$ and $\bar{\vartheta}_2(l)$ for the countercurrent-flow heat exchanger ($\vartheta_{1i} = 100$ °C, $\vartheta_{2i} = 50$ °C, $v_1 = 1$ m/s, $v_2 = -0.2$ m/s and $v_2 = -0.1$ m/s).

From the obtained results it is possible to determine e.g. the outlet temperatures of the both fluids involved in the heat exchange. For example, for the parallel-flow configuration the outlet temperature $\bar{\vartheta}_2(L)$ of the heated fluid is about 82.5 °C and the outlet temperature $\bar{\vartheta}_1(L)$ of the heating fluid is 93.5 °C (Fig. 1). Reducing the flow rate v_2 of the heated fluid from 0.2 m/s to 0.1 m/s increases its outlet temperature $\bar{\vartheta}_2(L)$ to about 92.5 °C and also causes a slight increase in the outlet temperature of the heating fluid. As is apparent from Fig. 2, the change in the flow configuration causes further increase in the temperature of the heated fluid as compared to the parallel-flow mode. For example, when changing the flow rate v_2 from -0.2 m/s to -0.1 m/s its outlet temperature $\bar{\vartheta}_2(0)$ reaches 95 °C and is lower only by 5 °C than the inlet temperature $\bar{\vartheta}_1(0)$ of the heating fluid. As mentioned in Section 1, the derived analytical expressions not only make it possible to determine the outlet temperatures of the fluids but also allow an analysis of the temperature profiles along the heat exchanger, which may be of great importance from a technological point of view.

To sum up, the counter-flow mode of operation has several advantages as compared to parallel-flow one. The outlet temperature of the heated fluid can approach the inlet temperature of the heating fluid. The more uniform temperature difference between the two fluids prevents thermal stresses in the exchanger material. The other advantage is

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that the more uniform difference between temperatures $\bar{\vartheta}_1(l)$ and $\bar{\vartheta}_2(l)$ has an effect of more uniform heat transfer rate. The maximum amount of heat or mass transfer that can be obtained is higher with the countercurrent than parallel exchange because the first one maintains a slowly declining difference in temperature. In the parallel-flow mode the initial gradient is higher but falls off quickly, leading to wasted potential of thermal energy. The above simulation results has been fully confirmed by the industrial practice [20, 34].

5. Conclusion

The paper has addressed the problem of the general steady-state representation for a class of distributed parameter systems of hyperbolic type. The analytical expressions for the steady-state distribution of the state variables have been derived, both in the exponential and in the hyperbolic form. The influence of the location of the boundary inputs on the steady-state solutions has been demonstrated. The considerations have been illustrated with a practical example of a shell and tube heat exchanger operating in parallel- and countercurrent-flow modes, which correspond to the two types of the boundary inputs discussed in the paper.

The generalization of the results presented here has been recently proposed in our paper [7], which is related to the transfer function representation of the considered class of hyperbolic systems. The approach presented there allows to determine also their frequency- as well as time-domain responses, which is of great importance from the viewpoint of the control theory.

References

[1] F. Ancona and G.M. Coclite: On the boundary controllability of first-order hyperbolic systems. *Nonlinear Analysis: Theory, Methods & Applications*, **63**(5-7), (2005), e1955-e1966.

[2] M.A. Arbaoui, L. Vernieres-Hassimi, D. Seguin and M.A. Abdelghani-Idrissi: Counter-current tubular heat exchanger: Modeling and adaptive predictive functional control. *Applied Thermal Engineering*, **27**(13), (2007), 2332-2338.

[3] G.M. Bahaa and M.M. Tharwat: Optimal control problem for infinite variables hyperbolic systems with time lags. *Archives of Control Sciences*, **21**(4), (2010), 373-393.

[4] K. Bartecki: Comparison of frequency responses of parallel- and counter-flow type of heat exchanger. In *Proc. of the 13th IEEE IFAC Int. Conf. on Methods and*

10.2478/acsc-2013-0018

Models in Automation and Robotics, Szczecin, Poland, (2007), 411–416.

[5] K. BARTECKI: Frequency- and time-domain analysis of a simple pipeline system. In *Proc. of the 14th IEEE IFAC Int. Conf. on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, (2009), 366-371.

[6] K. BARTECKI: PCA-based approximation of a class of distributed parameter systems: classical vs. neural network approach. *Bulletin of the Polish Academy of Sciences – Technical Sciences*, **60**(3), (2012). 651-660.

[7] K. BARTECKI: A general transfer function representation for a class of hyperbolic distributed parameter systems. *Int. J. of Applied Mathematics and Computer Science*, **23**(2), (2013), 291-307.

[8] K. BARTECKI and R. ROJEK: Instantaneous linearization of neural network model in adaptive control of heat exchange process. In *Proc. of the 11th IEEE Int. Conf. on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, (2005), 967-972.

[9] H. BOUNIT: The stability of an irrigation canal system. *Int. J. of Applied Mathematics and Computer Science*, **3**(4), (2003), 453-468.

[10] F. M. CALLIER and J. WINKIN: Infinite dimensional system transfer functions. In *Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems*, ser. Lecture Notes in Control and Information Sciences, C.R.F., A. Bensoussan, and J.L. Lions, Eds. Springer Berlin Heidelberg, **185** (1993), 75-101.

[11] B. CHENTOUF and J.M. WANG: Boundary feedback stabilization and Riesz basis property of a 1-d first order hyperbolic linear system with L^∞-coefficients. *J. of Differential Equations*, **246**(3), (2009), 1119-1138.

[12] P.D. CHRISTOFIDES and P. DAOUTIDIS: Robust control of hyperbolic PDE systems. *Chemical Engineering Science*, **53**(1), (1998), 85-105.

[13] R. CURTAIN and K. MORRIS: Transfer functions of distributed parameters systems: A tutorial. *Automatica*, **45**(5), (2009), 1101-1116.

[14] J. CZECZOT, M. METZGER and M. N. J. BABARY: Monitoring and control of a class of distributed parameter bioreactors with application of the substrate consumption rate. *Archives of Control Sciences*, **11**(1/2), (2001), 5-22.

[15] C. C. DELNERO, D. DREISIGMEYER, D.C. HITTLE, P.M. YOUNG, C.W. ANDERSON and M.L. ANDERSON: Exact solution to the governing PDE of a hot water-to-air finned tube cross-flow heat exchanger. *HVAC&R Research*, **10**(1), (2004), 21-31.

10.2478/acsc-2013-0018

[16] A. DIAGNE, G. BASTIN and J.-M. CORON: Lyapunov exponential stability of 1-d linear hyperbolic systems of balance laws. *Automatica*, **48**(1), (2012), 109-114.

[17] J.C.I. DOOGE and J.J. NAPIORKOWSKI: The effect of the downstream boundary conditions in the linearized St Venant equations. *The Quarterly J. of Mechanics and Applied Mathematics*, **40**(2), (1987), 245-256.

[18] V. DOS SANTOS, G. BASTIN, J.-M. CORON and B. D'ANDRÉA NOVEL: Boundary control with integral action for hyperbolic systems of conservation laws: Stability and experiments. *Automatica*, **44**(5), (2008), 1310-1318.

[19] L.C. EVANS: Partial Differential Equations. Providence: American Mathematical Society, 1998.

[20] J.C. FRIEDLY: Dynamic Behaviour of Processes. New York: Prentice Hall, 1972.

[21] K. FUJAREWICZ: Identification and suboptimal control of heat exchanger using generalized back propagation through time. *Archives of Control Sciences*, **10**(3/4), (2001), 167-183.

[22] H. GÓRECKI, S. FUKSA, P. GRABOWSKI and A. KORYTOWSKI: *Analysis and Synthesis of Time Delay Systems*. New York: Wiley, 1989.

[23] D.D. GVOZDENAC: Transient response of the parallel flow heat exchanger with finite wall capacitance. *Archive of Applied Mechanics*, **60** (1990), 481-490.

[24] A. KOWALEWSKI: Time-optimal control of infinite order hyperbolic systems with time delays. *Int. J. of Applied Mathematics and Computer Science*, **19** (2009), 597-608.

[25] A. KOWALEWSKI and A. KRAKOWIAK: Optimal distributed control problems of retarded parabolic systems. *Archives of Control Sciences*, **19**(3), (2009), 279-294.

[26] J. KUKAL and O. SCHMIDT: Useful approximation of discrete transcendent transfer function. *Archives of Control Sciences*, **17**(1), (2007), 5-13.

[27] A. MAIDI, M. DIAF and J.-P. CORRIOU: Boundary control of a parallel-flow heat exchanger by input-output linearization. *J. of Process Control*, **20**(10), (2010), 1161-1174.

[28] R.M.M. MATTHEIJ, S.W. RIENSTRA and J.H.M. TEN THIJE BOONKKAMP: Partial Differential Equations: modeling, analysis, computation. Philadelphia: Society for Industrial and Applied Mathematics (SIAM), 2005.

[29] D.A. SPAULDING: Synthesis of rational transfer function approximations using a tapped distributed RC line with feedback. *The Bell System Technical J.*, **47**(9), (1968), 2051-2070.

10.2478/acsc-2013-0018

[30] J.C. SUTHERLAND and C.A. KENNEDY: Improved boundary conditions for viscous, reacting, compressible flows. *J. of Computational Physics*, **191**(2), (2003), 502-524.

[31] P. TATJEWSKI: Advanced Control of Industrial Processes: Structures and Algorithms, ser. Advances in Industrial Control. Springer, 2007.

[32] W. WU and C.-T. LIOU: Output regulation of two-time-scale hyperbolic PDE systems. *J. of Process Control*, **11**(6), (2001), 637-647.

[33] C.-Z. XU and G. SALLET: Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, **7** (2002), 421-442.

[34] A. ZAVALA-RIO, C.M. ASTORGA-ZARAGOZA and O. HERNÁNDEZ-GONZÁLEZ: Bounded positive control for double-pipe heat exchangers. *Control Engineering Practice*, **17**(1), (2009), 136-145.

[35] H. ZWART: Transfer functions for infinite-dimensional systems. *Systems and Control Letters*, **52**(3-4), (2004), 247-255.