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The wave transformer – analysis and application of time digital filters and the integral-derivative operators of ½ order

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Abstract: In this paper the way of modeling phenomena occurring during the voltage and current waves passing through a point connection of two lines, with different wave impedance operators, is presented. This connection point is called „the wave transformer". The analyzes and the resulting formulas concern not the frequency domain, but the time domain. The appropriate transition matrices of waves through the wave transformer are defined. This matrices are the convolution integral-derivative operators of fractional order (the digital filters). For a lossless line the wave transition matrices through the wave transformer become number type instead of operator type. All matrix multiplications occurring in the formulas should be understood in convolution way.

Key words: wave transformer, integral-derivative operators of fractional order, convolution operators, time domain, digital filters

1. Introduction

The ideal wave transformer is a simple connection of two lossless transmission lines. Waves: voltage and current, passing through such a point, change in step way, as it happens in a winding transformer for sinusoidal signals of voltage and current. However, detailed analysis of the wave transformer in the time domain is much more complicated, taking into account the distributed losses of waves energy along the line. This is because the appropriate transition matrices of waves through the wave transformer become the convolution integral-derivative operators of fractional order. The analysis of such cases in the frequency domain is not effective, especially for rapidly changing waveforms of voltage and current waves. It much better to use the time domain, but such an analysis has not been previously applied, by using the fractional order digital filters [1-3, 7]. This article meets the task and for the first time resolves the issue. The transition matrix of waves through the wave transformer is composed of threes of convolution integral-derivative operators of ½ order. This allows to accurately analyze the waveforms of voltage and current after passing through a direct connection of two lines with different wave impedance operators. The discussion uses a discrete time domain, which enabled the use of recursion process, which is not in continuous time domain.

2. The transmission line and the integral-derivative operators of ½ order

The connection of two transmission lines with different wave impedance operators will be called the wave transformer. In the time domain, wave impedance operators, are the integral-derivative operators of ½ order with real zeros and poles. The analysis of such a case, taking into account the distribution of losses along the transmission line, has not been previously done.

The differential equation of a transmission line in partial derivatives has the form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2\beta \frac{\partial u}{\partial t} + RGu = 0 \tag{1}$$

and ordinary differential equation with operators

$$\frac{\partial}{\partial t} \to s \to \frac{1}{\tau}(1 - z) \tag{2}$$

can be reduced to the form

$$\frac{d^2 u}{dx^2} - A^2 u = 0, \tag{3}$$

where:

$$A = (s^2 + 2\beta s + RG)^{\frac{1}{2}} = \left((s + R\rho^{-1})(s + G\rho)\right)^{\frac{1}{2}} = \frac{1}{\tau}(a - z)^{\frac{1}{2}}(b - z)^{\frac{1}{2}} \tag{4}$$

$$a = 1 + \tau R\rho^{-1}, \quad b = 1 + \tau G\rho.$$

$$t \to \frac{t}{\sqrt{LC}}, \quad 2\beta = R\rho^{-1} + G\rho, \quad \rho = \sqrt{\frac{L}{C}}.$$

The general solution of the differential equation (3) has the form:

$$\sqrt{2}u^x = e^{-x\mathbf{A}} f^+ + e^{x\mathbf{A}} f^-,$$
$$\sqrt{2}i^x = Ye^{-x\mathbf{A}} f^+ - Ye^{x\mathbf{A}} f^-, \tag{5}$$

where: f^+, f^- – any time signals characterized by sequences $\{f_n^+\}_{n=-\infty}^{\infty}$, $\{f_n^-\}_{n=-\infty}^{\infty}$,

$$\hat{Z} = \rho\sqrt{\frac{a-z}{b-z}} \quad Y = \gamma\sqrt{\frac{b-z}{a-z}}$$

the integral-derivative operators of ½ order with real zeros and poles *a,b,* $\rho\gamma = 1$, $\hat{Z}$ is the wave impedance operator, and Y – the wave admittance operator of the line.

The solution of (5) can be written as the equation:

$$\begin{bmatrix} u^x \\ i^x \end{bmatrix} = \mathbf{K}^U \begin{bmatrix} e^{-x\mathbf{A}} & 0 \\ 0 & e^{x\mathbf{A}} \end{bmatrix} \begin{bmatrix} f^+ \\ f^- \end{bmatrix}. \tag{6}$$

Equation (6), at a distance "*x*" of transmission line, can be written as:

$$\begin{bmatrix} u \\ i \end{bmatrix} = \mathbf{K}^U \begin{bmatrix} u^+ \\ u^- \end{bmatrix} = \mathbf{K}^I \begin{bmatrix} i^+ \\ i^- \end{bmatrix}, \tag{7}$$

where:

$$\begin{gathered} \mathbf{K}^U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ Y & -Y \end{bmatrix}; \qquad \mathbf{K}^I = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{Z} & -\hat{Z} \\ 1 & 1 \end{bmatrix} \\ \left(\mathbf{K}^U\right)^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \hat{Z} \\ 1 & -\hat{Z} \end{bmatrix}; \quad \left(\mathbf{K}^I\right)^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} Y & 1 \\ -Y & 1 \end{bmatrix}, \end{gathered} \tag{8}$$

$u^+, u^-; i^+, i^-$ – the waves: voltage and current, the forward "+" and reverse "–" in place "*x*".

Relationships between the pairs of waves in locations *x, y* have the form:

$$\begin{gathered} \begin{bmatrix} u^+ \\ u^- \end{bmatrix}^x = \begin{bmatrix} e^{-(x-y)\mathrm{A}} & 0 \\ 0 & e^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix}^y \\ \begin{bmatrix} i^+ \\ i^- \end{bmatrix}^x = \begin{bmatrix} e^{-(x-y)\mathrm{A}} & 0 \\ 0 & e^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} i^+ \\ i^- \end{bmatrix}^y \\ \begin{bmatrix} u^+ \\ u^- \end{bmatrix}^x = \hat{Z} \begin{bmatrix} e^{-(x-y)\mathrm{A}} & 0 \\ 0 & -e^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} i^+ \\ i^- \end{bmatrix}^y \\ \begin{bmatrix} i^+ \\ i^- \end{bmatrix}^x = Y \begin{bmatrix} e^{-(x-y)\mathrm{A}} & 0 \\ 0 & -e^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix}^y \end{gathered} \tag{9}$$

The last two formulas in Equations (9) can be called *distributed Ohm's wave law*, and their particular form for $x = y$ (point):

$$\begin{bmatrix} u^+ \\ u^- \end{bmatrix} = \hat{Z}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i^+ \\ i^- \end{bmatrix}$$
$$\begin{bmatrix} i^+ \\ i^- \end{bmatrix} = Y\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u^+ \\ u^- \end{bmatrix} \tag{10}$$

can be called *point Ohm's wave law.*

The samples-convolution implementations of operators: A – double differential operator of ½ order, A^{-1} – double integral operator of ½ order, $\hat{Z}$, Y – the integral-derivative operators of ½ order with real zeros and poles have the form [5, 6]:

$$A_n = \frac{\sqrt{ab}}{\tau} a^{-n}\alpha_n, \quad A_n^{-1} = \frac{\tau}{\sqrt{ab}} a^{-n}\beta_n$$
$$\alpha_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m h_{n-m}h_m, \quad \beta_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m k_{n-m}k_m$$
$$\hat{Z}_n = \rho\sqrt{\frac{a}{b}}a^{-n}\rho_n, \quad Y_n = \gamma\sqrt{\frac{b}{a}}a^{-n}\gamma_n$$
$$\rho_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m h_{n-m}k_m, \quad \gamma_n = \sum_{m=0}^{n}\left(\frac{a}{b}\right)^m h_m k_{n-m}, \tag{11}$$

where: $\{k_n\}$, $\{h_n\}$ – integral-derivative operators of ½ order, defined by the formulas:

$$k_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{7}{8}\ldots\frac{2n-1}{2n}$$
$$h_n = -\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{5}{8}\ldots\frac{2n-3}{2n} \tag{12}$$

$k_0 = h_0 = 1, \quad n = 0, 1, 2, \ldots$

Exponential operator appearing in Equations (9) is obtained from the recursive Equation [4, 6]:

$$e_n^{xA} = x\sum_{m=1}^{n}\frac{m}{n}A_m e_{n-m}^{xA} \tag{14}$$

with the initial condition $e_0^{xA} = e^{xA_0}$.

3. Wave transformer as a lumped barrier in the transmission line

In Figure 1 the transmission line diagram with inserted a lumped four-terminal as the chain matrix **A** is shown.

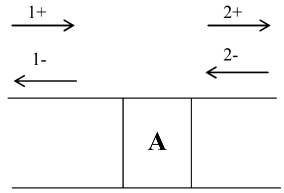

Fig. 1. A lumped barrier in the transmission line – the four-terminal as the chain matrix **A**

The equations of the waves on the left and on the right of the barrier have the form:

$$\begin{bmatrix} u^{1+} \\ u^{1-} \end{bmatrix} = \mathbf{T}^{U} \begin{bmatrix} u^{2+} \\ u^{2-} \end{bmatrix} = \mathbf{T}^{UI} \begin{bmatrix} i^{2+} \\ i^{2-} \end{bmatrix}$$
$$\begin{bmatrix} i^{1+} \\ i^{1-} \end{bmatrix} = \mathbf{T}^{IU} \begin{bmatrix} u^{2+} \\ u^{2-} \end{bmatrix} = \mathbf{T}^{I} \begin{bmatrix} i^{2+} \\ i^{2-} \end{bmatrix}. \tag{15}$$

The four **T** matrices appearing in Equations (15) will be called: *the transition matrices*. The relationships between them have the form:

$$\mathbf{T}^{I} = \frac{\hat{Z}^{2}}{\hat{Z}^{1}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{T}^{U} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\mathbf{T}^{IU} = \frac{1}{\hat{Z}^{1}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{T}^{U} = \frac{1}{\hat{Z}^{2}} \mathbf{T}^{I} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{16}$$
$$\mathbf{T}^{UI} = \hat{Z}^{2} \mathbf{T}^{U} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \hat{Z}^{1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{T}^{I},$$

where: $\hat{Z}^{1}, \hat{Z}^{2}$ – impedance operators on appropriate sides of the line

Using Equations (7) and (15) can be obtained the formulas:

$$\mathbf{T}^{U} = \left(\mathbf{K}^{U1}\right)^{-1} \mathbf{A} \mathbf{K}^{U2} \qquad \mathbf{T}^{UI} = \left(\mathbf{K}^{U1}\right)^{-1} \mathbf{A} \mathbf{K}^{I2}$$
$$\mathbf{T}^{IU} = \left(\mathbf{K}^{I1}\right)^{-1} \mathbf{A} \mathbf{K}^{U2} \qquad \mathbf{T}^{I} = \left(\mathbf{K}^{I1}\right)^{-1} \mathbf{A} \mathbf{K}^{I2}. \tag{17}$$

The connection of two transmission lines with different operators of wave impedance (without using the four-terminal i.e. **A** = **1**) will be called the wave transformer. Then the four transition matrices are determined by the formulas:

$$
\begin{aligned}
\mathbf{T}^{U} &= \frac{1}{\hat{Z}^{2}}\begin{bmatrix} \overline{Z} & \Delta\hat{Z} \\ \Delta\hat{Z} & \overline{Z} \end{bmatrix} \\
\mathbf{T}^{UI} &= \begin{bmatrix} \overline{Z} & -\Delta\hat{Z} \\ \Delta\hat{Z} & -\overline{Z} \end{bmatrix} \\
\mathbf{T}^{IU} &= \begin{bmatrix} \overline{Y} & -\Delta Y \\ \Delta Y & -\overline{Y} \end{bmatrix} \\
\mathbf{T}^{I} &= \frac{1}{Y^{2}}\begin{bmatrix} \overline{Y} & \Delta Y \\ \Delta Y & \overline{Y} \end{bmatrix}
\end{aligned}
\tag{18}
$$

and inverse matrices

$$
\begin{aligned}
(\mathbf{T}^{U})^{-1} &= \frac{1}{\hat{Z}^{1}}\begin{bmatrix} \overline{Z} & -\Delta\hat{Z} \\ -\Delta\hat{Z} & \overline{Z} \end{bmatrix} \\
(\mathbf{T}^{UI})^{-1} &= \begin{bmatrix} \overline{Y} & \Delta Y \\ -\Delta Y & \overline{Y} \end{bmatrix} \\
(\mathbf{T}^{IU})^{-1} &= \begin{bmatrix} \overline{Z} & \Delta\hat{Z} \\ -\Delta\hat{Z} & -\overline{Z} \end{bmatrix} \\
(\mathbf{T}^{I})^{-1} &= \frac{1}{Y^{1}}\begin{bmatrix} \overline{Y} & -\Delta Y \\ -\Delta Y & \overline{Y} \end{bmatrix},
\end{aligned}
\tag{19}
$$

where Y^1, Y^2 – admitance operators on appropriate sides of the line

$$\overline{Z} = \frac{1}{2}\left(\hat{Z}^{1} + \hat{Z}^{2}\right); \quad \overline{Y} = \frac{1}{2}\left(Y^{1} + Y^{2}\right) \text{ – the average operators}$$

$$\Delta\hat{Z} = \frac{1}{2}\left(\hat{Z}^{2} - \hat{Z}^{1}\right); \quad \Delta Y = \frac{1}{2}\left(Y^{2} - Y^{1}\right) \text{ – the difference operators}$$

All obtained operators are discrete-time convolutions here. Thus, each matrix operators (18) and (19), which occur in Equations (15) consist of four convolution operators defined by sequences:

$$\{\overline{Z}\}_{n=0}^{\infty};\ \{\overline{Y}\}_{n=0}^{\infty};\ \{\Delta\hat{Z}\}_{n=0}^{\infty};\ \{\Delta Y\}_{n=0}^{\infty}$$

and inversions:

$$\{(\hat{Z}^1)^{-1}\}_{n=0}^{\infty};\ \{(\hat{Z}^2)^{-1}\}_{n=0}^{\infty};\ \{(Y^1)^{-1}\}_{n=0}^{\infty};\ \{(Y^2)^{-1}\}_{n=0}^{\infty}.$$

The inversion algorithm is obtained by the convolution algorithm of one (the algorithm of convolution one). For any causal operator B, characterized by a series $\{B_n\}_{n=0}^{\infty}$, the inversion $B^{-1} \leftrightarrow \{B_n^{-1}\}_{n=0}^{\infty}$ satisfies the convolution equation

$$BB^{-1} = 1$$

or

$$\sum_{m=0}^{n} B_m B_{n-m}^{-1} = \delta_n = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

hence the recursive formula:

$$B_n^{-1} = \frac{1}{B_0} \text{ dla } n = 0,$$

$$B_n^{-1} = -\frac{1}{B_0}\sum_{m=1}^{n} B_m B_{n-m}^{-1} \text{ dla } n = 1, 2, ...$$

4. Conclusions

For a lossless line, the transition matrices of waves through the wave transformer, defined by formulas (18) and (19) become number type instead of operator type. Instead of convolution operators of wave impedance and admittance $\hat{Z}$, Y occur in these matrices positive real numbers – wave resistance and conductance ρ, γ. Using the transition matrices of waves (18) and (19) much more accurate analyzing of time waveforms of voltage and current waves on both sides of the wave transformer can be done, taking into account losses distributed along the line, because in the previous theory of transmission line operators, was not considered distributed attenuation (lumped attenuation was considered only). The approach in the article is able to take into account the distributed attenuation of the waves, which means that it is more accurate with respect to the current state of the art.

All matrix multiplications occurring in the formulas should be understood in convolution way.

For example, the relationship between voltage waves (mods) at places *x, y* (formulas (9)) should be understood as:

$$\begin{bmatrix} u^+ \\ u^- \end{bmatrix}_n = \sum_{m=0}^{\mathrm{n}} \begin{bmatrix} e_m^{-(x-y)\mathrm{A}} & 0 \\ 0 & e_m^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix}_{n-m},$$

but the formula for the "distributed Ohm's wave law" (also formulas (9)) takes the form of double-convolution:

$$\begin{bmatrix} u^{+} \\ u^{-} \end{bmatrix}_n = \sum_{m=0}^{n} \begin{bmatrix} \sum_{p=0}^{m} \hat{Z}_{m-p} e_p^{-(x-y)\mathrm{A}} & 0 \\ 0 & -\sum_{p=0}^{m} \hat{Z}_{m-p} e_p^{(x-y)\mathrm{A}} \end{bmatrix} \begin{bmatrix} i^{+} \\ i^{-} \end{bmatrix}_{n-m} .$$

The convolution of exponential operator with wave impedance operator occurs within the transition matrix. The exponential operator is determined using the recursive convolution formula (14). Also, formulas (15) are complex convolution relationships determined at the joining point of the lines:

$$\begin{bmatrix} u^{1+} \\ u^{1-} \end{bmatrix}_n = \sum_{m=0}^{n} \sum_{p=0}^{m} (\hat{Z}^2)^{-1}_{m-p} \begin{bmatrix} \overline{Z}_p & \Delta\hat{Z}_\mathrm{p} \\ \Delta\hat{Z}_\mathrm{p} & \overline{Z}_p \end{bmatrix} \begin{bmatrix} u^{2+} \\ u^{2-} \end{bmatrix}_{n-m} .$$

$(\hat{Z}^2)^{-1}$ – the convolution inversion of wave operator $\hat{Z}^2$.

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