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# Application of methods for area calculation of geodesic polygons on Polish administrative units

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**Abstract:** The paper presents methods of area calculation, which may be applied for big geodesic polygons on the ellipsoid. Proposal developed by the authors of this paper is discussed. The proposed methods are compared with other, alternative methods of area calculation of such polygons. Test calculations are performed for administrative units in Poland. The obtained results are also compared with areas of those units registered in statistical annals.

Utilisation of the equal-area map projections of the ellipsoid onto a plane seems to be the best solution for the discussed task. In the case of small distances between points we may expect accurate results of calculations, since the area size is influenced by the projection reductions only, which are small in such cases. In some cases their influence on results of calculations may be neglected. Then, only re-calculation of co-ordinates from the GRS80 ellipsoid to the cartographic, equal-area projection is required.

**Keywords:** equal-area map projections, geodetic polygons, calculation area

## 1. Introduction

The ability to calculate areas of geodesic polygons, i.e. such polygons, the sides of which are parts of geodesic lines, is important in geodetic and cartographic practice. One of the basic tasks performed by geodesists and cartographers is to calculate areas of various surface objects, such as municipalities, voivodships, arable areas, etc. It is a relatively simple task, when calculations are performed for a sphere or a plane. It becomes complicated, if an ellipsoid of revolution is assumed as the reference surface for the physical Earth surface, since analytical accurate formulas, which would allow for performing such calculations, do not exist. Only approximate formulas, which may be applied for small areas, are known. Difficulties are related, in particular, to the cases of big polygons, located on the ellipsoid.

The paper presents methods of area calculation, which may be applied for big geodesic polygons on the ellipsoid. Proposal developed by the authors of this paper is discussed. The proposed methods are compared with other, alternative methods of area calculation of such polygons. Test calculations will be performed for administrative units in Poland. The obtained results are also compared with areas of those units registered in statistical annals.

## 2. Characteristics of selected methods of calculation of areas of geodesic polygons

Two interesting methods of area calculation was proposed by Sjöberg (2006). First method is based on recursive procedure. Sjöberg started from Gauss-Bennett's formula:

$$\varepsilon_g = \iint_T \kappa (B)\, dT \tag{1}$$

where $\varepsilon$is the geodetic excess, $\kappa$– the Gaussian curvature, $B$– geodesic latitude, $T$– area of a geodetic polygon. He proposed following formula to calculate area of geodesic polygons:

$$T = b^2 \varepsilon_g + \Delta T \tag{2}$$

where $b$ is the semi-minor axis of the ellipsoid, and $\Delta T$ is a correction to the approximate area $T^0$ determined by the geodesic excess.

He also presented a direct method for numerical integration of the area under the geodesic line. Another solution was proposed by Karney (2011). He extended the method of Danielsen (1989) to higher order of series. In formulating the problem, he followed Sjöberg (2006). Karney applied the auxilary sphere of the radius $R_q$, which area is equal to the area of the ellipsoid:

$$R_q^2 = \frac{a^2}{2} + \frac{b^2}{2} \frac{\tanh^{-1} e}{e} \tag{3}$$

where $a, b$ are the semi-axes of the ellipsoid, and $e$ is the eccentricity.

Karney transformed the Sjöberg formula for area calculation into the following form:

$$S_{12} = S(\sigma_2) - S(\sigma_1) \tag{4}$$

where:

$$S(\sigma) = R_q^2 \alpha + e^2 a^2 \cos \alpha_0 \sin \alpha_0 I_4 (\sigma) \tag{5}$$

$$\sin \alpha_0 = \sin \alpha \cos \beta \tag{6}$$

$$I_4 (\sigma) = \sum_{l=0}^{\infty} C_{4l} \cos ((2l + 1)\, \sigma) \tag{7}$$

$$\cos\sigma = \cos\beta\cos\omega \tag{8}$$

$$\omega = \frac{1}{\sqrt{1 + e'^2\cos^2 B}} \tag{9}$$

$$k = e'\cos\alpha_0 \tag{10}$$

$e'$ is a second eccentricity of an ellipsoid, $f$ – a flattening, $\beta$ – the reduced latitude.

Summing up areas for each geodesic line results in its total area, under the condition, that the area does not include the pole.

A similar approach to the problem was proposed by Balcerzak and Pędzich (2006). This method is based on calculation of ellipsoidal, meridian-and-parallel trapezoids. The area of such a trapezoid is calculated from the well-known formula:

$$P = \frac{1}{2}a^2\left(1 - e^2\right)(L_2 - L_1)\left(\frac{\sin B}{1 - e^2\sin^2 B} + \frac{1}{2e}\ln\frac{1 + e\sin B}{1 - e\sin B}\right)\Bigg|_{B_1}^{B_2} \tag{11}$$

The method consists of projection of vertices of the polygon onto a selected parallel, for example $B_{min}$, along the meridian line (Fig. 1a). The resulting set of areas of curvilinear trapezoids, after summing up with correct signs (depending on the integration direction), similarly to area calculation on the xOy plane, results in the required area of the polygon $P$. We may also project vertices of the polygon on the selected meridian, for example $L_{\min}$ (Fig. 1b). The polygon area is calculated by summing up areas limited by parallels and meridians and arcs of geodesic lines. These areas are calculated by summing up, depending on the assumption of the applicable increment $\Delta L$ in the first projection and $\Delta B$ in the second projection.

Since the part of the geodesic line between two points of the polygon may contain the point of turn of this line, the possible point of turn should be checked and determined; if such a point exists, the given part should be divided into two parts, which will form two trapezoids.

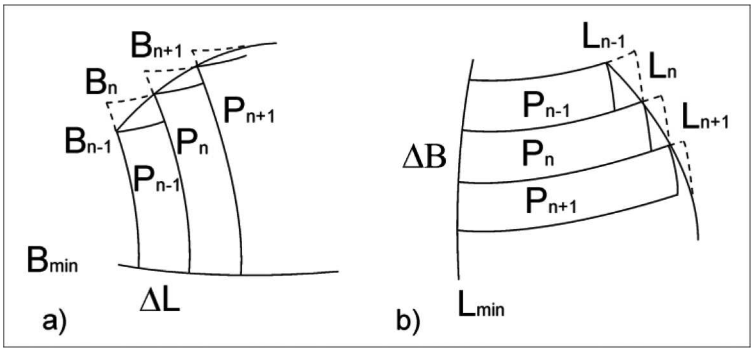


Fig. 1. Division of the area into meridian-and-parallel trapezoids (Balcerzak and Pędzich, 2006)

The basic issue in this method is to investigate the location of the geodesic line and to build an appropriate series of divisions of the given arc of the geodesic line.

The knowledge of methods of transferring co-ordinates on the ellipsoid, as well as methods of determination of geodesic co-ordinates of the point of turn of the geodesic line, is required.

When determining areas of curvilinear ellipsoidal polygons, it should be noticed that – besides areas of meridian-and-parallel trapezoids, they also include areas of triangles $\Delta P_n$ located between the upper base of meridian or parallel trapezoid and the geodesic line. In order to calculate the area of such a triangle, the area of meridian or parallel trapezoid $\Delta \mathrm{P}_{n,n+1}$, limited by parallels $B_n$ and $B_{n+1}$ and then the area $\Delta \mathrm{P}_n$ of a curvilinear triangle should be calculated, from the approximate formula

$$\Delta P_n = \frac{N_n \cos B_n}{N_n \cos B_n + N_{n+1} \cos B_{n+1}} \Delta P_{n,n+1} \tag{12}$$

Balcerzak and Pędzich (2006) also proposed to apply the method consisting of application of the equal-area projection of the ellipsoid onto a sphere.

The basis of this variant is application of the projection function of the form:

$$\phi(B) = \arcsin\left\{\frac{a^2\left(1-e^2\right)}{2R^2}\left[\frac{\sin B}{1-e^2\sin^2 B} + \frac{1}{2e}\ln\left(\frac{1+e\sin B}{1-e\sin B}\right)\right]\right\}, \quad \lambda = L. \tag{13}$$

The radius $R$ of the sphere is selected in such a way that the parallel $B_S$ crossing the centre of the geodesic polygon is isometrically projected. Value of radius is determined from the relation:

$$\left(R^2\right)^2 - \left(N^2\cos^2 B_S\right)R^2 - \frac{1}{4}a^4\left(1-e^2\right)^2\left[\frac{\sin B_S}{1-e^2\sin^2 B_S} + \frac{1}{2e}\ln\left(\frac{1+e\sin B_S}{1-e\sin B_S}\right)\right]^2 = 0 \tag{14}$$

The geodesic ellipsoidal polygon is substituted by the spherical geodesic polygon and then apex angles are determined in this polygon $\alpha_i$ i=1,2,3...n, where n is the number of vertices. The area of the spherical polygon is calculated according to the following formula:

$$P = \left(\sum_{i=1}^{n} \alpha_i - \pi(n-2)\right)R^2 \tag{15}$$

## 3. Application of equal-area map projections to area calculations

The authors of this paper propose to apply equal-area map projections of the ellipsoid into a plane for calculations of areas of geodesic polygons. Then it is necessary to transform co-ordinates of vertices of geodesic polygons, from the reference ellipsoid onto the plane of the equal-area projection. When calculating areas of polygons, which sides are sections of straight lines, it should be remembered in various equal-area projections, that – although we deal with equal-area projections – high differences may be obtained between areas of polygons, based on the same vertices and calculated in those projections (Pędzich and Kuźma 2010). This results from the fact that in map projections the geodesics are projected onto certain curve lines, but not onto straight lines. Therefore, using equal-area projections for area calculations of those polygons, it should be realised that the area of an ellipsoidal geodesic polygon is preserved, but its corresponding, image polygon is a certain curvilinear polygon. In order to obtain convergence of calculated areas between the original polygon and its image in cartographic projections, in the projection plane, areas of curvilinear polygons should be calculated. Thus, it is necessary to consider reductions of areas, resulting from curvilinearity of images of geodesic lines in cartographic projections. This problem may be also solved by shortening the distance between vertices and by selection of projection parameters, in order to minimise projection deformations and thus the value of area reductions (Pędzich and Kuźma 2011).

Authors of the paper used a Witkowski's conical projections for area calculation. Conic map projection based on Witkowski's criterion is determined from the equation:

$$\vec{r} = \begin{bmatrix} x = \gamma(B_0, c) - \rho(B, c)\cos\left[c(L - L_0)\right] \\ y = \rho(B, c)\sin\left[c(L - L_0)\right] \end{bmatrix}, c \in (0, 1) \tag{16}$$

where:

$L_0$ – fixed value of the parameter L (central meridian of a given area),

$B_0$ – fixed value of the parameter B (central parallel of a given area),

$c$ – constant of conic projection, $c$ is between (0,1).

$$\rho = \sqrt{\frac{2C}{c} - \frac{2}{c}S(B)} \tag{17}$$

$$S(B) = \frac{a^2(1-e^2)}{2e}\left(\frac{\sin B}{1 - e^2\sin^2 B} + \frac{1}{2e}\ln\left|\frac{1 + e\sin B}{1 - e\sin B}\right|\right) \tag{18}$$

Constant $c$ and $C$ (constant of integration) are described the following formulas:

$$C = \frac{N_N^2\cos^2 B_N S(B_S) - N_S^2\cos^2 B_S S(B_N)}{N_N^2\cos^2 B_N - N_S^2\cos^2 B_S} \tag{19}$$

$$c = \frac{2N_S\cos B_S\ N_0\cos B_0}{N_0\cos B_0\,(C - S(B_S)) + N_S\cos B_S\,(C - S(B_0))} \tag{20}$$

where: $B_N$ and $B_S$ – fixed value of the parameter $B$ (maximum and minimum value of latitude for a given area).

The projection reduction of an area $\Delta F$ between the image of an arc of the geodesic line, considered as the circular arc and the section of the straight line, connecting the points $P_1'P_2'$ is expressed by the formula:

$$\Delta F = \frac{s''^2}{4}\left(\frac{\delta - 0.5\sin 2\delta}{\sin^2\delta}\right) \tag{21}$$

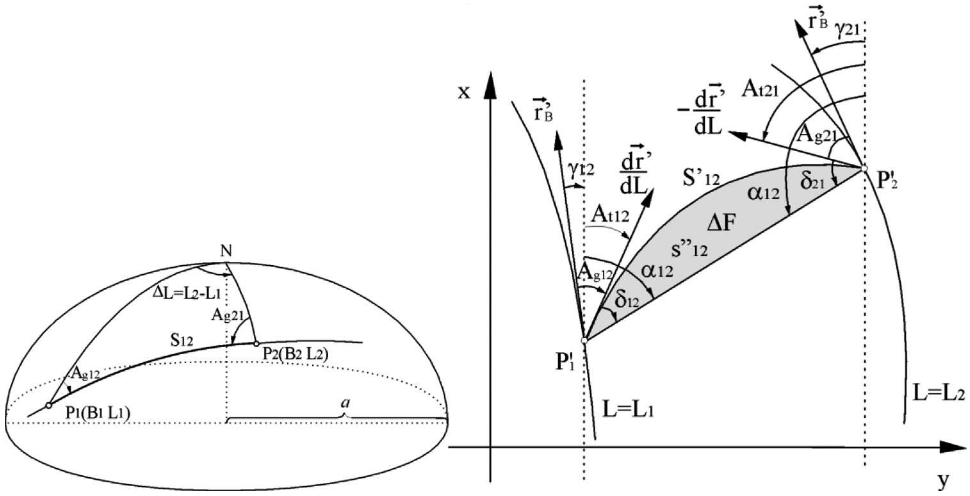


Fig. 2. Reduction of an area $\Delta F$ (Balcerzak and Pędzich, 2006)

where $s''$ – the chord length $P_1$' $P_2$' and $\delta = 0.5(\delta_{12} + \delta_{21})$, $\delta_{12}$ and $\delta_{21}$ are the, so-called, reduction angles, i.e. angles between the image of the arc of the geodesic line and the straight line section, connecting the points $P'_1 P'_2$ (Fig. 2).

It is the approximate formula, which results from approximation of the image of a geodesic line by the circular arc. It is applicable for short (1-2 kilometres) sections of geodesic lines (Pędzich Kuźma 2011).

The additional task in this method is to calculate reduction angles. The knowledge of methods of transferring co-ordinates along the geodesic line, in the reverse aspect, is required in this case.

## 4. Methods of calculations of areas of the Polish administrative units

In Poland information concerning areas of the basic, three-level territorial division of the country and of administrative units, as well as sea areas of the Republic of Poland are stored in the State Register of Borders (PRG). Areas are calculated on the surface of the GRS80 ellipsoid, spatial data of the PRG are stored in the State Coordinate System "1992" and in the geodetic reference system GRS80.

In Poland there are used 2 coordinate systems: State Coordinate System "1992" and State Coordinate System "2000". Both are based on Gauss-Krűger projection. "2000" system consists of four 3-degrees meridional zones and it is applied in geodetic surveying. "1992" system consists of one zone for entire area of Poland and is applied to topographic maps and topographic data bases.

The method of area calculation applied in the PRG (Radzio, 2006) consists of division of a given area in the plane of the 1992 System into squares – blocks of 1 km sides. The size of the given area is calculated as the sum of squares, which are completely covered by the given area (S_inner=1000·1000· $Q$ in $m^2$) and the sum of parts of squares limited by the border of the area (S_edge=1000·1000·$Q$·$Z$ in m$^2$), where $Q$ – inverse scale of area, due to distortion and $Z$ – a part of the given block, covered by the area.

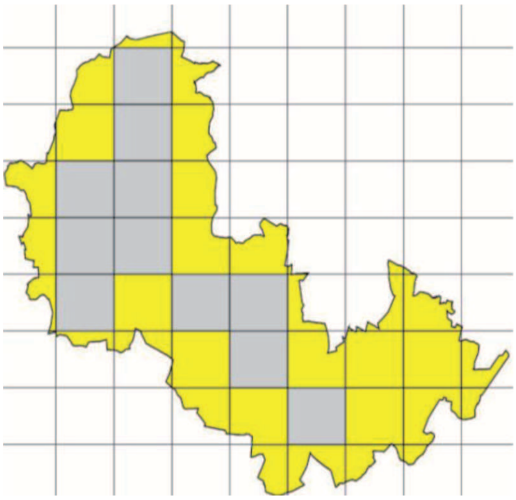

Fig. 3. Division of the area into the grid of squares of 1 km sides

$$Q = \frac{1}{m^2} \tag{22}$$

where $m$ – a scale of linear distortion is calculated as the mean value in four vertices and in the square centre.

The size of the given area, calculated in this way, is stored in the PRG and made accessible by the Main Statistical Office (GUS) in statistical annals.

Another way of area calculation is presented in the *G-5 Instruction. The lands and buildings register.* (based on Technical Guidelines G-1.10, item 5.9); it should be specified in hectares, with the accuracy of 0,0001ha. The area ($P$), as the part of the surface of the GRS80 ellipsoid, is expressed by the following formula:

$$P = P_0 - \Delta P_0 \tag{23}$$

where $P_0$ – the area calculated based on rectangular, plane co-ordinates in the 2000 System, $\Delta P_0$ – the area projection correction.

$$P_0 = \frac{1}{2}\sum_{i=1}^{n} X_i\,(Y_{i+1} - Y_{i-1}) \tag{24}$$

$$\Delta P_0 = P_0 \cdot \tau \tag{25}$$

$\tau$ – is a coefficient of area correction.

$$\tau = 1 - m^2 \tag{26}$$

$m$ – a scale of linear distortion.

The area of the register unit is the sum of areas of parcels included in this register unit.

In presented methods of calculations of areas of administrative units, the issue of projection reduction of an area $\Delta F(21)$ between the image of an arc of the geodesic line and the section of the straight line is neglected.

## 5. Comparison of obtained results of area calculations for selected Polish administrative units

In this task, areas of selected administrative units, of various sizes, have been calculated; they are: Warszawa, Gostynin District, Lubelskie Province, Małopolskie Province and Lubuskie Province. Methods described in the paper have been used for calculations; they are: the method of ellipsoidal trapezoids (M1) proposed by Balcerzak and Pędzich, the Karney method (M2), the methods based on the projection of the ellipsoid onto a sphere (M3) proposed by Balcerzak and Pędzich, the method based on the equal-area, conical projection of the ellipsoid onto a plane according to Witkowski criterion, in two variants: without reduction (M4) and with reductions (M5). Results were also compared with the values of areas specified in statistical annals for 2011 (Table 1).

Table 1. Results of area calculation for Polish administrative units

| **Gostynin District** | Area, following the annals for 2011<br>61481 ha |
|---|---|
| M1 | 61 480.6575 ha |
| M2 | 61 480.6533 ha |
| M3 | 61 487.7860 ha |
| M4 | 61 480.6540 ha |
| M5 | 61 480.6540 ha |
| Maximum distance between points | 952 m |
| Mean distance between points | 65 m |
| Number of points | 3013 |
| **Lubelskie Province** | Area, following the annals for 2011<br>2 512 246 ha |
| M1 | 2 512 248.1086 |
| M2 | 2 512 248.0400 |
| M3 | 2 512 312.0283 |
| M4 | 2 512 248.0630 |
| M5 | 2 512 248.0640 |
| Maximum distance between points | 1340 m |
| Mean distance between points | 45 m |
| Number of points | 26265 |

| **Lubuskie Province** | Area, following the annals for 2011<br>1 398 788 ha |
|---|---|
| M1 | 1 398 790.4872 |
| M2 | 1 398 790.4775 |
| M3 | 1 398 801.0100 |
| M4 | 1 398 790.4846 |
| M5 | 1 398 790.4848 |
| Maximum distance between points | 1 772 m |
| Mean distance between points | 68 m |
| Number of points | 13 860 |
| **Małopolskie Province** | Area, following the annals for 2011<br>1 518 279 ha |
| M1 | 1 518 286.3925 |
| M2 | 1 518 286.4140 |
| M3 | 1 518 274.3232 |
| M4 | 1 518 286.3995 |
| M5 | 1 518 286.3993 |
| Maximum distance between points | 1 126 m |
| Mean distance between points | 35 m |
| Number of points | 27 946 |
| **Warsaw** | Area, following the annals for 2011<br>51 724 ha |
| M1 | 51 721.5076 |
| M2 | 51 721.8198 |
| M3 | 51 722.0255 |
| M4 | 51 721.6252 |
| M5 | 51 721.6248 |
| Maximum distance between points | 4 158 m |
| Mean distance between points | 75 m |
| Number of points | 1 797 |

The following conclusions may be drawn based on the obtained results:

1. The biggest differences occur between the results obtained using the method based on projection of the ellipsoid onto the sphere and other results of other methods. Therefore we may consider this solution as less accurate and propose to improve the discussed method.
2. Differences between results obtained by other methods do not exceed 1 hectare. The higher convergence of results could be obtained by the division of geodesic lines into smaller parts.
3. Differences are smaller for smaller areas and shorter distances between points. The example of Warszawa (the smallest area and the longest distances between points)

shows, that results of calculations are highly influenced by the distances between points.

4. Values of projection reductions are small in the case of the selected Witkowski's conical projection – they reach several square metres; they may be neglected, when the high accuracy of results is not required.
5. Differences between calculated areas and data acquired from statistical annals reach several hectares. They may be the result of many reasons: data acquired in 2005 from digitization of topographic maps at 1:10 000 scale were used by authors for calculations. Due to the high costs of more accurate data from the State Register of Borders were not used. Besides, it may be expected that methods applied in the State Register of Borders may also lead to results which differ from the real areas of polygons.

## 6. Concluding remarks

Performed calculations prove the high difficulty of area calculations on the ellipsoid. Considerable discrepancies between obtained results may be observed. Applied algorithms allow for obtaining approximate results only. It is a very difficult task to estimate the accuracy of particular methods. On the other hand, many interesting conclusions may be drawn based on comparison of obtained results.

Utilisation of the equal-area projection of the ellipsoid onto a plane seems to be the best solution for calculation of polygons with short sides (such as administrative units). In the case of small distances between points we may expect accurate results of calculations, since the area size is influenced by the projection reductions only (Pędzich and Kuźma 2011), which are small in such cases. In some cases their influence on results of calculations may be neglected (for example in statistical annals areas are specified with the accuracy of 1 hectare). Then, only re-calculation of co-ordinates from the GRS80 ellipsoid to the cartographic, equal-area projection is required.

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## Zastosowanie metod obliczania pól powierzchni wieloboków geodezyjnych na przykładzie jednostek administracyjnych w Polsce

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**Streszczenie**

Umiejętność obliczania pól wieloboków geodezyjnych, czyli takich, których bokami są odcinki linii geodezyjnych, ma istotne znaczenie w praktyce geodezyjnej i kartograficznej. Jednym z podstawowych zadań wykonywanych przez geodetów i kartografów jest obliczanie pól różnych obiektów powierzchniowych takich jak gmina, województwo, obszary użytków gruntowych itp. Jeżeli zadanie sprowadza się tylko do powierzchni kuli lub płaszczyzny to rozwiązanie jest stosunkowo łatwe. Zadanie komplikuje się, jeżeli za powierzchnię odniesienia fizycznej powierzchni Ziemi przyjmiemy elipsoidę obrotową, ponieważ nie ma ścisłych wzorów, które pozwalałyby na realizację takiego zadania; są jedynie wzory przybliżone mające zastosowanie dla niewielkich obszarów. Trudności pojawiają się szczególnie w przypadku dużych wieloboków zlokalizowanych na elipsoidzie obrotowej spłaszczonej.

W artykule przedstawiono metody obliczania pól powierzchni, które mogą być stosowane dla dużych wieloboków geodezyjnych na elipsoidzie. Opisano propozycje autorów niniejszego artykułu. Zaproponowane sposoby skonfrontowano z innymi alternatywnymi metodami obliczania pól powierzchni tych wieloboków. Obliczenia testowe przeprowadzono dla jednostek administracyjnych obszaru Polski. Otrzymane wyniki porównano z powierzchniami tych jednostek zapisanymi w rocznikach statystycznych.