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# Decomposition method and its application to the extremal problems

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In the article solution of the problem of extremal value of $x(\tau)$ is presented, for the $n$-th order linear systems. The extremum of $x(\tau)$ is considered as a function of the roots $s_1$, $s_2$, ... $s_n$ of the characteristic equation. The obtained results give a possibility of decomposition of the whole $n$-th order system into a set of 2-nd order systems.

**Key words:** extremal problems, decomposition, characteristic equation, transmittance.

## 1. Introduction

Let us consider the differential equation with constant and real parameters $a_i > 0$, $i = 1, 2, \dots, n$

$$\frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx(t)}{dt} + a_n x(t) = 0 \tag{1}$$

with the initial conditions $x^{(i-1)}(0) = c_i \neq 0$, $i = 1, 2, \dots, n$. The solution of equation (1) takes the following form:

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t} \tag{2}$$

where $s_k$ are the simple roots of the characteristic equation

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0. \tag{3}$$

**Theorem 3** *The explicit form of the coefficient* $A_1$ *is as follows* [2]*:*

$$A_1 = \frac{c_n - \left(\sum_{j\neq 1}^{n} s_j\right) c_{n-1} + \left(\sum_{i,j\neq i=1}^{n} s_i s_j\right) c_{n-2} + \cdots + (-1)^{n-1} \prod_{i=1}^{n} s_i c_1}{(s_n - s_1)(s_{n-1} - s_1) \cdots (s_2 - s_1)}, \tag{4}$$

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*then the coefficients $A_2$, $A_3$, ..., $A_n$ can be obtained by the sequential change of the indices of $s_i$ according to the rule*

$$s_1 \to s_2 \to s_3 \to \cdots s_{n-1} \to s_n \to s_1.$$

## 2. Problem formulation

Let us determine the extremal times $\tau_1$, $\tau_2$, ..., $\tau_{n-1}$ at which the solution $x(t)$ of the equation (1) assumes extremal values $x_1(\tau_1)$, $x_2(\tau_2)$, ..., $x_{n-1}(\tau_{n-1})$. The conditions for the extremum of $x(t)$ are

$$x^{(1)}(\tau) = 0 \tag{5}$$

$$x^{(2)}(\tau) \neq 0. \tag{6}$$

We consider $x(\tau)$, representing dynamic error of the system, as the function of the roots $s_1,\ s_2,\ \ldots,\ s_n$ and look for necessary conditions for $x[\tau(s_1,\ s_2,\ \ldots,\ s_n)]$ to have an extremum with respect to $(s_1,\ s_2,\ \ldots,\ s_n)$.

## 3. Solution of the problem

**Theorem 4** *In the paper* [1] *it is proved that the necessary condition for $x[\tau(s_1,\ s_2,\ \ldots,\ s_n)]$ to have an extremum with respect to $(s_1,\ s_2,\ \ldots,\ s_n)$ is*

$$(-1)^n \tau^n \prod_{k=1}^{n} A_k = 0. \tag{7}$$

It is concluded from (7) that either

$$\tau = 0 \tag{8}$$

which means that

$$c_2 = 0 \tag{9}$$

or

$$A_k = 0 \tag{10}$$

for some values of $k$ from $[1, 2, \ldots, n]$.

The relation (10) gives some relations between roots $s_i$ and initial conditions $c_i$. From the relation (4) we have that the coefficients $A_k$ are given by

$$A_k = \frac{c_n - \sum_{v=1, v\neq k}^{n} c_{n-1} s_v + \sum_{v=1, v\neq k}^{n} s_v s_k c_{n-2} + \cdots + (-1)^{n-1} c_1 \prod_{v=1, v\neq k}^{n} s_v}{\prod_{v=1, v\neq k}^{n} (s_v - s_k)}. \tag{11}$$

From (11) it is evident that for $s_v \neq s_k$, $A_k = 0$ if

$$c_n - \sum_{v=1,v\neq k}^{n} c_{n-1}s_v + \sum_{v=1,v\neq k}^{n} s_v s_k c_{n-2} + \cdots + (-1)^{n-1}c_1 \prod_{v=1,v\neq k}^{n} s_v = 0 \tag{12}$$

for some $k = 1,2,\ldots,n$.

The relation (12) can be transformed, using Vietta's formulae, to show the dependence between one root $s_1$ and initial conditions $c_i$, $i = 1,2,\ldots,n$, so

$$a_1 = -\sum_{v=1}^{n} s_v = -s_1 - \sum_{v=2}^{n} s_v \tag{13}$$

from which we have

$$\sum_{v=2}^{n} s_v = -(a_1 + s_1) \tag{14}$$

$$\sum_{v=2,v\neq k}^{n} s_v s_k = a_2 + s_1(a_1 + s_1) \tag{15}$$

$$\prod_{v=2,v\neq 1}^{n} s_2 s_v \cdots s_n = (-1)^n \frac{a_n}{s_1}. \tag{16}$$

Substituting (14),(15),(16) into relation (12) we obtain the following basic theorem.

**Theorem 5** *The root $s_1$ can be calculated as a common root of the equation*

$$\begin{aligned} &s_1^{n-1}c_{n-2} + s_1^{n-2}(c_{n-1} + a_1c_{n-2}) + s_1^{n-3}(c_n + a_1c_{n-1} + a_2c_{n-2} + a_3c_{n-3}) + \\ &\cdots + a_nc_1 = 0 \end{aligned} \tag{17}$$

*and the characteristic equation for $s = s_1$*

$$s_1^n + a_1s_1^{n-1} + a_2s_1^{n-2} + \cdots + a_{n-1}s_1 + a_n = 0.$$

**Theorem 6** *The equation* (17) *can be obtained directly using the Laplace-transform of the equation* (1)

$$X(s) = \frac{s_1^{n-1}c_{n-2} + s_1^{n-2}(c_{n-1} + a_1c_{n-2}) + s_1^{n-3}(c_n + a_1c_{n-1} + a_2c_{n-2} + a_3c_{n-3}) + \cdots + a_nc_1}{s_1^n + a_1s_1^{n-1} + a_2s_1^{n-2} + \cdots + a_{n-1}s_1 + a_n}. \tag{18}$$

Taking into account the Theorem 2 and the relations (11), (12) and (18) we obtain the following theorem.

**Theorem 7** *The vanishing of one of the coefficients $A_k$ in the relation* (2) *is possible if the numerator and denominator of the transform $X(s)$ have a common root.*

For the calculation the common root $s_1$ the algorithm of Euclid can be used and the necessary and sufficient conditions for the existence of the common root of the two equations.

In what follows we will use two theorems:

**Theorem 8** [4] *Two polynomials*

$$P_1 = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{19}$$

$$P_1 = s^m + d_1 s^{m-1} + \cdots + d_{m-1}s + d_m, \quad m \leqslant n \tag{20}$$

*are not relative prime if the rest $R(s)$ of the division of the polynomial $P_1(s)$ by the polynomial $P_2(s)$ is equal to zero.*

**Theorem 9** [4] *The necessary and sufficient condition for the two polynomials* (19) *and* (20) *to have a common root is that their discriminant $D$ is equal to zero.*

$$D = \begin{vmatrix} 1 & a_1 & \cdots & \cdots & a_n & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & a_{n-1} & a_n & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_1 & \cdots & a_n & 0 \\ 1 & d_1 & \cdots & \cdots & d_m & 0 & 0 & \cdots \\ 0 & 1 & d_1 & \cdots & \cdots & d_m & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & d_1 & \cdots & d_m \end{vmatrix} = 0. \tag{21}$$

In general we can conclude these considerations in the following theorem.

**Theorem 10** [2] *The relations*

$$x(t) = \sum_{i=1}^{n} A_i e^{s_i t} \tag{22}$$

*or*

$$x^{(1)}(t) = \sum_{i=1}^{n} s_i A_i e^{s_i t} \tag{23}$$

*or*

$$x^{(2)}(t) = \sum_{i=1}^{n} s_i^2 A_i e^{s_i t}$$

*can be decomposed into a system of relations containing a set of relations which have only two terms.*

The set contains

$$\begin{pmatrix} n \\ n-2 \end{pmatrix} = \frac{1}{2}n(n-1) \tag{24}$$

relations with only two exponential terms. This can be obtained under the restriction that the two coefficients $A_j \neq 0$, $A_k \neq 0$, $(j,k = 1,\ldots,n)$ and the remaining coefficients $A_i = 0$, $(i \neq j,\ i \neq k)$.

## 4. Calculation of the extremal time $\tau$

Using the necessary condition for the extremum $x(\tau)$, i.e.

$$\begin{aligned} x^{(1)}(\tau) &= 0 \\ x^{(2)}(\tau) &\neq 0 \end{aligned} \tag{25}$$

and the Theorem 8 we can calculate the extremal time $\tau$.

**Theorem 11** *Let the roots of the characteristic equation be ordered in the following way*

$$s_n < s_{n-1} < s_{n-2} < \cdots < s_1 < 0 \tag{26}$$

*and the coefficients satisfy* $A_i = 0$, $i = 1,2,\ldots,n$, $i \neq k$, $i \neq l$, *then*

$$s_k A_k e^{s_k \tau} + s_l A_l e^{s_l \tau} = 0. \tag{27}$$

Let us denote

$$\begin{aligned} x^{(p-1)}(0) &= c_p = s_k^{p-1} A_k + s_l^{p-1} A_l, \quad k = 1,2,\ldots,n, \quad p = 1,2,\ldots,n \\ x^{(q-1)}(0) &= c_q = s_k^{q-1} A_k + s_l^{q-1} A_l, \quad q > p. \end{aligned} \tag{28}$$

Then from (28) we obtain

$$A_k = \frac{c_p s_l^{q-1} - c_q s_l^{p-1}}{s_k^{p-1} s_l^{q-1} - s_k^{q-1} s_l^{p-1}} \tag{29}$$

$$A_l = -\frac{c_p s_k^{q-1} - c_q s_k^{p-1}}{s_k^{p-1} s_l^{q-1} - s_k^{q-1} s_l^{p-1}}. \tag{30}$$

Usually $p = 1$, $q = 2$ and

$$A_1 = \frac{c_1 s_2 - c_2}{s_2 - s_1} \tag{31}$$

$$A_2 = -\frac{c_1 s_1 - c_2}{s_2 - s_1}. \tag{32}$$

From (27) using (31), (32) we have

$$\tau = \frac{1}{s_k - s_l} \ln\left[-\frac{s_l A_l}{s_k A_k}\right] \tag{33}$$

and

$$x(\tau) = A_k e^{s_k \tau} + A_l e^{s_l \tau} \tag{34}$$

has minimum value.

## 5. Illustrative example

Let us consider a 3rd order equation

$$a_0 x^{(3)}(t) + a_1 x^{(2)}(t) + a_2 x^{(1)}(t) + a_3 x(t) = 0 \tag{35}$$

with initial conditions

$$x(0) = c_1', \quad x^{(1)}(0) = c_2', \quad x^{(2)}(0) = c_3'. \tag{36}$$

The coefficients $a_0, a_1, \ldots, a_n$ represent some parameters. The characteristic equation of the equation (35) is

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0. \tag{37}$$

We assume that the coefficients $a_0, a_1, \ldots, a_n$ fulfill the Hurwitz stability conditions and the roots of equation (37) are different and nonzero, so $s_1 \neq s_2 \neq s_3 \neq 0$.

After dividing equation (37) by $a_0 > 0$ we obtain

$$s^3 + \frac{a_1}{a_0} s^2 + \frac{a_2}{a_0} s + \frac{a_3}{a_0} = 0. \tag{38}$$

Putting

$$s = \sqrt[3]{\frac{a_3}{a_0}} z \tag{39}$$

and dividing the equation (38) by $\frac{a_3}{a_0}$ we obtain the equation with only two parameters $b_1$, $b_2$ as follows

$$z^3 + b_1 z^2 + b_2 z + 1 = 0 \tag{40}$$

where

$$\left. \begin{aligned} b_1 &= \frac{a_1}{a_3} \sqrt[3]{\left(\frac{a_3}{a_0}\right)^2} \\ b_2 &= \frac{a_2}{a_3} \sqrt[3]{\frac{a_3}{a_0}} \end{aligned} \right\} \tag{41}$$

and

$$c_1 = \frac{c_1'}{\sqrt[3]{\frac{a_3}{a_0}}}, \quad c_2 = \frac{c_2'}{\sqrt[3]{\frac{a_3^2}{a_0^2}}}, \quad c_3 = \frac{c_3'}{\frac{a_3}{a_0}}.$$

The solution of the equation (35) is

$$x(t) = \sum_{k=1}^{3} A_k e^{z_k t} \tag{42}$$

where $z_k$, $k = 1,2,3$ are the roots of equation (40) and coefficients $A_k$ are equal

$$A_1 = \frac{c_3 - (z_2 + z_3)c_2 + z_2 z_3 c_1}{(z_2 - z_1)(z_3 - z_1)} \tag{43}$$

$$A_2 = \frac{c_3 - (z_3 + z_1)c_2 + z_1 z_3 c_1}{(z_3 - z_2)(z_1 - z_2)} \tag{44}$$

$$A_3 = \frac{c_3 - (z_1 + z_2)c_2 + z_1 z_2 c_1}{(z_1 - z_3)(z_2 - z_3)}. \tag{45}$$

The equation (18) in this case has the form

$$X(z) = \frac{c_1 z^2 + z(c_2 + b_1 c_1) + c_3 + b_1 c_2 + b_2 c_1}{(z^3 + b_1 z^2 + b_2 z + 1)} \tag{46}$$

where

$$z^3 + b_1 z^2 + b_2 z + 1 = 0$$

is the characteristic equation of the equation (35).

The common root of the equation (40) and

$$c_1 z^2 + z(c_2 + b_1 c_1) + c_3 + b_1 c_2 + b_2 c_1 = 0 \tag{47}$$

is obtained using Euclid algorithm. The first division of equation (40) by equation (47) gives that

$$\frac{z}{c_1} - \frac{c_2}{c_1^2} = 0$$

and from this relation we have

$$z_1 = \frac{c_2}{c_1}, \quad c_1 \neq 0, \quad c_2 \neq 0. \tag{48}$$

The division of the numerator of (46) by $(z - z_1)$ gives

$$(c_1 z^2 + z(c_2 + b_1 c_1) + c_3 + b_1 c_2 + b_2 c_1) \div \left(z - \frac{c_2}{c_1}\right) = c_1 z + (2c_2 + b_1 c_1) \tag{49}$$

and the rest

$$\frac{b_2 c_1^2 + c_1 c_3 + 2c_1 c_2 b_1 + 2c_2^2}{c_1} = 0 \tag{50}$$

which must be zero.

The division of denominator of (46) by $(z-z_1)$ gives

$$(z^3+b_1z^2+b_2z+1)\div\left(z-\frac{c_2}{c_1}\right)=z^2+\frac{(c_1c_2+b_1c_1^2)}{c_1^2}z+\frac{b_2c_1^2+c_1c_2b_1+c_2^2}{c_1^2} \tag{51}$$

and the rest of which must be zero

$$\frac{b_2c_1^2c_2+b_1c_1c_2^2+c_1^3+c_2^3}{c_1^3}=0. \tag{52}$$

From equations (50) and (52) we obtain coefficients

$$b_2=\frac{c_2c_3-2c_1^2}{c_1c_2},\quad c_1c_2\neq 0 \tag{53}$$

$$b_1=\frac{c_1^3-c_2^3-c_1c_2c_3}{c_1c_2^2},\quad c_1c_2\neq 0. \tag{54}$$

The transform (46) takes now the form

$$X(z)=\frac{c_1^2(c_1z+b_1c_1+2c_2)}{c_1^2z^2+(c_1^2b_1+c_1c_2)z+b_2c_1^2+b_1c_1c_2+c_2^2}. \tag{55}$$

The characteristic equation becomes

$$z^2+\frac{c_1^2b_1+c_1c_2}{c_1^2}z+\frac{b_2c_1^2+b_1c_1c_2+c_2^2}{c_1^2}=0,\quad c_1\neq 0. \tag{56}$$

From (56) and taking into account (48) we obtain

$$z_2=-\frac{1}{2}(b_1+z_1)+\frac{1}{2}\sqrt{b_1^2-2z_1b_1-3z_1^2-4b_2} \tag{57}$$

$$z_3=-\frac{1}{2}(b_1+z_1)-\frac{1}{2}\sqrt{b_1^2-2z_1b_1-3z_1^2-4b_2}. \tag{58}$$

After substitution (53) and (54) we calculate

$$z_2=\frac{1}{2}\,\frac{c_1c_2c_3-c_1^3+\sqrt{\frac{c_1^2(4c_1c_2^3+c_2^2c_3^2-2c_1^2c_2c_3+c_1^4)}{c_2^4}}\,c_2^2}{c_2^2c_1} \tag{59}$$

$$z_3=\frac{1}{2}\,\frac{c_1c_2c_3-c_1^3-\sqrt{\frac{c_1^2(4c_1c_2^3+c_2^2c_3^2-2c_1^2c_2c_3+c_1^4)}{c_2^4}}\,c_2^2}{c_2^2c_1}. \tag{60}$$

The solution of (55) is

$$X(t)=\frac{c_1(b_1c_1+2c_2+z_2c_1)}{2z_2c_1+b_1c_1+c_2}e^{z_2t}+\frac{c_1(b_1c_1+2c_2+z_3c_1)}{2z_3c_1+b_1c_1+c_2}e^{z_3t}. \tag{61}$$

### 5.1. Stability analysis

We apply the Hurwitz stability criterion to the characteristic equation (40)

$$z^3 + b_1 z^2 + b_2 z + 1 = 0.$$

1. $b_1 > 0,\ b_2 > 0$ (62)
2. $\Delta = \begin{vmatrix} b_1 & 1 & 0 \\ 1 & b_2 & b_1 \\ 0 & 0 & 1 \end{vmatrix} > 0.$ (63)

Explicitly

$$b_1 b_2 - 1 > 0. \tag{64}$$

The limit of stability is obtained when

$$b_1 b_2 - 1 = 0. \tag{65}$$

Using the relation (48), the characteristic equation (56) can be written in the following form

$$z^2 + (b_1 + z_1)z + (b_2 + b_1 z_1 + z_1^2) = 0. \tag{66}$$

From the relations (57) and (58) it is evident that the stability limit is obtained when

$$z_1 = -b_1 \tag{67}$$

$$z_{2,3} = \pm i\sqrt{b_2}. \tag{68}$$

Using the relations (53) and (54) we have

$$\frac{c_1}{c_2} = -b_2, \quad c_2 \neq 0 \tag{69}$$

$$\frac{c_3}{c_2} = \frac{b_2}{b_1}, \quad c_2 \neq 0. \tag{70}$$

Finally using the relation (65) we obtain

$$\frac{c_1}{c_2} = -\frac{1}{b_1} \tag{71}$$

$$\frac{c_3}{c_2} = \frac{1}{b_1^2} \tag{72}$$

and

$$\frac{c_3}{c_2} = \left(\frac{c_1}{c_2}\right)^2. \tag{73}$$

In Fig. 1 this relation is shown, where $b_1$ is the parameter.

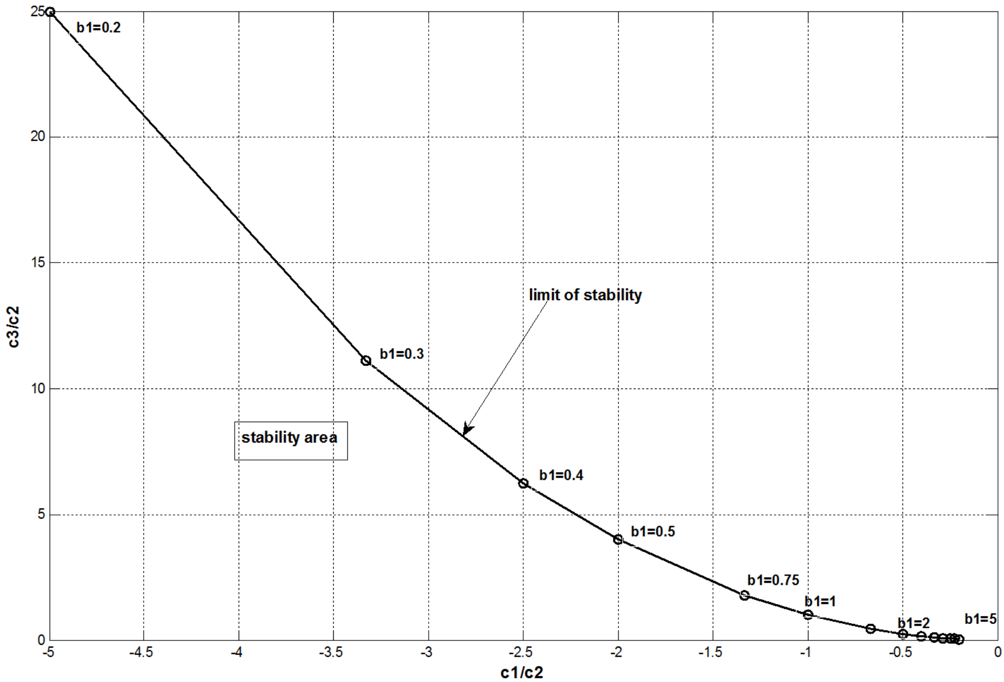


Figure 1. Limit of stability

### 5.2. Calculation of the extremal value of $\tau > 0$ and the extremal value of $x(\tau)$ for $z_2 \neq z_3$ real and negative

Let us consider relations (43), (44) and (45) assuming $A_1 = 0$. The necessary conditions for extremal time $\tau$ are

$$x^{(1)}(\tau) = 0 \tag{74}$$

and $\Delta > 0$, where $\Delta$ is the discriminant of the equation (56). Substitution (43) and (44) into (74) gives

$$z_2 A_2 e^{z_2\tau} + z_3 A_3 e^{z_3\tau} = 0. \tag{75}$$

From (75) we obtain that for real $z_1$, $z_2$, $z_3$

$$\tau = \frac{1}{z_2 - z_3} \ln\left(-\frac{z_3 A_3}{z_2 A_2}\right) \tag{76}$$

or explicitly using (57) and (58)

$$\tau = \frac{1}{z_2 - z_3} \ln\left[-\frac{-z_3 \frac{c_3-(z_1+z_2)c_2+z_1 z_2 c_1}{(z_1-z_3)(z_2-z_3)}}{z_2 \frac{c_3-(z_3+z_1)c_2+z_3 z_1 c_1}{(z_3-z_2)(z_1-z_2)}}\right].$$

After elimination of $c_3$ using (43) and $A_1 = 0$ we have finally

$$\tau = \frac{1}{z_2 - z_3} \ln \left[ \frac{z_3 \left( z_2 - \frac{c_2}{c_1} \right)}{z_2 \left( z_3 - \frac{c_2}{c_1} \right)} \right], \quad c_1 \neq 0 \tag{77}$$

where $z_2$, $z_3$ are described by the relations (59), (60).

Using the relations (59), (60) we can express $\tau$ as the function of $\frac{c_3}{c_2}$ and $\frac{c_1}{c_2}$, $c_2 \neq 0$. Note that for $c_2 = 0$, $\tau = 0$. The extremal value of

$$x(\tau) = A_2 e^{z_2 \tau} + A_3 e^{z_3 \tau} \tag{78}$$

may be obtained explicitly using the relation in [7], pp.101

$$x^2(\tau) e^{(b_1 + z_1)\tau} = c_1^2 + \frac{(b_1 + z_1) c_1 c_2}{b_2 + z_1 (b_1 + z_1)} + \frac{c_2^2}{b_2 + z_1 (b_1 + z_1)}. \tag{79}$$

### 5.3. Existence of the extremal time $\tau > 0$

Let us consider the case when the roots $z_1$, $z_2$, $z_3$ of the characteristic equation are negative, different and real. It is always possible to arrange these roots as follows

$$z_3 < z_2 < 0. \tag{80}$$

The relation (77) may be expressed in the form

$$e^{(z_2 - z_3)\tau} = \frac{z_3 \left( z_2 - \frac{c_2}{c_1} \right)}{z_2 \left( z_3 - \frac{c_2}{c_1} \right)}, \quad c_1 > 0. \tag{81}$$

**Theorem 12** *The necessary and sufficient conditions for existence of* $\tau > 0$ *are*

$$z_2 - z_3 > 0 \tag{82}$$

*and*

$$\frac{c_2}{c_1} < z_3. \tag{83}$$

**Proof** The inequality (82) results from the assumption (80). Taking into account (82) and $\tau > 0$ it is evident that

$$e^{(z_2 - z_3)\tau} > 1. \tag{84}$$

From (84) and (81) we obtain

$$\frac{z_3 \left( z_2 - \frac{c_2}{c_1} \right)}{z_2 \left( z_3 - \frac{c_2}{c_1} \right)} > 1. \tag{85}$$

Let

$$z_3\left(z_2-\frac{c_2}{c_1}\right)<0 \tag{86}$$

then

$$\left(z_2-\frac{c_2}{c_1}\right)>0 \tag{87}$$

because $z_3<0$. Also we must put

$$z_2\left(z_3-\frac{c_2}{c_1}\right)<0. \tag{88}$$

Then

$$\left(z_3-\frac{c_2}{c_1}\right)>0. \tag{89}$$

because $z_2<0$. From (87) we have

$$\frac{c_2}{c_1}<z_2. \tag{90}$$

Similarly from (89)

$$\frac{c_2}{c_1}<z_3. \tag{91}$$

But $z_3<z_2$, so finally we have

$$\frac{c_2}{c_1}<z_3. \tag{92}$$

which ends the proof. □

If we assume that inequalities (86) and (88) both change their signs, then the basic assumption is not fulfilled. From the stability condition we know that

$$z_1=\frac{c_2}{c_1}<0 \tag{93}$$

$$z_2=\frac{1}{2}\left[-\left(b_1+\frac{c_2}{c_1}\right)+\sqrt{\Delta}\right] \tag{94}$$

$$z_3=\frac{1}{2}\left[-\left(b_1+\frac{c_2}{c_1}\right)-\sqrt{\Delta}\right] \tag{95}$$

where

$$\Delta=\left[\left(b_1+\frac{c_2}{c_1}\right)\right]^2-4\left[b_2+\frac{c_2}{c_1}b_1+\left(\frac{c_2}{c_1}\right)^2\right] \tag{96}$$

$$z_2-z_3=\sqrt{\Delta}>0 \tag{97}$$

and from (92)

$$\frac{c_2}{c_1} - z_3 < 0. \tag{98}$$

In a particular case when $z_3 = z_2 = z_1 = z$ it is easy to prove that inequality (87) holds and

$$\frac{c_2}{c_1} < z. \tag{99}$$

In this case

$$x(t) = [c_1 + (c_2 - zc_1)t]e^{zt} \tag{100}$$

$$\frac{dx(t)}{dt} = [c_1 + (c_2 - zc_1)zt]e^{zt}. \tag{101}$$

From the necessary condition we obtain using (101) that

$$\tau = \frac{\frac{c_2}{c_1}}{\left(z - \frac{c_2}{c_1}\right)z} \tag{102}$$

but $\frac{c_2}{c_1} < 0$ and $z < 0$. Finally we obtain the condition for $\tau > 0$, so

$$\frac{c_2}{c_1} < z. \tag{103}$$

The third case arises when the roots $z_2$, $z_3$ are complex conjugate. The relation (74) after substitution $A_2$ and $A_3$ from (44) and (45) takes the form

$$\frac{dz}{dt} = \frac{(z_3c_1 - c_2)z_2}{z_3 - z_2}e^{z_2\tau} - \frac{(z_2c_1 - c_2)z_2}{z_3 - z_2}e^{z_3\tau} = 0 \tag{104}$$

where

$$\begin{aligned} z_2 &= \alpha + j\omega\tau \\ z_3 &= \alpha - j\omega\tau. \end{aligned} \tag{105}$$

Substitution (105) into (104) gives

$$\frac{\omega c_2 \cos(\omega\tau) - [c_1(\alpha^2 + \omega^2) - c_2\alpha]\sin(\omega\tau)}{\omega}e^{\alpha\tau} = 0. \tag{106}$$

From (106) we obtain that

$$\tau = \frac{1}{\omega}\left[\operatorname{arctg}\frac{\omega\frac{c_2}{c_1}}{(\alpha^2 + \omega^2) - \alpha\frac{c_2}{c_1}} + k\pi\right], \quad c_1 \neq 0, \ k = 0, 1, 2, \ldots \tag{107}$$

and

$$x(\tau) = \frac{[c_1\omega\cos(\omega\tau) - (c_1\alpha - c_2)\sin(\omega\tau)]}{\omega}e^{\alpha\tau}. \tag{108}$$

Domains of the different kinds of roots $z_1$, $z_2$, $z_3$ and the extremal time $\tau$ are presented in Fig. 2.

Extremal transients of the error in terms of the initial conditions from different domains of the roots are presented in Figs. 3, 4, 5, 6 and 7.

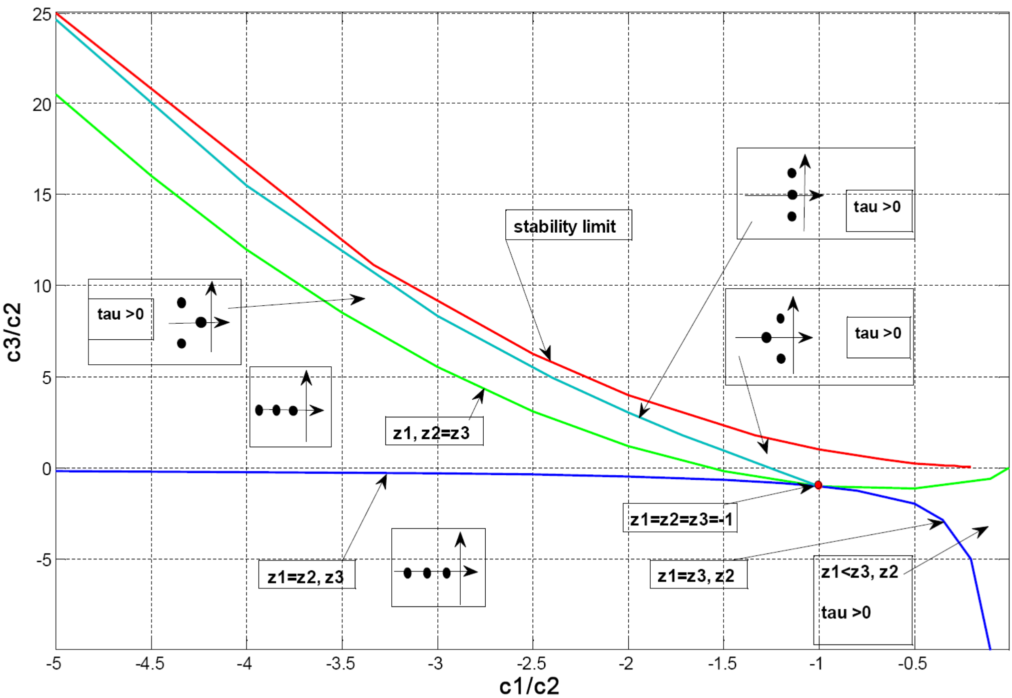


Figure 2. Domains of the different kinds of roots

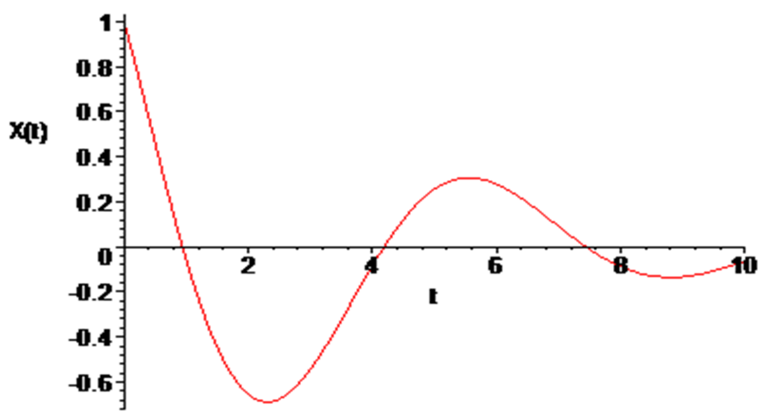


Figure 3. Transient of the error for $\left\{\frac{c_1}{c_2} = -1,\ \frac{c_3}{c_2} = 0.5\right\} \rightarrow \{z_1 = -1,\ z_{2,3} = -0.25 \pm j0.9682458365\}$. From (107) $\tau_1 = 2.302983683$, $x(\tau_1) = -0.688656$.

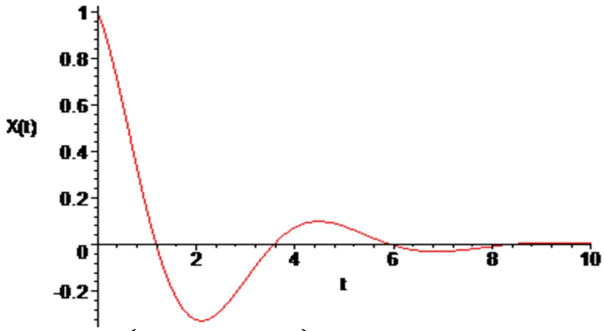


Figure 4. Transient of the error for $\left\{\frac{c_1}{c_2}=-2,\ \frac{c_3}{c_2}=3\right\} \rightarrow \{z_1=-0.5,\ z_{2,3}=-0.5\pm j1.322875656\}$. From (107) $\tau_1=2.101652954$, $x(\tau_1)=-0.327066$.

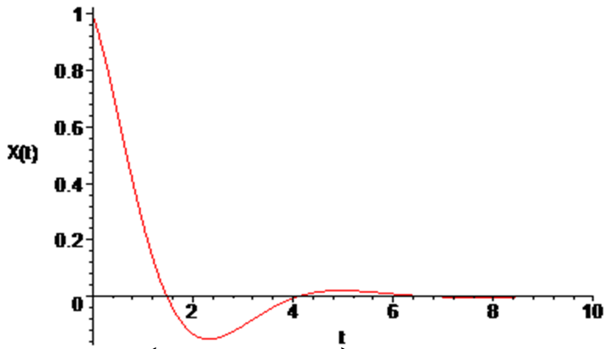


Figure 5. Transient of the error for $\left\{\frac{c_1}{c_2}=-2,\ \frac{c_3}{c_2}=2.5\right\} \rightarrow \{z_1=-0.5,\ z_{2,3}=-0.75\pm j1.19895788\}$. From (107) $\tau_1=2.32549596$, $x(\tau_1)=-0.151379$.

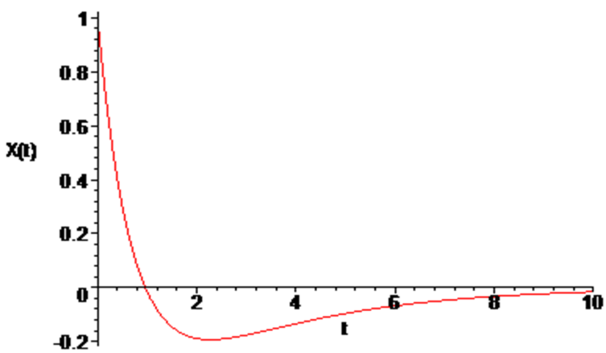


Figure 6. Transient of the error for $\left\{\frac{c_1}{c_2}=-0.5,\ \frac{c_3}{c_2}=-1.5\right\} \rightarrow \{z_1=-2,\ z_2=-0.3596117969,\ z_3=-1.390388203\}$. From (107) $\tau=2.272247399$, $x(\tau_1)=-0.19366$.

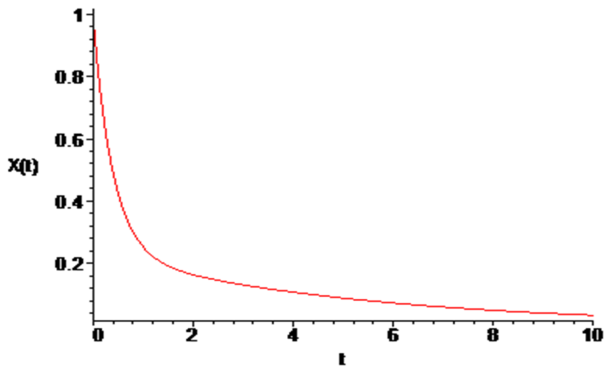


Figure 7. Transient of the error for $\left\{\frac{c_1}{c_2} = -0.5,\ \frac{c_3}{c_2} = -2.5\right\} \rightarrow \{z_1 = -2,\ z_2 = -0.19575235,\ z_3 = -2.55424764\}$, $z_1 > z_3$, $\tau < 0$.

## 6. Practical example

Let us consider the optimal choice of gain and time constant of the differential network of the compensator, Fig.8.

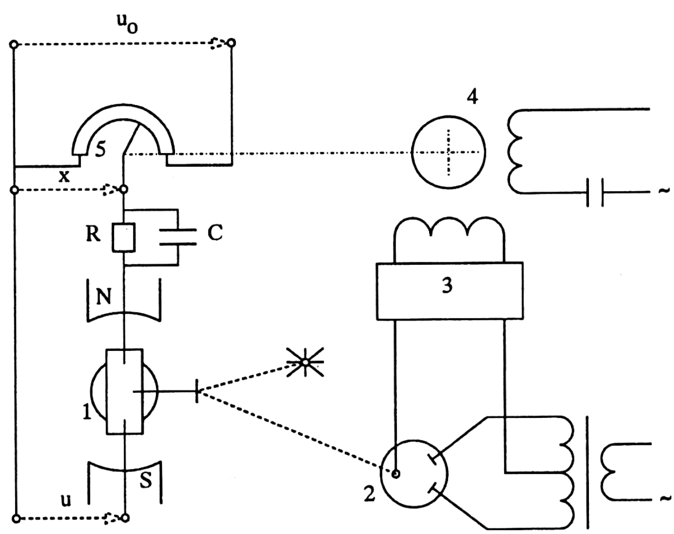


Figure 8. Voltage compensator: 1– galvanometer, 2 – photocell, 3 – amplifier, 4 – motor, 5 – potentiometer

Optimal choice ensures the minimal value of dynamic error. The difference between the measured voltage $u$ and that on the potentiometer 5 is fed through the network RC to the galvanometer 1. The light signal from the galvanometer mirror is sent to the photocell

2, which through the amplifier 3 supplies the two-phase motor 4. The motor rotates the potentiometer 5 until the balance between the voltage $x$ on the potentiometer and the measured voltage $u$ is reached.

The compensator is described by the following equations

$$(T_1 s + 1)X_1(s) = k_1(1 + T_2 s)[U(s) - X(s)]$$

$$(2\xi T_3 s + 1)X_2(s) = k_2 X_1(s)$$

$$sX(s) = k_3 X_2(s)$$

where:

$U(s)$, $X(s)$ – Laplace transforms of the voltages,
$X_1(s)$ – Laplace transform of the galvanometer current,
$X_2(s)$ – Laplace transform of the angle of the galvanometer frame,
$\frac{1}{T_3}$ – natural frequency of the galvanometer,
$\xi$ – damping coefficient of the galvanometer,
$T_1 = \frac{R_g R}{R_g + R} C$,
$T_2 = RC$,
$R + R_g$ – resistance of the galvanometer circuit,
R,C – resistance and capacitance of the correction-circuit,
$k_1$, $k_2$, $k_3$ – gain coefficients.

Taking into account that the resistance $R_g$ is very small we can neglect the time constant $T_1$. Assuming that $u(t)$ is the unit step function, we can write the transform of the output as

$$X(s) = \frac{1}{s} \frac{K + KT_2 s}{a_0 s^3 + a_1 s^2 + a_2 s + a_3}$$

where

$K = k_1 k_2 k_3, \quad a_0 = T_3^2, \quad a_1 = 2\xi T_3$
$a_2 = (1 + KT_2), \quad a_3 = K.$

The steady state error is equal to zero as

$$\lim_{s \to 0} sX(s) = \frac{K}{a_3} = 1.$$

The transform of the error is

$$E(s) = \frac{1}{s} - \frac{K + KT_2 s}{a_3 + a_2 s + a_1 s^2 + a_0 s^3} \frac{1}{s} = \frac{1 + a_1 s + a_0 s^2}{a_3 + a_2 s + a_1 s^2 + a_0 s^3}$$

$$E(s) = \frac{1 + 2\xi T_3 s + T_3^2 s^2}{K + (1 + KT_2)s + 2\xi T_3 s^2 + T_3^2 s^3}.$$

We look for optimal $K$ and $T_2$, for which dynamic error $x(\tau)$ assumes a minimal value. Putting

$$s = \sqrt[3]{\frac{K}{T_3^2}}\, z$$

we obtain characteristic equation

$$z^3 + b_1 z^2 + b_2 z + 1 = 0$$

where

$$\begin{cases} b_1 = \frac{2\xi T_3}{K} \sqrt[3]{\left(\frac{K}{T_3^2}\right)^2} \\ b_2 = \frac{1+KT_2}{K} \sqrt[3]{\frac{K}{T_3^2}}. \end{cases}$$

Comparing with the corresponding initial conditions

$$b_1 = \left(\frac{c_1}{c_2}\right)^2 - \frac{c_2}{c_1} - \frac{c_3}{c_2}$$

and

$$b_2 = \frac{c_3}{c_1} - 2\frac{c_1}{c_2} = \frac{c_3}{c_2}\frac{c_2}{c_1} - 2\frac{c_1}{c_2}$$

we obtain that

$$K = \frac{(2\xi)^3}{T_3\left[\left(\frac{c_1}{c_2}\right)^2 - \frac{c_2}{c_1} - \frac{c_3}{c_2}\right]^3}$$

$$T_2 = \left\{ \frac{\left[\frac{c_3}{c_2}\frac{c_2}{c_1} - 2\frac{c_1}{c_2}\right] K}{\sqrt[3]{\frac{K}{T_3^2}}} - 1 \right\} \frac{1}{K}.$$

In our example we take $T_3 = 0.1s$, $\xi = 0.75$, then $a_1 = 0.15$, $a_0 = 0.01$.

For $\left\{\frac{c_1}{c_2} = -2 \text{ and } \frac{c_3}{c_2} = 2\right\}$ the optimal $K = 2.16$ and the optimal coefficient of the derivative $T_2 = 0.037037$.

For $\left\{\frac{c_1}{c_2} = -2 \text{ and } \frac{c_3}{c_2} = 3\right\}$ the optimal $K = 10$ and the optimal coefficient of the derivative $T_2 = 0.15$.

For $\left\{\frac{c_1}{c_2} = -2 \text{ and } \frac{c_3}{c_2} = 3.5\right\}$ the optimal $K = 33.75$ and the optimal coefficient of the derivative $T_2 = 0.12037$.

**Remark.** In the article [5] the solution of the extremal value of $\tau(s_1, \ldots, s_n)$ as the function of the roots $s_1, \ldots, s_n$ has been presented, with the assumption that the roots are real and negative. In the next article [6] this problem has been solved for the complex-conjugate roots of the characteristic equation.

## 7. Conclusions

In the article it is proved that, from the condition $A_k = 0$ for the extremum of $x(\tau)$, the method results for decomposition of the $n$th order system into the set of 2nd order subsystems. It is also proved that the condition $A_k = 0$ is equivalent to the condition that the numerator and denominator of the transmittance $X(s)$ have a common root.

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